COMMUNICATIONS IN ALGEBRA®
https://doi.org/10.1080/00927872.2026.2642996

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Uncountable groups whose large subnormal subgroups are close to normal

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ABSTRACT
Let G be a group. A subgroup X of G is said to be nearly normal if it has finite index in its normal closure X^G, and X is called normal-by-finite if it is finite over its core X_G. In this paper, we investigate the behavior of uncountable periodic soluble groups G of regular cardinality $\aleph$ in which the section H^G/H_G is finite (of bounded order) for all subnormal subgroups H of G having cardinality $\aleph$.

ARTICLE HISTORY
Received 13 September 2025
Communicated by Attila Maroti

KEYWORDS
Core; normal closure; subnormal subgroup; uncountable group

2020 MATHEMATICS SUBJECT CLASSIFICATION
20E15; 20E26; 20F14

1. Introduction

A group G is said to have *finite rank* if there exists a positive integer r such that every finitely generated subgroup of G can be generated by at most r elements; if such an r does not exist, we say that the group G has *infinite rank*. In a long series of papers, it has been shown that the structure of a group having infinite rank is strongly influenced by that of its proper subgroups of infinite rank (see for instance [5], where a full reference list on this subject can be found). Results of this type suggest that the behavior of *small* subgroups in a *large* group is neglectable, at least for a right choice of the definition of largeness and within a suitable universe. This point of view is adopted also in the papers [7–10], by considering uncountable groups G whose subgroups of the same cardinality as G are normal or satisfy other relevant embedding properties that generalize normality.

The aim of this paper is to provide a further contribution to this topic, by investigating the behavior of uncountable groups G in which all subnormal subgroups equipotent to G are close to normal, with the only obstruction of a finite section. The corresponding problem in the case of groups of infinite rank has been solved in [6].

Recall that a subgroup X of a group G is said to be *nearly normal* if it has finite index in its normal closure X^G. In a famous paper of 1955, B. H. Neumann [16] proved that groups admitting only nearly normal subgroups are precisely those with finite commutator subgroup. This result was later refined by I. D. Macdonald [15], who showed that if G is a group in which every subgroup has index at most the same positive integer m in its normal closure, then the commutator subgroup G' of G is finite of order bounded by a function of m. Recall that a group G is called a *T-group* if normality in G is a transitive relation, i.e. if all subnormal subgroups of G are normal. The structure of soluble T-groups has been described by W. Gaschütz [12] in the finite case and by D. J. S. Robinson [18] for arbitrary groups. Motivated by Neumann's and Macdonald's results, and taking as a model the known theory of T-groups, C. Casolo

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investigated the class T^* (respectively, T_m) including all groups in which every subnormal subgroup has finite index (at most m, respectively) in its normal closure (see [2, 3]). Recall also that a subgroup X of a group G is said to be *normal-by-finite* if it is finite over its core X_G. The structure of groups in which all subgroups are normal-by-finite has been investigated by Neumann in a joint paper with J. Buckley, J. C. Lennox, H. Smith, and J. Wiegold, where it was proved that any locally finite group with this property is abelian-by-finite (see [1]). In this case, the corresponding investigation of the class T_* of groups in which every subnormal subgroup is normal-by-finite was carried out by S. Franciosi, F. de Giovanni and M. L. Newell in [11]. Finally, the class U (respectively, U_m), consisting of all groups G in which the section X^G/X_G is finite (of order at most m, respectively) for all subnormal subgroups X of G, has been considered by Casolo in [2, 3].

Let G be an uncountable group of regular cardinality $\aleph$. Then, G will be called a T^--*group* if every subnormal subgroup of cardinality $\aleph$ of G is nearly normal; furthermore, we shall say that G belongs to the class T_- if all subnormal subgroups of cardinality $\aleph$ of G are normal-by-finite. The first section of this paper is devoted to the study of uncountable groups that belong to the classes T^- and T_-.

Finally, let G be an uncountable group of regular cardinality $\aleph$, and let $\mathcal{V}$ be the set of all normal subgroups of G having cardinality $\aleph$. The group G is called a $T_\aleph$-*group* if all its subnormal subgroups of cardinality $\aleph$ are normal and

$$\bigcap_{V\in\mathcal{V}} V' = \{1\}.$$

$T_\aleph$-groups have been investigated by M. De Falco, F. de Giovanni, C. Musella, and Y. P. Sysak in [8], where it was proved that every hyperabelian periodic $T_\aleph$-group is a T-group. Let $\mathcal{U}$ denote the set of all subnormal subgroups of G having cardinality $\aleph$. Motivated by the notion of $T_\aleph$-groups, we consider here uncountable groups G in which the section H^G/H_G is finite of order bounded by a positive integer m for every $H \in \mathcal{U}$, and the intersection

$$\bigcap_{H\in\mathcal{U}} (H')^G$$

is finite. Our main result can be stated as follows.

Theorem 1.1. *Let G be an uncountable periodic soluble group of regular cardinality $\aleph$ having large commutator subgroup G'. Let $\mathcal{U}$ be the set of all subnormal subgroups of G having cardinality $\aleph$, and assume that there exists a positive integer m such that $|H^G/H_G| \leq m$ for every $H \in \mathcal{U}$. Then, the second commutator subgroup G'' of G is finite if and only if the intersection $\bigcap_{H\in\mathcal{U}} (H')^G$ is finite.*

It is easy to see that the hypothesis of solubility also allows us to avoid the existence of uncountable groups G having all proper subgroups of cardinality strictly smaller than G, the so-called *Jónsson groups* (see for instance [13]), which represent the main obstacle in the study of groups of large cardinality. Relevant examples of Jónsson groups of cardinality $\aleph_1$ have been constructed by S. Shelah [20] and V. N. Obraztsov [17].

Throughout this paper, $\aleph$ will denote an uncountable regular cardinal number, and if G is a group of cardinality $\aleph$, a subgroup X of G will be called *large* if it likewise has cardinality $\aleph$, and *small* otherwise. Most of our notation is standard and can be found in [19].

2. Some properties of the classes T^- and T_-

It is clear that T^* and T_* are (proper) subclasses of T^- and T_-, respectively. Our first result shows that, within the universe of uncountable periodic groups of regular cardinality, the classes T^* and T^- coincide, provided the existence of a large abelian subnormal subgroup. A corresponding statement is established for the classes T_* and T_-.

Lemma 2.1. *Let G be an uncountable periodic group of regular cardinality $\aleph$ in the class T^-. If G contains a large abelian subnormal subgroup, then G is a T^*-group.*

Proof. Let A be a large abelian subnormal subgroup of G. Then the index $|A^G : A|$ is finite, so that A^G is abelian-by-finite and hence it contains an abelian characteristic subgroup B of finite index. In particular, B is a large normal subgroup of G. Clearly, also the socle S of B is a large normal subgroup of G, and it is a direct product of subgroups of prime order. Let X be a small subnormal subgroup of G, and let S_0 be a subgroup of S such that

$$S = (X \cap S) \times S_0.$$

Then S_0 is a large subnormal subgroup of G, and hence it has finite index in its normal closure $N = S_0^G$. As the index $|X \cap N : X \cap S_0|$ is finite, it follows that the intersection $X \cap N$ is also finite. Clearly, N contains large subgroups U and V such that

$$\frac{N}{X \cap N} = \left(\frac{U}{X \cap N}\right) \times \left(\frac{V}{X \cap N}\right).$$

In particular, the intersection $U \cap V = X \cap N$ is finite.
Put $H = U^G$ and $K = V^G$. Since U and V are nearly normal in G, it follows that the index $|H \cap K : U \cap V|$ is finite, so that $H \cap K$ is also finite. Moreover, we have

$$X \cap HK \leq X \cap N \leq H \cap K.$$

Therefore, H and K are normal subgroups of G such that $H \cap K$ is finite and contains $X \cap HK$, and hence the index $|XH \cap XK : X|$ is finite (see [6, Lemma 1]). On the other hand, XH and XK are subnormal subgroups of G having cardinality $\aleph$, so that each of them has finite index in its normal closure, thus the index $|X^G : X|$ is also finite. Therefore X is nearly normal and G is a T^*-group. □

Lemma 2.2. *Let G be an uncountable group of regular cardinality $\aleph$ in the class T_-. If G contains a large abelian subnormal subgroup, then G is a T_*-group.*

Proof. Let X be a small subnormal subgroup of G, and let A be an abelian subnormal subgroup of G having cardinality $\aleph$. It is known (see for instance [14, Lemma 3.2]) that A contains large subgroups U and V such that

$$\langle U, V\rangle = U \times V$$

and

$$X \cap \langle U, V\rangle = \{1\}.$$

As G belongs to the class T_-, the indices $|U : U_G|$ and $|V : V_G|$ are finite, so that U_G and V_G have cardinality $\aleph$. Therefore, the large subnormal subgroups XU_G and XV_G are normal-by-finite, and hence also

$$X = XU_G \cap XV_G$$

is finite over its core X_G. □

Let G be an uncountable group of regular cardinality $\aleph$ in the class T^- (respectively, T_-). It is easy to show that the classes T^- and T_- are closed with respect to forming large subnormal subgroups and homomorphic images; of course, every quotient of G by a large normal subgroup belongs to the class T^* (respectively, T_*). As a consequence of Lemma 2.1 (respectively, Lemma 2.2), we can further extend the quotient belonging to T^* (respectively, T_*).

Corollary 2.3. *Let G be an uncountable group of regular cardinality $\aleph$, and let X be a normal subgroup of cardinality $\aleph$ of G.*

1. *If G is a periodic group in the class T^{-}, then G/X' is a T^{*}-group.*
2. *If G belongs to the class T_{-}, then G/X' is a T_{*}-group.*

Proof. 1. Clearly, X' is a normal subgroup of G and the statement is obvious if X' has cardinality $\aleph$. Suppose now that X' has cardinality strictly smaller than $\aleph$. Then X/X' is a large abelian normal subgroup of G/X', so that G/X' is a T^{*}-group by Lemma 2.1.
2. By applying the same arguments as above, the statement is a consequence of Lemma 2.2.

□

Recall that a group G is called a *Baer* group if it is generated by abelian subnormal subgroups, or equivalently if all finitely generated subgroups of G are subnormal. It is known that a Baer group in the class T_{*} is abelian-by-finite (see [11, Corollary 3.3]), while a Baer group in the class T^{*} is nilpotent (see [3, Lemma 3.1]). Observe also that a nilpotent T^{*}-group G has finite commutator subgroup G', since all its subgroups are nearly normal; furthermore, every soluble T^{*}-group G is finite-by-metabelian (see [3, Corollary 3.7]), so that it has finite second commutator subgroup G''.
Note that if G is an uncountable group of regular cardinality $\aleph$ in the class T^{-} and it contains a finite normal subgroup N such that G/N belongs to T^{*}, then G itself belongs to the class T^{*}. Indeed, if X is a small subnormal subgroup of G, then the quotient XN/N is a subnormal subgroup of the T^{*}-group G/N, so that XN is nearly normal in G and also X is nearly normal in G.

Proposition 2.4. *Let G be an uncountable group of regular cardinality $\aleph$.*

1. *If G is a periodic group in the class T^{-} and the Fitting subgroup of G is a T^{*}-group of cardinality $\aleph$, then G is a T^{*}-group.*
2. *If G belongs to the class T_{-} and the Fitting subgroup of G is a T_{*}-group of cardinality $\aleph$, then G is a T_{*}-group.*

Proof. 1. The Fitting subgroup F of G is a Baer group in the class T^{*}, so that its commutator subgroup F' is finite. Moreover, Corollary 2.3 yields that G/F' is a T^{*}-group, thus G itself is a T^{*}-group.
2. The Fitting subgroup of G is a large abelian-by-finite subgroup of G. The statement follows from Lemma 2.2.

□

Let G be an uncountable periodic group of regular cardinality $\aleph$ in the class T^{-}. As established in Lemma 2.1, the presence of a large abelian subnormal subgroup in G forces all subnormal subgroups of G to be nearly normal. In particular, if G is also a Baer group, we obtain a nilpotent group having finite commutator subgroup G'. By weakening the assumption on the existence of a large abelian subnormal subgroup and instead assuming the presence of a large nilpotent normal subgroup, it turns out that the group G is nilpotent with small commutator subgroup G'.

Corollary 2.5. *Let G be an uncountable periodic Baer group of regular cardinality $\aleph$ in the class T^{-}. If G contains a large nilpotent normal subgroup N, then G is nilpotent and the commutator subgroup G' of G has cardinality strictly smaller than $\aleph$.*

Proof. The Baer group G/N' is a T^{*}-group by Corollary 2.3, and hence G/N' is nilpotent. Now, by a well known nilpotency criterion of P. Hall (see [19, 5.2.16]), it follows that G itself is nilpotent, thus all large subgroups of G are nearly normal. The statement follows from [10, Proposition 2.3]. □

Recall here that subgroup H of a group G is called *almost normal* if it has only finitely many conjugates in G, or equivalently if its normalizer $N_G(H)$ has finite index in G. Furthermore, H is said to be *almost subnormal* if it is subnormal in a finite-index subgroup of G, i.e. if there exists a subgroup K of G such that H is subnormal in K and the index $|G : K|$ is finite. Finally, H is called *nearly subnormal* if it has finite index in a subnormal subgroup of G, i.e., if there exists a subnormal subgroup K of G containing H such that the index $|K : H|$ is finite.

Our next results show that an uncountable group G belongs to the class T^- (respectively, T_-) if and only if the relation of being a large nearly normal (respectively, normal-by-finite) subgroup is transitive in G.

Theorem 2.6. *Let G be an uncountable group of regular cardinality $\aleph$. Then the following statements are equivalent.*

1. *G belongs to the class T^-.*
2. *Every large nearly subnormal subgroup of G is nearly normal.*
3. *Every large almost subnormal subgroup of G is nearly normal.*
4. *If K and H are large subgroups of G such that K is nearly normal in H and H is nearly normal in G, then K is nearly normal in G.*

Proof. (1) $\Rightarrow$ (2): Let H be a large nearly subnormal subgroup of G; then the index $|H_n : H|$ is finite, for some term H_n of the normal closure series of H in G. As H_n is a large subnormal subgroup of the T^--group G, it follows that H_n has finite index in its normal closure $H_n^G = H^G$. Therefore, the index $|H^G : H| = |H_n^G : H_n||H_n : H|$ is finite and H is nearly normal in G.

(2) $\Rightarrow$ (3): Let H be a large almost subnormal subgroup of G; then there exists a large subgroup L of G such that H is subnormal in L and the index $|G : L|$ is finite. In particular, also the core L_G of L in G has finite index in G. We will prove the statement proceeding by induction on the defect n of H in L. Clearly, if $n = 1$ it follows that $H \trianglelefteq L$ and hence $H \cap L_G \trianglelefteq L_G \trianglelefteq G$, thus $H \cap L_G$ is a subnormal subgroup of G. Furthermore, as the index $|H : H \cap L_G|$ is finite, $H \cap L_G$ is a large nearly subnormal subgroup of G, so that it has finite index in its normal closure $W = (H \cap L_G)^G$. Since $N_G(HW) \geq L$ and L has finite index in G, the index $|G : N_G(HW)|$ is finite; moreover, we have that $|HW/W| \leq |L/L_G|$ is finite. Therefore, it follows from the well-known Dietzmann's Lemma that $(HW/W)^{G/W}$ is finite (see [19, 14.5.7]), and so in particular the index $|(HW)^G : HW|$ is finite. The equality $|HW : H| = |W : H \cap L_G|$ ensures that $|(HW)^G : H|$ is finite, thus $|H^G : H|$ is also finite and H is nearly normal in G.

Assume now that $n > 1$, and put $T = H^L$. By the first part of the proof, we obtain that the index $|T^G : T|$ is finite, so that H is almost subnormal in T^G; moreover, the defect of H in T is equal to $n-1$. As condition (2) is inherited by large normal subgroups, it follows from inductive hypothesis that $|H^{T^G} : H|$ is finite, thus H is a large nearly subnormal of G and hence it is nearly normal in G.

(3) $\Rightarrow$ (1): Obvious. (1) $\Rightarrow$ (4): Let H, K be large subgroups of G such that the indices $|K^H : K|$ and $|H^G : H|$ are finite. As $H \leq N_{H^G}(K^H)$, it follows that the index $|H^G : N_{H^G}(K^H)|$ is finite, thus K^H is almost normal in H^G. In particular, K^H is a large almost subnormal subgroup of the T^--group H^G, so that condition (3) ensures that K^H has finite index in its normal closure $(K^H)^{H^G} = K^{H^G}$. Therefore, K has finite index in the subnormal subgroup K^{H^G} of G and hence K is a large nearly subnormal subgroup of G. By condition (2), K is nearly normal in G.

(4) $\Rightarrow$ (1): Let H be a large subnormal subgroup of G; then there exists a finite series

$$H \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G$$

from H to G, in which every term has cardinality $\aleph$. We argue by induction on the defect n of H in G. Clearly, the statement is obvious if $n = 1$, so that we may assume $n > 1$. By inductive hypothesis, H is nearly normal in the normal subgroup H_{n-1} of G, thus the transitivity of nearly normality for large subgroups ensures that H is nearly normal in G. □

Theorem 2.7. *Let G be an uncountable group of regular cardinality $\aleph$. G belongs to the class T_{-} if and only if, whenever H is a large normal-by-finite subgroup of G and K is a large normal-by-finite subgroup of H, then K is normal-by-finite in G.*

Proof. Assume that G lies in the class T_{-}, and let H be a large normal-by-finite subgroup of G and K a large normal-by-finite subgroup of H. As the indices $|H : H_G|$ and $|K : K_H|$ are finite, it follows that $K_H \cap H_G$ has finite index in K and has cardinality $\aleph$. Clearly, $K_H \cap H_G$ is a normal subgroup of H_G, so that $K_H \cap H_G$ is a large subnormal subgroup of G, and hence $(K_H \cap H_G)/(K_H \cap H_G)_G$ is finite. Therefore, K/K_G is also finite and K is normal-by-finite in G.

The converse can be proved using an argument analogous to the implication (4) $\Rightarrow$ (1) of Lemma 2.6, replacing the assumption of nearly normality by that of being a large normal-by-finite subgroup. □

Through the study of uncountable groups in which all large subgroups are nearly normal, it turns out that such a group G has finite commutator subgroup G', provided that G contains a large abelian subgroup (see [9, proof of Theorem 3.6]). In our situation, the following can be proved.

Lemma 2.8. *Let G be an uncountable soluble group of regular cardinality $\aleph$ in the class T^{-}. If G contains a large abelian subnormal subgroup, then the second commutator subgroup G'' of G is finite.*

Proof. As G contains a large abelian subnormal subgroup, there exists in G an abelian subgroup of the form $A = A_1 \times A_2$, where both A_1 and A_2 have cardinality $\aleph$ and are subnormal in G (see for instance [9, Corollary 3.2]). Then the factor G/A_i^G is a soluble T^*-group and hence its second commutator subgroup $G''A_i^G/A_i^G$ is finite, for $i = 1, 2$. Consequently, the index $|G''A_1^G \cap G''A_2^G : A_1^G \cap A_2^G|$ is finite. On the other hand, as the indices $|A_1^G : A_1|$ and $|A_2^G : A_2|$ are finite, the intersection $A_1^G \cap A_2^G$ is also finite, so G'' is likewise finite. □

Lemma 2.9. *Let G be an uncountable group of regular cardinality $\aleph$ in the class T^{-}. If the second commutator subgroup G'' of G has cardinality $\aleph$, then the factor G''/G''' is finite.*

Proof. Assume that the abelian group G''/G''' has cardinality $\aleph$, so that it contains a proper subgroup X/G''' such that both X/G''' and G''/X have cardinality $\aleph$ (see for instance [13, Lemma 2.4]). Thus, X is a subnormal subgroup of cardinality $\aleph$ of G, and hence X is nearly normal in G. As the normal closure X^G of X in G is a large subgroup of G such that both X^G/G''' and G''/X^G have cardinality $\aleph$, we may replace X by X^G and assume that X is normal in G. Therefore, the factor G/X is a soluble T^*-group and hence its second commutator subgroup G''/X is finite, which is clearly impossible. This contradiction shows that G''/G''' has cardinality strictly smaller than $\aleph$, and so G''' has cardinality $\aleph$. It follows that the factor group G/G''' is a soluble T^*-group, thus its second commutator subgroup G''/G''' is finite. □

Corollary 2.10. *Let G be an uncountable group of regular cardinality $\aleph$ in the class T^{-}. If G is soluble of derived length at most 3, then the second commutator subgroup G'' of G has cardinality strictly smaller than $\aleph$.*

By adding the hypothesis of periodicity, we obtain that in an uncountable periodic soluble T^{-}-group, the second commutator subgroup is always small, independently on the derived length. Note that the second commutator subgroup exhibits the same behavior also in the case of uncountable soluble groups in which every large subnormal subgroup is normal (see [8, Corollary 3.4]).

Lemma 2.11. *Let G be an uncountable periodic soluble group of regular cardinality $\aleph$ in the class T^{-}. Then the second commutator subgroup G'' of G has cardinality strictly smaller than $\aleph$.*

Proof. As the uncountable group G is soluble, it contains a normal proper subgroup X having cardinality $\aleph$ (see [14, Lemma 2.1]). In particular, there exists a positive integer i such that $X^{(i)}$ has cardinality $\aleph$ and $X^{(i+1)}$ has cardinality strictly smaller than $\aleph$. An application of Corollary 2.3 ensures that the factor $G/X^{(i+1)}$ is a soluble T^*-group, so that its second commutator subgroup $G''X^{(i+1)}/X^{(i+1)}$ is finite, and hence G'' is small. □

3. Main Theorem

Recall here that a group G is an *FC-group* if all its elements admit only finitely many conjugates, i.e. if the centralizer $C_G(g)$ has finite index in G for each element g of G. It is well known that groups having finite commutator subgroup are *FC*-groups, and abelian-by-finite *FC*-groups are finite over their center. Central-by-finite groups have been characterized by Neumann [16] as precisely those groups in which every subgroup is almost normal. On the other hand, Casolo [2, 3] investigated the class V (respectively, V_m) consisting of all groups in which every subnormal subgroup admits only finitely many conjugates (at most a positive integer m, respectively).

Our first result of this section describes a general property of soluble groups in the class $V \cap T^*$, which is of independent interest.

Lemma 3.1. *Let G be a soluble group in the class $V \cap T^*$. Then the commutator subgroup G' of G is central-by-finite.*

Proof. As G is a soluble T^*-group, its second commutator subgroup G'' is finite. Therefore, G' is a soluble V-group having finite commutator subgroup, so that G' is an abelian-by-finite *FC*-group (see [4, Lemma 3.6]), and hence G' is central-by-finite. □

From now on, we will impose bounds on the indices involving large subnormal subgroups. Revisiting Lemma 2.8 through the imposition of the same bound on the index of every large subnormal subgroup in its normal closure, we obtain an analogue of a result due to Casolo (see [2, Lemma 2]), that we state here for the reader's convenience.

Lemma 3.2. *Let G be a soluble group. If there exists a positive integer m such that the index $|H^G : H| \leq m$ for every subnormal subgroup H of G, then the second commutator subgroup G'' of G is finite and its order is bounded by a function of m.*

Lemma 3.3. *Let G be an uncountable soluble group of regular cardinality $\aleph$. If G contains a large abelian subnormal subgroup and there exists a positive integer m such that the index $|H^G : H| \leq m$ for every large subnormal subgroup H of G, then the second commutator subgroup G'' of G is finite and its order is bounded by a function of m.*

Proof. As G contains a large abelian subnormal subgroup, there exists in G an abelian subgroup of the form $A = A_1 \times A_2$, where both A_1 and A_2 have cardinality $\aleph$ and are subnormal in G (see for instance [9, Corollary 3.2]). Then all subnormal subgroups of the soluble quotient G/A_i^G are nearly normal with the same bound as in G, so it follows from Lemma 3.2 that the second commutator subgroup $G''A_i^G/A_i^G$ of G/A_i^G is finite of order at most a positive integer k depending only on m, for $i = 1, 2$. Consequently, the index $|G''A_1^G \cap G''A_2^G : A_1^G \cap A_2^G|$ is finite of order at most k^2. On the other hand, as $|A_1^G : A_1| \leq m$ and $|A_2^G : A_2| \leq m$, we have that $|A_1^G \cap A_2^G| \leq m^2$. Therefore, the second commutator subgroup G'' has order at most m^2k^2. □

Lemma 3.4. *Let G be an uncountable soluble group of regular cardinality $\aleph$. If there exists a positive integer m such that the index $|H^G : H| \leq m$ for every large subnormal subgroup H of G, then the index $|G'' : G'' \cap H|$ is finite and bounded by a function of m, for every large subnormal subgroup H of G.*

Proof. Let H be a large subnormal subgroup of G. The factor G/H^G is a soluble group having all subnormal subgroups of index at most m in their normal closures, thus Lemma 3.2 ensures the existence of a positive integer k depending only on m such that the second commutator subgroup $G''H^G/H^G$ of G/H^G is finite of order less or equal to k. As the index $|H^G : H| \leq m$, it follows that $|G'' : G'' \cap H| \leq mk$. □

We are now in a position to prove our main result.

Proof. Assume first that the second commutator subgroup G'' of G is finite. Clearly, as the commutator subgroup G' has cardinality $\aleph$, the intersection $\bigcap_{H\in\mathcal{U}} (H')^G$ is finite. Conversely, assume now that the intersection $\bigcap_{H\in\mathcal{U}} (H')^G$ is finite, and let H be a large subnormal subgroup of G. Suppose first that the normal closure $(H')^G$ of H' in G has cardinality $\aleph$; then the factor group $\overline{G} = G/(H')^G$ belongs to U_m, so in particular it lies in $V_m \cap T_m$ (see [3]). Assume now that $(H')^G$ has cardinality strictly smaller than $\aleph$. As $\overline{G}$ contains the large abelian subnormal subgroup $H(H')^G/(H')^G$, an application of Lemmas 2.1 and 2.2 yields that $|\overline{H}^{\overline{G}} : \overline{H}| \leq m^5$ and $|\overline{H} : \overline{H}_{\overline{G}}| \leq m^2$, for every subnormal subgroup $\overline{H}$ of $\overline{G}$. In any case, independently on the cardinality of $(H')^G$, the soluble group $\overline{G}$ belongs to the class $U_{m^7} \subseteq V_{m^7} \cap T_{m^7}$, and hence its commutator subgroup $\overline{G'} = G'/(H')^G$ is central-by-finite by Lemma 3.1. Let $\mathcal{X}$ be the set of all subnormal subgroups X of G' such that the index $|G' : X| < \aleph$. As the intersection of all subgroups of small index of the uncountable abelian group $Z(\overline{G'})$ is trivial, we conclude that

$$\bigcap_{\substack{(H')^G \leq X \\ X\in\mathcal{X}}} X = (H')^G,$$

and hence the intersection

$$\bigcap_{X\in\mathcal{X}} X \leq \bigcap_{H\in\mathcal{U}} (H')^G$$

is finite. Moreover, for every $X \in \mathcal{X}$, it follows that X is a large subnormal subgroup of G, thus Lemma 3.4 yields that the index $|G'' : G'' \cap X|$ is finite and bounded by a function of m. Let K be an element of $\mathcal{X}$ such that the index $|G'' : G'' \cap K|$ is the largest possible. If X is any element of $\mathcal{X}$, $X \cap K$ also belongs to $\mathcal{X}$, therefore

$$|G'' : G'' \cap X \cap K| \leq |G'' : G'' \cap K|.$$

It follows that $G'' \cap X \cap K = G'' \cap K$, and hence $G'' \cap K \leq X$. Thus,

$$G'' \cap K \leq \bigcap_{X\in\mathcal{X}} X$$

is finite, so that G'' is likewise finite. □

Corollary 3.5. *Let G be an uncountable periodic soluble group of regular cardinality $\aleph$ having large commutator subgroup G', and let $\mathcal{U}$ be the set of all subnormal subgroups of G having cardinality $\aleph$. If the intersection $\bigcap_{H\in\mathcal{U}} (H')^G$ is finite and there exists a positive integer m such that $|H^G/H_G| \leq m$ for every $H \in \mathcal{U}$, then G is finite-by-U_{m^7}.*

Proof. The second commutator subgroup G'' of G is finite by Theorem 1.1, thus the factor group $\overline{G} = G/G''$ contains the large abelian normal subgroup G'/G''. An application of Lemma 2.1 and Lemma 2.2 yields that $|\overline{H}^{\overline{G}} : \overline{H}| \leq m^5$ and $|\overline{H} : \overline{H}_{\overline{G}}| \leq m^2$, for every subnormal subgroup $\overline{H}$ of $\overline{G}$. Therefore, $\overline{G}$ lies in the class U_{m^7} and the statement is proved. □

Acknowledgments

The author is a member of both GNSAGA (INdAM) and AGTA – Advances in Group Theory and Applications (www.advgrouptheory.com), as there was no financial support for this work.

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