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THE $\dot{H}$-JOIN OPERATION OF SIGNED GRAPHS CONSTRAINED BY INDEXING MAPS

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Abstract. Let H be a graph of order k and let $\mathcal{F} = \{G_1, G_2, \ldots, G_k\}$ be a family of k graphs. Then the H-join of the family $\mathcal{F}$ is obtained by replacing each vertex v_i of H with the graph G_i of $\mathcal{F}$ and respecting the adjacencies existing in H. To generalise this graph operation we consider a signed variant with the addition of fixing $m \in \mathbb{N}$ and introducing indexing maps to define the $\dot{H}_m$-join. Once having done so we can determine the characteristic polynomials and spectra of the compound graphs produced as pertaining to many graph matrices, such as the adjacency, Laplacian, universal adjacency, net Laplacian, and A_α. Furthermore we show that the $\dot{H}_m$-join remains stable under switching of $\dot{H}$.

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1. Introduction

A signed graph is a generalisation of a simple graph where the edges are labelled as either positive or negative. Let $G = (V, E)$ be an ordinary unsigned graph on n vertices and denote by V and E its vertex set and its edge set, respectively. In formal terms, a signed graph denoted by $\dot{G} = (G, \sigma)$ is a pair consisting of the graph G, which we usually call the underlying graph, alongside a signature map $\sigma : E \to \{\pm 1\}$. When an edge between two vertices $u, v \in V$ exists, that is, $uv \in E$, we write $u \sim v$. A signed graph has an edge set which is made up of subsets of positive and negative edges. We think of an unsigned graph G as the all-positive signed graph $\dot{G} = (G, +)$ whose signature assigns the value $+1$ to all of its edges. In the same way, by $\dot{G} = (G, -)$ we are denoting a signed graph whose signature assigns the value -1 to all of its edges. In general we have $-\dot{G} = (G, -\sigma)$. The graphs to be considered in this paper shall all be simple.

Informally, for ordinary graphs, if we have a graph H on k vertices along with a set $\{G_i : 1 \leq i \leq k\}$ of graphs, one for each vertex of H, then the H-join operation is performed by substituting the vertex v_i in H for the graph G_i and making adjacencies between these newly introduced graphs precisely where they existed previously in H. For signed graphs this operation is known as the $\dot{H}$-join and the basic idea remains the same, only with the added step of respecting the signatures of the relevant graphs. We shall be exploring a version of this operation on signed graphs which is constrained by a set of indexing maps.

Keywords. Signed graphs, adjacency spectrum, Laplacian, H-join, universal adjacency.

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The adjacency matrix of a signed graph $\dot{G} = (G, \sigma)$ on n vertices is a square matrix of order n that is denoted by $A_{\dot{G}} = (a_{ij})_{n\times n}$ and defined by

$$a_{ij} = \begin{cases} \sigma(e_{ij}) & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

Generally speaking, the spectrum of $\dot{G}$ consists of the eigenvalues of its adjacency matrix $A_{\dot{G}}$ and when we say eigenvalues of $\dot{G}$ we are also referring to the eigenvalues of $A_{\dot{G}}$. In this paper however, since we consider several different graph matrices, it may be necessary to specify that these particular eigenvalues are the adjacency eigenvalues, making up the adjacency spectrum.

Let the degree of vertex v_i in G be represented by $d_i = d_G(v_i)$, then the $n \times n$ diagonal matrix with entries $d_1, d_2, \ldots, d_n$ is known as the degree matrix of G and is denoted by D_G. For a signed graph $\dot{G} = (G, \sigma)$, the Laplacian matrix $L_{\dot{G}}$ is determined by $L_{\dot{G}} = D_G - A_{\dot{G}}$.

A vertex v_i in a signed graph $\dot{G}$ is said to have net degree $d_i^{\pm} = d_{\dot{G}}^{\pm}(v_i)$ and this is defined by the equation $d_{\dot{G}}^{\pm}(v_i) = d_{\dot{G}}^{+}(v_i) - d_{\dot{G}}^{-}(v_i)$ with $d_{\dot{G}}^{+}(v_i)$ and $d_{\dot{G}}^{-}(v_i)$ respectively denoting the total number of positive edges and the total number of negative edges incident with v_i. We define the net degree matrix $D_{\dot{G}}^{\pm}$ in the natural way, as the $n \times n$ diagonal matrix with entries $d_1^{\pm}, d_2^{\pm}, \ldots, d_n^{\pm}$. Hence, for a signed graph $\dot{G} = (G, \sigma)$, by $L_{\dot{G}}^{\pm} = D_{\dot{G}}^{\pm} - A_{\dot{G}}$ we define its net Laplacian matrix $L_{\dot{G}}^{\pm}$.

For an ordinary graph G, its universal adjacency matrix, denoted by U_G, is a linear combination of its adjacency matrix, the identity matrix, the all-one matrix, and its degree matrix along with real coefficients. Written explicitly, it is of the form $U_G = \alpha A_G + \beta I + \gamma J + \delta D_G$ with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha \neq 0$. Generalising this in the natural way gives us the universal adjacency matrix of a signed graph $\dot{G}$ as $U_{\dot{G}} = \alpha A_{\dot{G}} + \beta I + \gamma J + \delta D_G$ with the same conditions on the parameters. A multitude of graph matrices – such as the adjacency matrix, the Laplacian matrix, the A_α matrix as proposed by Nikoforov [11] – can be found as particular cases of the universal adjacency matrix.

For a matrix X of order n we denote its characteristic polynomial $\det(\lambda I_n - X)$ as $\phi_X(\lambda)$. For a signed graph $\dot{G}$ and a related graph matrix $M_{\dot{G}}$ we say that $\phi_{M_{\dot{G}}}(\lambda)$ is the M-characteristic polynomial of $\dot{G}$. The M-spectrum of $\dot{G}$ is the set of M-eigenvalues, as derived from the M-characteristic polynomial, with their multiplicities.

An important concept in signed graph theory depends upon switching operations. If we let $\dot{G}$ be an arbitrary signed graph and take a subset U of its vertices, so $U \subseteq V(G)$, then we can obtain from $\dot{G}$ a signed graph $\dot{G}'$ by reversing the signature placed upon the edges which have one endpoint in U and the other in $V \setminus U$. Two such signed graphs $\dot{G}$ and $\dot{G}'$, as well as their signatures σ and σ', are referred to as switching equivalent and we write $\dot{G} \sim \dot{G}'$.

It is a widely known fact that two signed graphs $\dot{G}_1 = (G_1, \sigma_1)$ and $\dot{G}_2 = (G_2, \sigma_2)$ on the same vertex set are switching equivalent if and only if their adjacency matrices satisfy $A_{\dot{G}_2} = S^{-1} A_{\dot{G}_1} S$ for some diagonal matrix S, known as the switching matrix, whose diagonal has entries only of either $+1$ or -1. Performing the operation of switching on a signed graph does not change its spectrum because there is no change made to the spectrum of a matrix after it has been conjugated. Consequently, any two signed graphs that are switching equivalent have exactly the same characteristic polynomials, eigenvalues, and spectra.

For any results or concepts concerning graph spectra or signed graphs used in this paper which have not been clearly stated the reader is directed to [9, 16].

In the simple setting many results relating to the characteristic polynomials and spectra of graphs produced by an H- or H-generalised join have been obtained. The first iteration of the operation we are studying was introduced as the generalised composition of graphs by Schwenk in 1974 in [13]. As an application of generalising Fiedler's lemma but without explicitly defining this procedure, Cardoso *et al.* found the adjacency spectra of the H-join, limited to H being a path graph, of a set of regular graphs in [3] in 2011. In 2013, Cardoso *et al.* reintroduced this operation formally as the H-join of graphs in [4], in which they also derived the spectra of the H-joins of families of regular graphs with H now being any graph. Also in 2013, in [5], a variation on the operation was presented as the H-generalised join of graphs; here the adjacencies between component graphs are restricted according to vertex subsets, and the spectra resulting from this operation were studied in the same paper and also in the paper [12] from 2021. Both the signless Laplacian and normalised Laplacian spectra of a graph procured from the performance of an H-join operation were determined by Wu *et al.* in [15]. The lexicographic product of two graphs is a special case of the H-join, where every G_i is the same, and their polynomials and spectra were studied in 2017, in [6]. In [7] the authors were able to characterise the universal adjacency spectrum of a graph obtained by an H-join operation, this time pertaining to a family of arbitrary graphs. More recently, Arunkumar and Ganeshbabu proposed a further variation on this operation, this time constraining adjacencies according to indexing maps on the vertices of the component graphs, and determined many results concerning several graph matrices and their respective characteristic polynomials and spectra [1].

Literature relating to the signed versions of these operations exists too albeit in a lesser quantity. In 2019, Brunetti *et al.* computed both the adjacency and Laplacian spectra of signed graphs garnered from the lexicographic product operation in [2]. For the $\dot{H}$-join operation on signed graphs, the adjacency and Laplacian spectra were characterised by Zhang *et al.* in [17] in 2021. With [10] in 2023, Li *et al.* produced signed graphs by performing the $\dot{H}$- and the $\dot{H}$-generalised join operations, and for various graph matrices they were able to determine their characteristic polynomials and spectra. Particularly, this saw the introduction of the universal adjacency matrix of a signed graph.

The remainder of this paper is structured in the following fashion. In Section 2 we shall assemble some preliminary facts and results for later use. In Section 3 we define the generalised $\dot{H}$-join operation constrained by indexing maps and determine the adjacency characteristic polynomial and spectrum as well as showing that the operation is stable under switching of $\dot{H}$. In Section 4 we investigate the universal adjacency matrix along with the net Laplacian matrix. In Section 5 we provide a brief summary of the results obtained in this paper.

2. Preliminaries

We shall use the following notations throughout the rest of this paper. The identity matrix is denoted by I_n when it is of order n or simply by I if the relevant size is clear. We denote the all-one matrix by $J_{m\times n}$ if it is an $m \times n$ matrix, by J_n if it is square of order n, or just by J. Likewise we denote the all-zero matrix by $O_{m\times n}$ if it is an $m \times n$ matrix, or by O_n or just O.

The following lemma is known as Schur's complement formula and appears in many textbooks, such as [9].

Lemma 2.1. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be a block matrix. Let A and D be square matrices. It follows that*

(a) *if A is an invertible matrix,* $\det(M) = \det(A)\det(D - CA^{-1}B)$*,*
(b) *if D is an invertible matrix,* $\det(M) = \det(D)\det(A - BD^{-1}C)$*.*

The subsequent result comes immediately from applying Schur's complement formula to particular matrices.

Lemma 2.2 ([1], Lem. 3)**.** *Let X and Y be $n \times m$ matrices and let M be an $n \times n$ invertible matrix. Then*

(a) $\det(I_n + XY^T) = \det(I_m + Y^T X)$*.*
(b) $\det(M + XY^T) = \det(M)\det(I_m + Y^T M^{-1} X)$*.*

A key element of the $\dot{H}$-join operation constrained by indexing maps is, unsurprisingly, the indexing maps. We will define these and their corresponding matrices as follows.

Definition 2.3. For a signed graph $\dot{G} = (G, \sigma)$ with vertex set $\{v_1, v_2, \dots, v_n\}$ and an indexing map $M : V(G) \to \{1, 2, \dots, m\}$, the corresponding $n \times m$ indexing matrix is denoted by $E = (e_{ij})_{n\times m}$ and defined by

$$e_{ij} = \begin{cases} 1 & \text{if } M(v_i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

The idea of the main function of a matrix was introduced by Saravanan *et al.* in [12] in order to generalise Fiedler's lemma in a new way. It is defined as follows.

Definition 2.4 ([12], Def. 1)**.** Let M be an $n \times n$ complex matrix and let X and Y be $n \times m$ complex matrices. The main function associated to the matrix M corresponding to X and Y is defined as

$$\Omega_M(X, Y) := Y^T(\lambda I_n - M)^{-1}X \in M_m(\mathbb{C}(\lambda)).$$

When $X = Y$ we simply write $\Omega_M(X, Y) = \Omega_M(X)$.

Let λ be an eigenvalue of the matrix $M(G)$ that is in correspondence with a graph G. When the eigenspace $\xi_{M(G)}(\lambda)$ is not orthogonal to $\mathbf{1}_n$, the vector of all ones of length n, λ is said to be a main eigenvalue. If this is not the case then it is called non-main. This notion was generalised by the authors in [1] to what they call E-main eigenvalues, with E being a rectangular matrix. This will go on to play a significant role in this paper.

Definition 2.5 ([1], Def. 3)**.** Let M be a square normal complex matrix of order n and let E be an $n \times m$ complex matrix. An eigenvalue λ of M is an E-main eigenvalue if its associated eigenspace $\xi_M(\lambda)$ is not orthogonal to the span of the columns of E. Otherwise, it is an E-non-main eigenvalue.

3. $\dot{H}_m$-join operation

We shall begin with the definition of the $\dot{H}_m$-join and after this we will detail both the structuring of its adjacency matrix and the computation of its spectrum. To conclude this section, a demonstration of the stability of the $\dot{H}_m$-join under switching of $\dot{H}$ is presented.

3.1. Definition

Definition 3.1. Let $\dot{H} = (H, \sigma_H)$ be a signed graph with vertex set $V(H) = \{v_i : 1 \leq i \leq k\}$ and let $\mathcal{F} = \{\dot{G}_i : 1 \leq i \leq k\}$, with $\dot{G}_i = (G_i, \sigma_i)$ for each i, be a set of signed graphs with vertex sets $V(G_i) = \{v_i^1, v_i^2, \ldots, v_i^{n_i}\}$. Fix $m \in \mathbb{N}$ and let $\mathcal{M} = \{M_i : 1 \leq i \leq k\}$ be a set of indexing maps $M_i : V(G_i) \to [m]$. Then the $\dot{H}_m$-join of the set of signed graphs $\mathcal{F}$, to be denoted with $\bigvee_{\dot{H}}^{\mathcal{F}\mathcal{M}}$, is the graph produced by first replacing the ith vertex of $\dot{H}$ with $\dot{G}_i$ and then, if $v_i \sim v_j$ in $\dot{H}$, making every pair (u, v) of vertices with the properties $u \in V(G_i)$, $v \in V(G_j)$, $M_i(u) = M_j(v)$ to be adjacent to one another in $\bigvee_{\dot{H}}^{\mathcal{F}\mathcal{M}}$ such that the signature σ of the compound graph satisfies

$$\sigma(uv) = \begin{cases} \sigma_i(uv) & \text{if } u, v \in V(G_i),\ u \sim v \text{ in } G_i, \\ \sigma_H(v_iv_j) & \text{if } u \in V(G_i),\ v \in V(G_j),\ M_i(u) = M_j(v),\ v_i \sim v_j \text{ in } H. \end{cases}$$

An example of such a graph obtained by this operation is provided in Figure 1.

3.2. Adjacency matrix and spectrum

Now we are able to define the adjacency matrix of a signed graph resulting from the $\dot{H}_m$-join procedure and subsequently determine its characteristic polynomial along with some facts about its spectrum.

Definition 3.2. Consider the signed graph $\dot{G} = \bigvee_{\dot{H}}^{\mathcal{F}\mathcal{M}}$. Denote the adjacency matrix of each signed graph $\dot{G}_i$ by $A_{\dot{G}_i}$, and let

$$\omega_{i,j} = \begin{cases} \sigma_H(v_iv_j) & \text{if } v_i \sim v_j \text{ in } H, \\ 0 & \text{otherwise.} \end{cases}$$

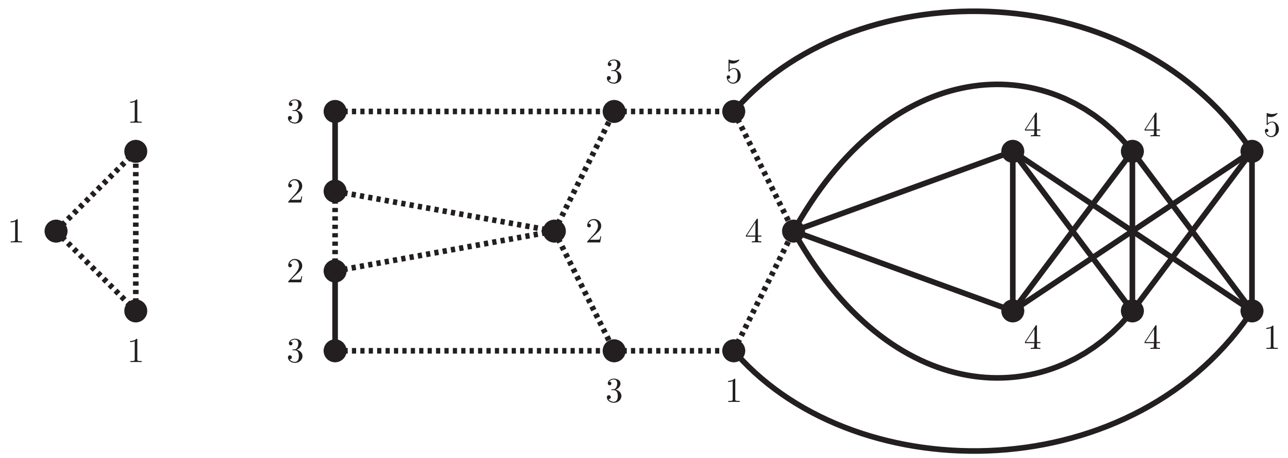

FIGURE 1. Let $\dot{H} = (P_4, \sigma_H)$ with v_1 adjacent to v_2, v_2 to v_3, v_3 to v_4, and let $\sigma_H(v_1v_2) = \sigma_H(v_3v_4) = 1$ and $\sigma_H(v_2v_3) = -1$. Then the diagram depicts the $\dot{H}_5$-join of the set $\{(C_3,-),(P_4,\sigma_H),(C_6,-),(K_{3,3},+)\}$ with respect to the indexing maps M_1, M_2, M_3, M_4 whose values are labelled on the vertices. Here and throughout, solid lines are used to represent positive edges and dashed lines are used to represent negative edges.

For all $1 \le i \le k$, let the map M_i correspond to the $n_i \times m$ indexing matrix E_i of $\dot{G}_i$. Then, when suitably ordering the vertices, the adjacency matrix of $\dot{G}$ is

$$A_{\dot{G}} = \begin{pmatrix} A_{\dot{G}_1} & \omega_{1,2}E_1E_2^T & \cdots & \omega_{1,k}E_1E_k^T \\ \omega_{2,1}E_2E_1^T & A_{\dot{G}_2} & \cdots & \omega_{2,k}E_2E_k^T \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{k,1}E_kE_1^T & \omega_{k,2}E_kE_2^T & \cdots & A_{\dot{G}_k} \end{pmatrix}.$$

Theorem 3.3. *Consider a signed graph $\dot{G} = \bigvee_{\dot{H}}^{\mathcal{FM}}$, where $\omega_{i,j}$ is defined as in* Definition 3.2 *for all $1 \le i,j \le k$. Define $\phi_i = \det(\lambda I_{n_i} - A_{\dot{G}_i})$ as the adjacency characteristic polynomial of $\dot{G}_i$ for each $1 \le i \le k$. Let $n = \sum_{i=1}^k n_i$, and denote by $\Omega_i = \Omega_{A_{\dot{G}_i}}(E_i)$. Then, the adjacency characteristic polynomial of $\dot{G}$ is*

$$\det(\lambda I_n - A_{\dot{G}}) = \left(\prod_{i=1}^k \phi_i\right) \det\left(\tilde{A}\right),$$

where

$$\tilde{A} = \begin{pmatrix} I_m & -\omega_{1,2}\Omega_1 & \cdots & -\omega_{1,k}\Omega_1 \\ -\omega_{2,1}\Omega_2 & I_m & \cdots & -\omega_{2,k}\Omega_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\omega_{k,1}\Omega_k & -\omega_{k,2}\Omega_k & \cdots & I_m \end{pmatrix}.$$

Proof. From Definition 3.2 we have

$$A_{\dot{G}} = \begin{pmatrix} A_{\dot{G}_1} & \omega_{1,2}E_1E_2^T & \cdots & \omega_{1,k}E_1E_k^T \\ \omega_{2,1}E_2E_1^T & A_{\dot{G}_2} & \cdots & \omega_{2,k}E_2E_k^T \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{k,1}E_kE_1^T & \omega_{k,2}E_kE_2^T & \cdots & A_{\dot{G}_k} \end{pmatrix}.$$

We will prove this by induction on k. In the case of $k = 2$ we can directly use Lemma 2.1 in order to obtain

$$\begin{aligned} \phi_{A_{\dot{G}}}(\lambda) &= \det \begin{pmatrix} \lambda I_{n_1} - A_{\dot{G}_1} & -\omega_{1,2}E_1E_2^T \\ -\omega_{2,1}E_2E_1^T & \lambda I_{n_2} - A_{\dot{G}_2} \end{pmatrix} \\ &= \det\left(\lambda I_{n_2} - A_{\dot{G}_2}\right) \det\left(\left(\lambda I_{n_1} - A_{\dot{G}_1}\right) - \omega_{1,2}E_1E_2^T\left(\lambda I_{n_2} - A_{\dot{G}_2}\right)^{-1}\omega_{2,1}E_2E_1^T\right) \\ &= \phi_2 \cdot \det\left(\left(\lambda I_{n_1} - A_{\dot{G}_1}\right) - \omega_{1,2}\omega_{2,1}E_1\Omega_2E_1^T\right). \end{aligned}$$

Applying Lemma 2.2(b) with $M = \lambda I_{n_1} - A_{\dot{G}_1}$, $X = E_1\Omega_2$, and $Y = E_1$, we then have

$$\begin{aligned} \phi_{A_{\dot{G}}}(\lambda) &= \phi_2 \cdot \det\left(\lambda I_{n_1} - A_{\dot{G}_1}\right) \det\left(I_m - \omega_{1,2}\omega_{2,1}E_1^T\left(\lambda I_{n_1} - A_{\dot{G}_1}\right)^{-1}E_1\Omega_2\right) \\ &= \phi_1 \cdot \phi_2 \cdot \det\left(I_m - \omega_{1,2}\omega_{2,1}\Omega_1\Omega_2\right). \end{aligned}$$

In reversing Schur's complement formula we can see that this in fact gives

$$\det\left(\lambda I_n - A_{\dot{G}}\right) = \phi_1 \cdot \phi_2 \cdot \det \begin{pmatrix} I_m & -\omega_{1,2}\Omega_1 \\ -\omega_{2,1}\Omega_2 & I_m \end{pmatrix} = \left(\prod_{i=1}^{2} \phi_i\right) \det(\tilde{A}).$$

Hence the claim is true for $k = 2$.

Now for $k \geq 3$. By Schur's complement formula we obtain

$$\begin{aligned} \det\left(\lambda I_n - A_{\dot{G}}\right) &= \det \begin{pmatrix} \lambda I_{n_1} - A_{\dot{G}_1} & -\omega_{1,2}E_1E_2^T & \cdots & -\omega_{1,k}E_1E_k^T \\ -\omega_{2,1}E_2E_1^T & \lambda I_{n_2} - A_{\dot{G}_2} & \cdots & -\omega_{2,k}E_2E_k^T \\ \vdots & \vdots & \ddots & \vdots \\ -\omega_{k,1}E_kE_1^T & -\omega_{k,2}E_kE_2^T & \cdots & \lambda I_{n_k} - A_{\dot{G}_k} \end{pmatrix} \\ &= \det\left(\lambda I_{n_k} - A_{\dot{G}_k}\right) \det(B) \end{aligned} \tag{1}$$

where

$$B = \begin{pmatrix} \lambda I_{n_1} - A_{\dot{G}_1} & -\omega_{1,2}E_1E_2^T & \cdots & -\omega_{1,k-1}E_1E_{k-1}^T \\ -\omega_{2,1}E_2E_1^T & \lambda I_{n_2} - A_{\dot{G}_2} & \cdots & -\omega_{2,k-1}E_2E_{k-1}^T \\ \vdots & \vdots & \ddots & \vdots \\ -\omega_{k-1,1}E_{k-1}E_1^T & -\omega_{k-1,2}E_{k-1}E_2^T & \cdots & \lambda I_{n_{k-1}} - A_{\dot{G}_{k-1}} \end{pmatrix}$$
$$- \begin{pmatrix} -\omega_{1,k}E_1E_k^T \\ -\omega_{2,k}E_2E_k^T \\ \vdots \\ -\omega_{k-1,k}E_{k-1}E_k^T \end{pmatrix} \left(\lambda I_{n_k} - A_{\dot{G}_k}\right)^{-1} \left(-\omega_{k,1}E_kE_1^T \quad -\omega_{k,2}E_kE_2^T \quad \cdots \quad -\omega_{k,k-1}E_kE_{k-1}^T\right).$$

From here we shall compute the determinant of B and show that we acquire the characteristic polynomial which we desire. To begin with, we have

$$B = \begin{pmatrix} \lambda I_{n_1} - A_{\dot{G}_1} & -\omega_{1,2}E_1E_2^T & \cdots & -\omega_{1,k-1}E_1E_{k-1}^T \\ -\omega_{2,1}E_2E_1^T & \lambda I_{n_2} - A_{\dot{G}_2} & \cdots & -\omega_{2,k-1}E_2E_{k-1}^T \\ \vdots & \vdots & \ddots & \vdots \\ -\omega_{k-1,1}E_{k-1}E_1^T & -\omega_{k-1,2}E_{k-1}E_2^T & \cdots & \lambda I_{n_{k-1}} - A_{\dot{G}_{k-1}} \end{pmatrix}$$
$$- \begin{pmatrix} \omega_{1,k}\omega_{k,1}E_1\Omega_kE_1^T & \omega_{1,k}\omega_{k,2}E_1\Omega_kE_2^T & \cdots & \omega_{1,k}\omega_{k,k-1}E_1\Omega_kE_{k-1}^T \\ \omega_{2,k}\omega_{k,1}E_2\Omega_kE_1^T & \omega_{2,k}\omega_{k,2}E_2\Omega_kE_2^T & \cdots & \omega_{2,k}\omega_{k,k-1}E_2\Omega_kE_{k-1}^T \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{k-1,k}\omega_{k,1}E_{k-1}\Omega_kE_1^T & \omega_{k-1,k}\omega_{k,2}E_{k-1}\Omega_kE_2^T & \cdots & \omega_{k-1,k}\omega_{k,k-1}E_{k-1}\Omega_kE_{k-1}^T \end{pmatrix}$$
$$= \begin{pmatrix} \lambda I_{n_1} - A_{\dot{G}_1} & O & \cdots & O \\ O & \lambda I_{n_2} - A_{\dot{G}_2} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \lambda I_{n_{k-1}} - A_{\dot{G}_{k-1}} \end{pmatrix}$$
$$- \begin{pmatrix} \omega_{1,k}\omega_{k,1}E_1\Omega_kE_1^T & E_1\big(\omega_{1,2} + \omega_{1,k}\omega_{k,2}\Omega_k\big)E_2^T & \cdots & E_1\big(\omega_{1,k-1} + \omega_{1,k}\omega_{k,k-1}\Omega_k\big)E_{k-1}^T \\ E_2\big(\omega_{2,1} + \omega_{2,k}\omega_{k,1}\Omega_k\big)E_1^T & \omega_{2,k}\omega_{k,2}E_2\Omega_kE_2^T & \cdots & E_2\big(\omega_{2,k-1} + \omega_{2,k}\omega_{k,k-1}\Omega_k\big)E_{k-1}^T \\ \vdots & \vdots & \ddots & \vdots \\ E_{k-1}\big(\omega_{k-1,1} + \omega_{k-1,k}\omega_{k,1}\Omega_k\big)E_1^T & E_{k-1}\big(\omega_{k-1,2} + \omega_{k-1,k}\omega_{k,2}\Omega_k\big)E_2^T & \cdots & \omega_{k-1,k}\omega_{k,k-1}E_{k-1}\Omega_kE_{k-1}^T \end{pmatrix}$$
$$= \begin{pmatrix} \lambda I_{n_1} - A_{\dot{G}_1} & O & \cdots & O \\ O & \lambda I_{n_2} - A_{\dot{G}_2} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \lambda I_{n_{k-1}} - A_{\dot{G}_{k-1}} \end{pmatrix}$$
$$- \left(\begin{pmatrix} \omega_{1,k}\omega_{k,1}E_1\Omega_k & E_1\big(\omega_{1,2} + \omega_{1,k}\omega_{k,2}\Omega_k\big) & \cdots & E_1\big(\omega_{1,k-1} + \omega_{1,k}\omega_{k,k-1}\Omega_k\big) \\ E_2\big(\omega_{2,1} + \omega_{2,k}\omega_{k,1}\Omega_k\big) & \omega_{2,k}\omega_{k,2}E_2\Omega_k & \cdots & E_2\big(\omega_{2,k-1} + \omega_{2,k}\omega_{k,k-1}\Omega_k\big) \\ \vdots & \vdots & \ddots & \vdots \\ E_{k-1}\big(\omega_{k-1,1} + \omega_{k-1,k}\omega_{k,1}\Omega_k\big) & E_{k-1}\big(\omega_{k-1,2} + \omega_{k-1,k}\omega_{k,2}\Omega_k\big) & \cdots & \omega_{k-1,k}\omega_{k,k-1}E_{k-1}\Omega_k \end{pmatrix} \cdot \begin{pmatrix} E_1^T & O & \cdots & O \\ O & E_2^T & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & E_{k-1}^T \end{pmatrix} \right).$$

Now we can apply Lemma 2.2(b) to obtain the determinant of B as

$$\det(B) = \left(\prod_{i=1}^{k-1}\phi_i\right)\cdot\det\left(I_{(k-1)m} - \begin{pmatrix} E_1^T & O & \cdots & O \\ O & E_2^T & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & E_{k-1}^T \end{pmatrix}\cdot\begin{pmatrix} \lambda I_{n_1} - A_{\dot{G}_1} & O & \cdots & O \\ O & \lambda I_{n_2} - A_{\dot{G}_2} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \lambda I_{n_{k-1}} - A_{\dot{G}_{k-1}} \end{pmatrix}^{-1}\right.$$

$$\left.\cdot\begin{pmatrix} \omega_{1,k}\omega_{k,1}E_1\Omega_k & E_1\left(\omega_{1,2}+\omega_{1,k}\omega_{k,2}\Omega_k\right) & \cdots & E_1\left(\omega_{1,k-1}+\omega_{1,k}\omega_{k,k-1}\Omega_k\right) \\ E_2\left(\omega_{2,1}+\omega_{2,k}\omega_{k,1}\Omega_k\right) & \omega_{2,k}\omega_{k,2}E_2\Omega_k & \cdots & E_2\left(\omega_{2,k-1}+\omega_{2,k}\omega_{k,k-1}\Omega_k\right) \\ \vdots & \vdots & \ddots & \vdots \\ E_{k-1}\left(\omega_{k-1,1}+\omega_{k-1,k}\omega_{k,1}\Omega_k\right) & E_{k-1}\left(\omega_{k-1,2}+\omega_{k-1,k}\omega_{k,2}\Omega_k\right) & \cdots & \omega_{k-1,k}\omega_{k,k-1}E_{k-1}\Omega_k \end{pmatrix}\right)$$

$$= \left(\prod_{i=1}^{k-1}\phi_i\right)\cdot\det\left(I_{(k-1)m}\right.$$

$$\left. - \begin{pmatrix} \omega_{1,k}\omega_{k,1}\Omega_1\Omega_k & \Omega_1\left(\omega_{1,2}+\omega_{1,k}\omega_{k,2}\Omega_k\right) & \cdots & \Omega_1\left(\omega_{1,k-1}+\omega_{1,k}\omega_{k,k-1}\Omega_k\right) \\ \Omega_2\left(\omega_{2,1}+\omega_{2,k}\omega_{k,1}\Omega_k\right) & \omega_{2,k}\omega_{k,2}\Omega_2\Omega_k & \cdots & \Omega_2\left(\omega_{2,k-1}+\omega_{2,k}\omega_{k,k-1}\Omega_k\right) \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{k-1}\left(\omega_{k-1,1}+\omega_{k-1,k}\omega_{k,1}\Omega_k\right) & \Omega_{k-1}\left(\omega_{k-1,2}+\omega_{k-1,k}\omega_{k,2}\Omega_k\right) & \cdots & \omega_{k-1,k}\omega_{k,k-1}\Omega_{k-1}\Omega_k \end{pmatrix}\right)$$

$$= \left(\prod_{i=1}^{k-1}\phi_i\right)\cdot\det\left(\begin{pmatrix} I_m & -\omega_{1,2}\Omega_1 & \cdots & -\omega_{1,k-1}\Omega_1 \\ -\omega_{2,1}\Omega_2 & I_m & \cdots & -\omega_{2,k-1}\Omega_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\omega_{k-1,1}\Omega_{k-1} & -\omega_{k-1,2}\Omega_{k-1} & \cdots & I_m \end{pmatrix}\right.$$

$$\left. - \begin{pmatrix} \omega_{1,k}\omega_{k,1}\Omega_1\Omega_k & \omega_{1,k}\omega_{k,2}\Omega_1\Omega_k & \cdots & \omega_{1,k}\omega_{k,k-1}\Omega_1\Omega_k \\ \omega_{2,k}\omega_{k,1}\Omega_2\Omega_k & \omega_{2,k}\omega_{k,2}\Omega_2\Omega_k & \cdots & \omega_{2,k}\omega_{k,k-1}\Omega_2\Omega_k \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{k-1,k}\omega_{k,1}\Omega_{k-1}\Omega_k & \omega_{k-1,k}\omega_{k,2}\Omega_{k-1}\Omega_k & \cdots & \omega_{k-1,k}\omega_{k,k-1}\Omega_{k-1}\Omega_k \end{pmatrix}\right)$$

$$= \left(\prod_{i=1}^{k-1}\phi_i\right)\cdot\det\left(\begin{pmatrix} I_m & -\omega_{1,2}\Omega_1 & \cdots & -\omega_{1,k-1}\Omega_1 \\ -\omega_{2,1}\Omega_2 & I_m & \cdots & -\omega_{2,k-1}\Omega_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\omega_{k-1,1}\Omega_{k-1} & -\omega_{k-1,2}\Omega_{k-1} & \cdots & I_m \end{pmatrix}\right.$$

$$\left. - \begin{pmatrix} \omega_{1,k}\Omega_1 \\ \omega_{2,k}\Omega_2 \\ \vdots \\ \omega_{k-1,k}\Omega_{k-1} \end{pmatrix}\left(I_m\right)\begin{pmatrix} \omega_{k,1}\Omega_k & \omega_{k,2}\Omega_k & \cdots & \omega_{k,k-1}\Omega_k \end{pmatrix}\right).$$

Using Schur's complement formula once again we obtain

$$\det(B) = \left(\prod_{i=1}^{k-1} \phi_i\right) \cdot \det \begin{pmatrix} I_m & -\omega_{1,2}\Omega_1 & \cdots & -\omega_{1,k}\Omega_1 \\ -\omega_{2,1}\Omega_2 & I_m & \cdots & -\omega_{2,k}\Omega_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\omega_{k,1}\Omega_k & -\omega_{k,2}\Omega_k & \cdots & I_m \end{pmatrix}.$$

Finally, from (1) we have

$$\det(\lambda I_n - A_{\dot{G}_i}) = \left(\prod_{i=1}^{k} \phi_i\right) \cdot \det \begin{pmatrix} I_m & -\omega_{1,2}\Omega_1 & \cdots & -\omega_{1,k}\Omega_1 \\ -\omega_{2,1}\Omega_2 & I_m & \cdots & -\omega_{2,k}\Omega_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\omega_{k,1}\Omega_k & -\omega_{k,2}\Omega_k & \cdots & I_m \end{pmatrix} = \left(\prod_{i=1}^{k} \phi_i\right) \det\left(\tilde{A}\right).$$

This proves the theorem. □

Consider the signed graph $\dot{G}_i$. For $1 \le i \le k$, let $\left\{\theta_i^1, \theta_i^2, \ldots, \theta_i^{s_i}\right\}$ be the set of its distinct eigenvalues and let $\left\{\theta_i^1, \theta_i^2, \ldots, \theta_i^{t_i}\right\}$, with $t_i \le s_i$, be its distinct E_i-main eigenvalues. Suppose that the eigendecomposition of the adjacency matrix of G_i is the sum $\sum_{j=1}^{s_i} \theta_i^j \pi_{\theta_i^j}$, in which $\pi_{\theta_i^j}$ represents the orthogonal projection of the space $\xi_{A_{\dot{G}_i}}(\theta_i^j)$ that corresponds to the eigenvalue θ_i^j. Then we have

$$(\lambda I - A_{\dot{G}_i})^{-1} = \sum_{j=1}^{s_i} \frac{\pi_{\theta_i^j}}{\lambda - \theta_i^j} \quad \text{and} \quad \Omega_i = E_i^T(\lambda I - A_{\dot{G}_i})^{-1}E_i = \sum_{j=1}^{s_i} \frac{E_i^T \pi_{\theta_i^j} E_i}{\lambda - \theta_i^j}.$$

Since θ_i^j is an E_i-main eigenvalue of $A_{\dot{G}_i}$ if and only if $E_i^T \pi_{\theta_i^j} E_i \neq 0$ we have the value of the main function associated to the indexing matrix E_i as $\Omega_i = \sum_{j=1}^{t_i} \frac{E_i^T \pi_{\theta_i^j} E_i}{\lambda - \theta_i^j}$. Then we can write

$$\Omega_i = \frac{f_i}{g_i}, \quad g_i = \prod_{j=1}^{t_i} \left(\lambda - \theta_i^j\right), \quad f_i \in M_m(\mathbb{C}(\lambda)).$$

Hence by Theorem 3.3

$$\det(\lambda I - A_{\dot{G}}) = \left(\prod_{i=1}^{k} \frac{\phi_i}{g_i}\right) \Phi(\lambda)$$

where

$$\Phi(\lambda) := \det \begin{pmatrix} g_1 I_m & -\omega_{1,2} f_1 & \cdots & -\omega_{1,k} f_1 \\ -\omega_{2,1} f_2 & g_2 I_m & \cdots & -\omega_{2,k} f_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\omega_{k,1} f_k & -\omega_{k,2} f_k & \cdots & g_k I_m \end{pmatrix}. \tag{2}$$

Following this we are able to establish some facts about how the eigenvalues of a graph $\dot{G}_i \in \mathcal{F}$ will appear in the spectrum of the graph $\dot{G}$.

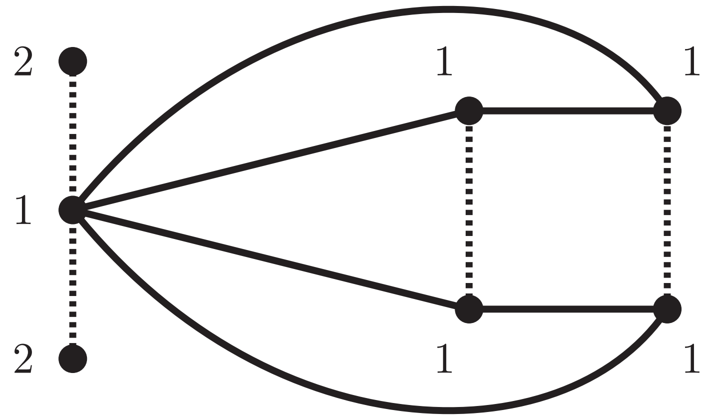

FIGURE 2. The $(P_2,+)_2$-join of $\dot{G}_1 = (P_3,-)$ and $\dot{G}_2 = (C_4,\sigma)$ with respect to the indexing maps M_1 and M_2 whose values are labelled on the vertices.

Corollary 3.4. *Let λ be an adjacency eigenvalue of the signed graph $\dot{G}_i$ and let $\mu_i(\lambda)$ be its multiplicity. There are two cases to consider: when λ is E_i-non-main and when λ is E_i-main. In the former, λ will be an adjacency eigenvalue of $\dot{G}$ with multiplicity no less than $\mu_i(\lambda)$. In the latter, λ will be an adjacency eigenvalue of $\dot{G}$ with multiplicity no less than $\mu_i(\lambda)-m$. The eigenvalues of $A_{\dot{G}}$ that have been left unaccounted for by these statements are exactly the roots of $\Phi(\lambda)$ in (2).*

Now we shall present an example to illustrate some of what has been discussed so far in this section.

Example 3.5. Consider a $(P_2,+)_2$-join of the two signed graphs $\dot{G}_1 = (P_3,-)$ and $\dot{G}_2 = (C_4,\sigma)$, in which each vertex has net degree equal to zero, along with their associated indexing maps M_1 and M_2. This compound graph is shown in Figure 2. With a suitable vertex ordering we have the indexing matrices, characteristic polynomials, and main functions of the component graphs as follows.

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \phi_1 = \lambda^3 - 2\lambda, \qquad \Omega_1 = E_1^T(\lambda I - A_{\dot{G}_1})^{-1}E_1 = \frac{1}{\lambda^2-2}\begin{pmatrix} \lambda & -2 \\ -2 & 2\lambda \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \phi_2 = \lambda^4 - 4\lambda^2, \qquad \Omega_2 = E_2^T(\lambda I - A_{\dot{G}_2})^{-1}E_2 = \frac{1}{\lambda}\begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}.$$

For $\dot{G}_1 = (P_3,-)$ the spectrum is $\{\sqrt{2}, 0, -\sqrt{2}\}$ and we have

$$\xi_{A_{\dot{G}_1}}(\sqrt{2}) = \operatorname{span}\left\{\begin{pmatrix} -\sqrt{2} \\ 1 \\ 1 \end{pmatrix}\right\}, \quad \xi_{A_{\dot{G}_1}}(0) = \operatorname{span}\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}, \quad \xi_{A_{\dot{G}_1}}(-\sqrt{2}) = \operatorname{span}\left\{\begin{pmatrix} \sqrt{2} \\ 1 \\ 1 \end{pmatrix}\right\}.$$

Hence 0 is an E_1-non-main eigenvalue and $\sqrt{2}$ and $-\sqrt{2}$ are E_1-main eigenvalues. This means that 0 will be an eigenvalue of $A_{\dot{G}}$ at least once.

For $\dot{G}_2 = (C_4,\sigma)$ the spectrum is $\{2, 0, 0, -2\}$ and we have

$$\xi_{A_{\dot{G}_2}}(2) = \operatorname{span}\left\{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}\right\}, \quad \xi_{A_{\dot{G}_2}}(0) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right\}, \quad \xi_{A_{\dot{G}_3}}(-2) = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}\right\}.$$

Hence 2 and -2 are E_2-non-main eigenvalues and 0 is an E_2-main eigenvalue. This means that 2 and -2 will each be eigenvalues of $A_{\dot{G}}$ at least once.

By Theorem 3.3 we have

$$\begin{aligned}\det(\lambda I - A_{\dot{G}}) &= \phi_1 \cdot \phi_2 \cdot \det\begin{pmatrix} I_2 & -\Omega_1 \\ -\Omega_2 & I_2 \end{pmatrix}\\ &= (\lambda^3 - 2\lambda)\cdot(\lambda^4 - 4\lambda^2)\cdot\frac{\lambda^2-6}{\lambda^2-4}\\ &= \lambda^7 - 10\lambda^5 + 24\lambda^3.\end{aligned}$$

The spectrum of $\dot{G}$ is $\{\sqrt{6}, 2, 0, 0, 0, -2, -\sqrt{6}\}$ and we can see that this satisfies the claims made about the E_i-non-main eigenvalues.

3.3. Stability under switching of $\dot{H}$

As previously mentioned, the operation of switching and the fact that any two signed graphs which are switching equivalent have identical characteristic polynomials, spectra, and eigenvalues is an important concept in signed graph theory. We shall prove that the $\dot{H}_m$-join operation remains stable under switching of $\dot{H}$, which is to say that if we perform the operation on any two switching equivalent signed graphs $\dot{H}$ and $\dot{H}'$, the two produced signed graphs $\dot{G}$ and $\dot{G}'$ shall also be switching equivalent.

Lemma 3.6. *Let $\dot{H} = (H, \sigma_H)$ and $\dot{H}' = (H, \sigma'_H)$ be switching equivalent. Then $\dot{G} = \bigvee_{\dot{H}}^{\mathcal{FM}}$ and $\dot{G}' = \bigvee_{\dot{H}'}^{\mathcal{FM}}$ are also switching equivalent.*

Proof. Switching equivalence between $\dot{H}$ and $\dot{H}'$ implies that their adjacency matrices are switching similar. Hence for some switching matrix $S = (s_{ij})$ we have $SA_{\dot{H}}S^{-1} = A_{\dot{H}'}$. Using this matrix to give the switching operation in terms of the signatures of $\dot{H}$ and $\dot{H}'$ we have $s_{ii}\sigma_H(v_iv_j)s_{jj} = \sigma'_H(v_iv_j)$. If $v_iv_j \notin E(H)$ then $\omega'_{i,j} = \omega_{i,j} = 0$ and if $v_iv_j \in E(H)$ we have

$$\omega'_{i,j} = \sigma'_H(v_iv_j) = s_{ii}\sigma_H(v_iv_j)s_{jj} = s_{ii}\omega_{i,j}s_{jj}.$$

Thus $\omega'_{i,j} = s_{ii}\omega_{i,j}s_{jj}$ for all $1 \le i, j \le k$.

Let $\dot{G} = \bigvee_{\dot{H}}^{\mathcal{FM}}$ and let $\dot{G}' = \bigvee_{\dot{H}'}^{\mathcal{FM}}$. By considering the adjacency matrix of $\dot{G}'$ we can see that

$$\begin{aligned} A_{\dot{G}'} &= \begin{pmatrix} A_{\dot{G}_1} & \omega'_{1,2}E_1E_2^T & \cdots & \omega'_{1,k}E_1E_k^T \\ \omega'_{2,1}E_2E_1^T & A_{\dot{G}_2} & \cdots & \omega'_{2,k}E_2E_k^T \\ \vdots & \vdots & \ddots & \vdots \\ \omega'_{k,1}E_kE_1^T & \omega'_{k,2}E_kE_2^T & \cdots & A_{\dot{G}_k} \end{pmatrix}\\ &= \begin{pmatrix} A_{\dot{G}_1} & s_{11}\omega_{1,2}s_{22}E_1E_2^T & \cdots & s_{11}\omega_{1,k}s_{kk}E_1E_k^T \\ s_{22}\omega_{2,1}s_{11}E_2E_1^T & A_{\dot{G}_2} & \cdots & s_{22}\omega_{2,k}s_{kk}E_2E_k^T \\ \vdots & \vdots & \ddots & \vdots \\ s_{kk}\omega_{k,1}s_{11}E_kE_1^T & s_{kk}\omega_{k,2}s_{22}E_kE_2^T & \cdots & A_{\dot{G}_k} \end{pmatrix}\\ &= \begin{pmatrix} s_{11}I_{n_1} & O & \cdots & O \\ O & s_{22}I_{n_2} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & s_{kk}I_{n_k} \end{pmatrix}\begin{pmatrix} A_{\dot{G}_1} & \omega_{1,2}E_1E_2^T & \cdots & \omega_{1,k}E_1E_k^T \\ \omega_{2,1}E_2E_1^T & A_{\dot{G}_2} & \cdots & \omega_{2,k}E_2E_k^T \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{k,1}E_kE_1^T & \omega_{k,2}E_kE_2^T & \cdots & A_{\dot{G}_k} \end{pmatrix}\end{aligned}$$

$$\times \begin{pmatrix} s_{11}I_{n_1} & O & \cdots & O \\ O & s_{22}I_{n_2} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & s_{kk}I_{n_k} \end{pmatrix}^{-1}.$$

Hence $A_{\dot{G}'} = SA_{\dot{G}}S^{-1}$ for some switching matrix S and so by definition $\dot{G} = \bigvee_{\dot{H}}^{\mathcal{F}\mathcal{M}}$ is switching equivalent to $\dot{G}' = \bigvee_{\dot{H}'}^{\mathcal{F}\mathcal{M}}$ and we are done. □

One way to think of the switching here is as follows: whenever we perform switching on a vertex $v_i \in V(H)$ it is the same operation as performing switching on every vertex of $V(G_i)$ in the compound graph produced by the $\dot{H}_m$-join operation.

Corollary 3.7. *Since the $\dot{H}$-join and $\dot{H}$-generalised join are special cases of the $\dot{H}_m$-join this also acts as a proof for those joins being stable under switching of $\dot{H}$.*

Remark 3.8. On the other hand it is generally not the case that the compound graph will remain switching equivalent if we substitute one or more of the G_i component signed graphs with a switching equivalent version. For example take the $\dot{H}_1$-join with $\dot{H} = (K_2, +)$, $\mathcal{F} = \{G_1 = (K_1, +), G_2 = (K_2, +)\}$. This will effectively give $\bigvee_{\dot{H}}^{\mathcal{F}\mathcal{M}} = (C_3, +)$, a balanced cycle on three vertices. Now if we replace $G_2 = (K_2, +)$ with the switching equivalent graph $G_2 = (K_2, -)$ we have $\bigvee_{\dot{H}}^{\mathcal{F}',\mathcal{M}} = (C_3, \sigma)$ where the compound graph is an unbalanced cycle on three vertices. Evidently the two graphs produced by the join operation are not switching equivalent.

4. Universal adjacency and net Laplacian matrices

Now we shall present some results regarding the universal adjacency and net Laplacian matrices of a graph that is obtained from the $\dot{H}_m$-join along with their characteristic polynomials and some mention of which eigenvalues may be carried through to the compound graph.

4.1. Universal adjacency matrix

Recall that for a signed graph $\dot{G}$ of order n, the universal adjacency matrix $U_{\dot{G}}$ is an $n \times n$ matrix given by

$$U_{\dot{G}} = \alpha A_{\dot{G}} + \beta I_n + \gamma J_n + \delta D_G$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha \neq 0$.

Definition 4.1. Using the notations we are by now familiar with, for $\dot{G} = \bigvee_{\dot{H}}^{\mathcal{F}\mathcal{M}}$ we have the universal adjacency matrix as

$$U_{\dot{G}} = \begin{pmatrix} U_{\dot{G}_1} + \delta\mathcal{D}_1 & \omega_{1,2}E_1E_2^T + \gamma J_{n_1\times n_2} & \cdots & \omega_{1,k}E_1E_k^T + \gamma J_{n_1\times n_k} \\ \omega_{2,1}E_2E_1^T + \gamma J_{n_2\times n_1} & U_{\dot{G}_2} + \delta\mathcal{D}_2 & \cdots & \omega_{2,k}E_2E_k^T + \gamma J_{n_2\times n_k} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{k,1}E_kE_1^T + \gamma J_{n_k\times n_1} & \omega_{k,2}E_kE_2^T + \gamma J_{n_k\times n_2} & \cdots & U_{\dot{G}_k} + \delta\mathcal{D}_k \end{pmatrix},$$

where for $1 \leq i \leq k$ we have

$$(\mathcal{D}_i)_{xy} = \begin{cases} \sum_{v_iv_j\in E(H)} \left|\{v_j^z \in V(G_j) : M_j(v_j^z) = M_i(v_i^x)\}\right| & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

In other words $\mathcal{D}_i$ is a diagonal matrix in which the xth diagonal entry corresponds to the vertex $v_i^x \in V(G_i)$ and takes the value equal to the total number of vertices v_i^x is adjacent to in $V(G_j)$ for all $i \neq j$ after the $\dot{H}_m$-join operation has been performed.

Theorem 4.2. *The universal characteristic polynomial of $\dot{G} = \bigvee_{\dot{H}}^{\mathcal{FM}}$ with $\gamma = 0$ is*

$$\det(\lambda I_n - U_{\dot{G}}) = \left(\prod_{i=1}^{k} \phi_{(U_{\dot{G}_i}+\delta\mathcal{D}_i)}\right)\det(\tilde{U}),$$

where

$$\tilde{U} = \begin{pmatrix} I_m & -\omega_{1,2}\Omega_{(U_{\dot{G}_1}+\delta\mathcal{D}_1)} & \cdots & -\omega_{1,k}\Omega_{(U_{\dot{G}_1}+\delta\mathcal{D}_1)} \\ -\omega_{2,1}\Omega_{(U_{\dot{G}_2}+\delta\mathcal{D}_2)} & I_m & \cdots & -\omega_{2,k}\Omega_{(U_{\dot{G}_2}+\delta\mathcal{D}_2)} \\ \vdots & \vdots & \ddots & \vdots \\ -\omega_{k,1}\Omega_{(U_{\dot{G}_k}+\delta\mathcal{D}_k)} & -\omega_{k,2}\Omega_{(U_{\dot{G}_k}+\delta\mathcal{D}_k)} & \cdots & I_m \end{pmatrix}.$$

Proof. If $\gamma = 0$ then we can follow the proof laid out for Theorem 3.3 in every step by replacing $A_{\dot{G}_i}$ with $U_{\dot{G}_i} + \delta\mathcal{D}_i$ for each $1 \leq i \leq k$. In doing so we have $\phi_{U_{\dot{G}_i}+\delta\mathcal{D}_i} = \det\left(\lambda I_{n_i} - (U_{\dot{G}_i} + \delta\mathcal{D}_i)\right)$ and $\Omega_{(U_{\dot{G}_i}+\delta\mathcal{D}_i)} = E_i^T\left(\lambda I_{n_i} - (U_{\dot{G}_i} + \delta\mathcal{D}_i)\right)^{-1} E_i$. Then we can make use of Lemmas 2.1, 2.2(b), and Definition 4.1 to prove the result by induction in the same way as in Theorem 3.3. □

4.2. Net Laplacian matrix

Recall that for a signed graph $\dot{G}$ of order n, the net Laplacian matrix $L_{\dot{G}}^{\pm}$ is an $n \times n$ matrix given by

$$L_{\dot{G}}^{\pm} = D_{\dot{G}}^{\pm} - A_{\dot{G}}$$

where $D_{\dot{G}}^{\pm}$ and $A_{\dot{G}}$ are the net degree and adjacency matrices of $\dot{G}$ respectively.

Definition 4.3. Once again using the notations we are by now familiar with, for $\dot{G} = \bigvee_{\dot{H}}^{\mathcal{FM}}$ we have the net Laplacian matrix as

$$L_{\dot{G}}^{\pm} = \begin{pmatrix} L_{\dot{G}_1}^{\pm} + \mathcal{D}_1^{\pm} & -\omega_{1,2}E_1E_2^T & \cdots & -\omega_{1,k}E_1E_k^T \\ -\omega_{2,1}E_2E_1^T & L_{\dot{G}_2}^{\pm} + \mathcal{D}_2^{\pm} & \cdots & -\omega_{2,k}E_2E_k^T \\ \vdots & \vdots & \ddots & \vdots \\ -\omega_{k,1}E_kE_1^T & -\omega_{k,2}E_kE_2^T & \cdots & L_{\dot{G}_k}^{\pm} + \mathcal{D}_k^{\pm} \end{pmatrix}.$$

where for $1 \leq i \leq k$ we have

$$(\mathcal{D}_i^{\pm})_{xy} = \begin{cases} \sum_{v_iv_j\in E(H)} \omega_{i,j}\left|\{v_j^z \in V(G_j) : M_j(v_j^z) = M_i(v_i^x)\}\right| & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Note the subtle difference with the inclusion of $\omega_{i,j}$ in the definition of $(\mathcal{D}_i^{\pm})_{xy}$ compared to that of $(\mathcal{D}_i)_{xy}$. In other words $\mathcal{D}_i^{\pm}$ is a diagonal matrix in which the xth diagonal entry corresponds to the vertex $v_i^x \in V(G_i)$ and takes the value equal to the difference between the number of vertices that are positively adjacent to v_i^x in $V(G_j)$ for all $i \neq j$ and the number of vertices that are negatively adjacent to v_i^x in $V(G_j)$ for all $i \neq j$ after the $\dot{H}_m$-join operation has been performed.

Much like the case of the universal adjacency matrix we can use a very similar proof to that of Theorem 3.3 to determine the net Laplacian spectrum.

Theorem 4.4. *The net Laplacian characteristic polynomial of $\dot{G} = \bigvee_{\dot{H}}^{\mathcal{FM}}$ is*

$$\det(\lambda I_n - L^{\pm}_{\dot{G}}) = \left(\prod_{i=1}^{k} \phi_{(L^{\pm}_{\dot{G}_i} + \mathcal{D}^{\pm}_i)}\right) \det\left(\tilde{L^{\pm}}\right),$$

where

$$\tilde{L^{\pm}} = \begin{pmatrix} I_m & -\omega_{1,2}\Omega_{(L^{\pm}_{\dot{G}_1}+\mathcal{D}^{\pm}_1)} & \cdots & -\omega_{1,k}\Omega_{(L^{\pm}_{\dot{G}_1}+\mathcal{D}^{\pm}_1)} \\ -\omega_{2,1}\Omega_{(L^{\pm}_{\dot{G}_2}+\mathcal{D}^{\pm}_2)} & I_m & \cdots & -\omega_{2,k}\Omega_{(L^{\pm}_{\dot{G}_2}+\mathcal{D}^{\pm}_2)} \\ \vdots & \vdots & \ddots & \vdots \\ -\omega_{k,1}\Omega_{(L^{\pm}_{\dot{G}_k}+\mathcal{D}^{\pm}_k)} & -\omega_{k,2}\Omega_{(L^{\pm}_{\dot{G}_k}+\mathcal{D}^{\pm}_k)} & \cdots & I_m \end{pmatrix}.$$

Proof. We can follow the proof laid out for Theorem 3.3 in every step by replacing $A_{\dot{G}_i}$ with $L^{\pm}_{\dot{G}_i} + \mathcal{D}^{\pm}_i$ for each $1 \le i \le k$. In doing so we have $\phi_{L^{\pm}_{\dot{G}_i}+\mathcal{D}^{\pm}_i} = \det(\lambda I_{n_i} - (L^{\pm}_{\dot{G}_i} + \mathcal{D}^{\pm}_i))$ and $\Omega_{(L^{\pm}_{\dot{G}_i}+\mathcal{D}^{\pm}_i)} = E_i^T(\lambda I_{n_i} - (L^{\pm}_{\dot{G}_i} + \mathcal{D}^{\pm}_i))^{-1}E_i$. Then we can make use of Lemmas 2.1, 2.2(b), and Definition 4.3 to prove the result by induction in the same way as in Theorem 3.3. □

4.3. Carried eigenvalues

We conclude this section with two corollaries regarding which eigenvalues are carried from the $\dot{G}_i$ component graphs to the graph produced by the $\dot{H}_m$-join operation.

Corollary 4.5. *Let λ be an eigenvalue of the matrix $\alpha A_{\dot{G}_i} + \beta I_{n_i} + \delta(D_{G_i} + \mathcal{D}_i)$ and let $\mu_i(\lambda)$ be its multiplicity.*

- *If λ is E_i-non-main then λ will be an eigenvalue of $\alpha A_{\dot{G}} + \beta I + \delta D_G$ with multiplicity no less than $\mu_i(\lambda)$.*
- *If λ is E_i-main then λ will be an eigenvalue of $\alpha A_{\dot{G}} + \beta I + \delta D_G$ with multiplicity no less than $\mu_i(\lambda) - m$.*

The leftover eigenvalues of $\alpha A_{\dot{G}} + \beta I + \delta D_G$ are exactly the roots of $\Phi_U(\lambda)$ which is to be similarly defined as in (2).

It is easy to see how one might select values for the parameters to obtain the results for well-known matrices such as the Laplacian of $\dot{G}$. As one might expect, we have a similar corollary pertaining to the net Laplacian matrix of $\dot{G}$.

Corollary 4.6. *Let λ be an eigenvalue of the matrix $L^{\pm}_{\dot{G}_i} + \mathcal{D}^{\pm}_i$ and let $\mu_i(\lambda)$ be its multiplicity.*

- *If λ is E_i-non-main then λ will be an eigenvalue of $L^{\pm}_{\dot{G}}$ with multiplicity no less than $\mu_i(\lambda)$.*
- *If λ is E_i-main then λ will be an eigenvalue of $L^{\pm}_{\dot{G}}$ with multiplicity no less than $\mu_i(\lambda) - m$.*

The leftover eigenvalues of $L^{\pm}_{\dot{G}}$ are exactly the roots of $\Phi_{L^{\pm}}(\lambda)$ which is to be similarly defined as in (2).

5. Summary

Over the course of this paper we have extended work previously done for ordinary graphs into the theory of signed graphs. We have defined the generalised $\dot{H}$-join operation constrained by indexing maps for signed graphs and determined the adjacency characteristic polynomial and spectrum as well as providing information about which eigenvalues may be carried from the component graphs to the compound graph. Furthermore we have shown that the operation is stable under switching of $\dot{H}$. In the final section we computed the characteristic polynomials and spectra of the universal adjacency matrix the net Laplacian matrix and again gave information about which eigenvalues will be carried to the graph produced by the join operation.

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Conflict of Interest

The author reports there are no competing interests to declare.

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