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ON THE LUZIN N-PROPERTY AND THE UNCERTAINTY PRINCIPLE FOR SOBOLEV MAPPINGS

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We say that a mapping $v:\mathbb{R}^n\to\mathbb{R}^d$ satisfies the (τ,σ)-N-property if $\mathcal{H}^\sigma(v(E))=0$ whenever $\mathcal{H}^\tau(E)=0$, where $\mathcal{H}^\tau$ means the Hausdorff measure. We prove that every mapping v of Sobolev class $W_p^k(\mathbb{R}^n,\mathbb{R}^d)$ with $kp>n$ satisfies the (τ,σ)-N-property for every $0<\tau\neq\tau_*:=n-(k-1)p$ with

$$\sigma=\sigma(\tau):=\begin{cases}\tau & \text{if } \tau>\tau_*,\\ p\tau/(kp-n+\tau) & \text{if } 0<\tau<\tau_*.\end{cases}$$

We prove also that for $k>1$ and for the critical value $\tau=\tau_*$ the corresponding (τ,σ)-N-property fails in general. Nevertheless, this (τ,σ)-N-property holds for $\tau=\tau_*$ if we assume in addition that the highest derivatives $\nabla^k v$ belong to the Lorentz space $L_{p,1}(\mathbb{R}^n)$ instead of L_p.

We extend these results to the case of fractional Sobolev spaces as well. Also, we establish some Fubini-type theorems for N-Nproperties and discuss their applications to the Morse–Sard theorem and its recent extensions.

1. Introduction

The classical Luzin N-property means that for a mapping $f:\mathbb{R}^n\to\mathbb{R}^n$ one has meas $f(E)=0$ whenever meas $E=0$. (Here meas E is the usual n-dimensional Lebesgue measure.)

This property plays a crucial role in classical real analysis and differentiation theory [Saks 1937]. It is very useful also in elasticity theory and in geometrical analysis, especially in the theory of quasiconformal mappings and, more generally, in the theory of mappings with bounded distortions, i.e., mappings $f:\Omega\subset\mathbb{R}^n\to\mathbb{R}^n$ of Sobolev class $W_n^1(\mathbb{R}^n)$ such that $\|f'(x)\|^n\le K\det f'(x)$ almost everywhere with some constant $K\in[1,+\infty)$. The notion of mappings with bounded distortion was introduced by Yu. G. Reshetnyak; see, e.g., his classical books [Reshetnyak 1989; 1994; Goldshtein and Reshetnyak 1990]. He proved that they satisfy the N-property and this was very helpful in his subsequent proofs of other basic topological properties of such mappings (openness, discreteness and etc.). Further this MBD theory was successfully developed by many mathematicians in both analytical and geometrical directions, and many interesting and deep results were obtained; see the monographs [Rickman 1993; Iwaniec and Martin 2001], for example.

The notion of mappings with bounded distortion leads to the theory of more general mappings with finite distortion (i.e., when K in the definition above depends on x and is not assumed to be uniformly bounded; see, e.g., the pioneering paper [Vodop'yanov and Goldshtein 1976], where the monotonicity,

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continuity and N-property of such mappings from the class W^1_n were established). This theory has been intensively developed in the last decades (see, e.g., the book [Hencl and Koskela 2014] for an overview), and studying the N-property constitutes one of the most important directions [Kauhanen et al. 2001; D'Onofrio et al. 2016].

Note that the belonging of a mapping to the Sobolev class $W^1_n(\mathbb{R}^n,\mathbb{R}^n)$ is crucial for N-properties. Indeed, every mapping of class $W^1_p(\mathbb{R}^n,\mathbb{R}^n)$ with $p>n$ is continuous and supports the N-property (it is a simple consequence of the Morrey inequality). But even if a mapping $f\in W^1_n(\mathbb{R}^n,\mathbb{R}^n)$ is continuous (which is not guaranteed in general), it may not have the N-property. On the other hand, the N-property holds for functions of the class $W^1_n(\mathbb{R}^n,\mathbb{R}^n)$ under some additional assumptions on its topological features, namely, for homeomorphic and open mappings [Reshetnyak 1987] (see also [Roskovec 2018]) and for quasimonotone[1] mappings [Vodop'yanov and Goldshtein 1976; Malý and Martio 1995].

The results above are very delicate and sharp: indeed, for any $p<n$ there are homeomorphisms $f\in W^1_p(\mathbb{R}^n,\mathbb{R}^n)$ without the N-property. This phenomenon was discovered by S. P. Ponomarev [1971]. In recent years his construction has been very refined and an example was constructed of a Sobolev homeomorphism with zero Jacobian a.e. which belongs simultaneously to all the classes $W^1_p(\mathbb{R}^n,\mathbb{R}^n)$ with $p<n$ [Hencl 2011; Černý 2011] — of course, this "strange" homeomorphism certainly fails to have the N-property.[2]

In the positive direction, it was proved in [Kauhanen et al. 1999], see also [Romanov 2008], that every mapping of the Sobolev–Lorentz class $W^1_{n,1}(\mathbb{R}^n,\mathbb{R}^n)$ (i.e., its distributional derivatives belong to the Lorentz space $L_{n,1}$; see Section 2 for the exact definitions) satisfies the N-property. Note that this space $W^1_{n,1}(\mathbb{R}^n,\mathbb{R}^n)$ is limiting in a natural sense between classes W^1_n and W^1_p with $p>n$.

Another direction is to study the N-properties with respect to Hausdorff (instead of Lebesgue) measures. One of the most elegant results was achieved for the class of plane quasiconformal mappings.

The famous area distortion theorem of K. Astala [1994] implies the following dimension distortion result: if $f:\mathbb{C}\to\mathbb{C}$ is a K-quasiconformal mapping (i.e., it is a plane homeomorphic mapping with K-bounded distortion) and E is a compact set of Hausdorff dimension $t\in(0,2)$, then the image $f(E)$ has Hausdorff dimension at most $t'=2Kt/(2+(K-1)t)$. This estimate is sharp; however, it leaves open the endpoint case: does $\mathcal{H}^t(E)=0$ imply $\mathcal{H}^{t'}(f(E))=0$? The remarkable paper [Lacey et al. 2010] gives an affirmative answer to Astala's conjecture (see also [Astala et al. 2013], where the further implication $H^t(E)<\infty\Rightarrow\mathcal{H}^{t'}(f(E))<\infty$ was considered).

Let us go to results which are closer to the present paper. It is more natural to discuss the topic in the scale of fractional Sobolev spaces, i.e., for *(Bessel)-potential space* $\mathscr{L}^\alpha_p$ with $\alpha>0$. Recall that a function $v:\mathbb{R}^n\to\mathbb{R}^d$ belongs to the space $\mathscr{L}^\alpha_p$ if it is a convolution of the Bessel kernel K_α with a

[1] Some of these results were generalised for the more delicate case of Carnot groups and manifolds; see, e.g., [Vodop'yanov 2003].

[2] Moreover, even the examples of bi-Sobolev homeomorphisms of class $W^1_p(\mathbb{R}^n,\mathbb{R}^n)$, $p<n-1$, with zero Jacobian a.e. were constructed recently; see, e.g., [D'Onofrio et al. 2014; Černý 2015]. Such homeomorphisms are impossible in the Sobolev class $W^1_{n-1}(\mathbb{R}^n,\mathbb{R}^n)$. Furthermore, Hencl and Vejnar [2016] constructed an example of a Sobolev homeomorphism $f\in W^1_1((0,1)^n,\mathbb{R}^n)$ such that the Jacobian $\det f'(x)$ changes its sign on sets of positive measure.

function $g \in L_p(\mathbb{R}^n)$, where $\widehat{K}_\alpha(\xi) = (1 + 4\pi^2\xi^2)^{-\alpha/2}$. It is well known that

$$\mathscr{L}_p^\alpha(\mathbb{R}^n) = W_p^\alpha(\mathbb{R}^n) \quad \text{if } \alpha \in \mathbb{N} \text{ and } 1 < p < \infty.$$

Recently H. Hencl and P. Honzík proved, in particular, the following assertion:

Theorem 1.1 [Hencl and Honzík 2015]. *Let $n, d \in \mathbb{N}$, $\alpha > 0$, $p > 1$, $\alpha p > n$, and $0 < \tau \le n$. Suppose that a mapping $f : \mathbb{R}^n \to \mathbb{R}^d$ belongs to the (fractional) Sobolev class $\mathscr{L}_p^\alpha$. Then for any set $E \subset \mathbb{R}^n$ with Hausdorff dimension $\dim_H E \le \tau$ the inequality $\dim_H f(E) \le \sigma(\tau)$ holds, where*

$$\sigma(\tau) := \begin{cases} \tau & \text{if } \tau \ge \tau_* := n - (\alpha - 1)p, \\ p\tau/(\alpha p - n + \tau) & \text{if } 0 < \tau < \tau_*. \end{cases} \tag{1-1}$$

But as above (see the discussion around the Astala theorem), this result raises a natural question. What happens in the limiting case, i.e., is it true that $\mathcal{H}^\tau(E) = 0$ implies $\mathcal{H}^{\sigma(\tau)}(f(E)) = 0$? Of course, such an *N*-property is much more precise and stronger than the assertion of Theorem 1.1.

Six years ago G. Alberti [2012] announced the validity of the following result, obtained in collaboration with M. Csörnyei, E. D'Aniello and B. Kirchheim.

Theorem 1.2. *Let $k, n, d \in \mathbb{N}$, $p > 1$, $kp > n$, and $0 < \tau \le n$. Suppose that a mapping $f : \mathbb{R}^n \to \mathbb{R}^d$ belongs to the Sobolev class W_p^k and $\tau \ne \tau_* = n - (k-1)p$. Then f has the (τ, σ)-N-property, where the value $\sigma = \sigma(\tau)$ is defined in* (1-1).

Here for convenience we use the following notation: a mapping $f : \mathbb{R}^n \to \mathbb{R}^d$ is said to satisfy the (τ, σ)-*N*-property if $\mathcal{H}^\sigma(f(E)) = 0$ whenever $\mathcal{H}^\tau(E) = 0$, $E \subset \mathbb{R}^n$.

We remark that in [Alberti 2012] the limiting case $\tau = \tau_* > 0$ is left as *an open question*. Further, as far as we know, proofs of the results announced have not been published (it was written in [Alberti 2012] that the work was still "in progress").

In the present paper we extend the assertion above to the case of fractional Sobolev spaces and also we cover the critical case $\tau = \tau_*$ as well.

Theorem 1.3. *Let $\alpha > 0$, $1 < p < \infty$, $\alpha p > n$, and $v \in \mathscr{L}_p^\alpha(\mathbb{R}^n, \mathbb{R}^d)$. Suppose that $0 < \tau \le n$. Then the following assertions hold*:

(i) *If $\tau \ne \tau_* = n - (\alpha - 1)p$, then v has the (τ, σ)-N-property, where the value $\sigma = \sigma(\tau)$ is defined in* (1-1).

(ii) *If $\alpha > 1$ and $\tau = \tau_* > 0$, then $\sigma(\tau) = \tau_*$ and the mapping v in general has* ***no*** *(τ_*, τ_*)-N-property*; *i.e., it could be that $\mathcal{H}^{\tau_*}(v(E)) > 0$ for some $E \subset \mathbb{R}^n$ with $\mathcal{H}^{\tau_*}(E) = 0$.*

Remark 1.4. We stress that there is no "competition" with Alberti, Csörnyei, D'Aniello and Kirchheim concerning Theorems 1.2–1.3. When we published our first paper on the topic [Bourgain et al. 2013], those authors contacted us and it was agreed that mutual citations would be provided (and indeed appeared in [Alberti 2012; Bourgain et al. 2013]). Similarly, when the present paper was finished, we contacted one of those authors. They told us that after [Alberti 2012] they had some further progress, especially for $\tau = \tau_*$. We came to an agreement that each research group could publish their results with independent proofs, respecting each other's activity in the subject.

Remark 1.5. If $\alpha = 1$ and $p > n$, then $\tau_* = n$ and $\mathscr{L}_p^\alpha(\mathbb{R}^n, \mathbb{R}^d) = W_p^1(\mathbb{R}^n, \mathbb{R}^d)$, and the validity of the (τ, σ)-N-property for all $\tau \in (0, n]$ and for all mappings of these spaces is a simple corollary of the classical Morrey inequality [Malý and Martio 1995].

Theorem 1.3 omits the limiting cases $\alpha p = n$ and $\tau = \tau_*$. It is possible to cover these cases as well using the Lorentz norms. Namely, denote by $\mathscr{L}_{p,1}^\alpha(\mathbb{R}^n, \mathbb{R}^d)$ the space of functions which can be represented as a convolution of the Bessel potential K_α with a function g from the Lorentz space $L_{p,1}$ (see the definition of these spaces in Section 2); that is,

$$\|v\|_{\mathscr{L}_{p,1}^\alpha} := \|g\|_{L_{p,1}}.$$

Theorem 1.6. *Let $\alpha > 0$, $1 < p < \infty$, $\alpha p \geq n$, and $0 < \tau \leq n$. Suppose that $v \in \mathscr{L}_{p,1}^\alpha(\mathbb{R}^n, \mathbb{R}^d)$. Then v is a continuous function satisfying the (τ, σ)-N-property, where again the value $\sigma = \sigma(\tau)$ is defined in* (1-1) (*i.e., the limiting case $\tau = \tau_*$ is* ***included***).

Remark 1.7. In the case $\alpha = k \in \mathbb{N}$, $kp = n$, $p \geq 1$, we have $\tau_* = p$ and the validity of the (τ, σ)-N-property for mappings of the corresponding Sobolev–Lorentz space $W_{p,1}^k(\mathbb{R}^n, \mathbb{R}^d)$ was proved in [Bourgain et al. 2015; Korobkov and Kristensen 2018].

1A. ***The counterexample for the limiting case $\tau = \tau_*$ in Theorem 1.3(ii).*** Suppose again that

$$n > (\alpha - 1)p > n - p.$$

Let us demonstrate that the positive assertion in Theorem 1.3(i) is very sharp: it fails in general for the limiting case

$$\tau = \tau_* = n - (\alpha - 1)p.$$

Take

$$n = 4, \quad \alpha = 2, \quad p = 3.$$

Then by definition

$$\tau_* = 1.$$

So we have to construct a function from the Sobolev space $\mathscr{L}_3^2(\mathbb{R}^4) = W_3^2(\mathbb{R}^4)$ which does not have the N-property with respect to $\mathcal{H}^1$-measure. Consider the restrictions (traces) of functions from $W_3^4(\mathbb{R}^4)$ to the real line. It is well known that the space of these traces coincides exactly with the Besov space $B_{3,3}^1(\mathbb{R})$; see, e.g., [Jonsson and Wallin 1984, Chapter 1, Theorem 4 on p. 20]. Consider the function of one real variable

$$f_\sigma(x) = e^{-x^2} \sum_{m=1}^{\infty} 5^{-m} m^{-\sigma} \cos(5^m x),$$

where

$$\tfrac{1}{3} < \sigma < \tfrac{1}{2}.$$

It is known that $f_\sigma \in B_{3,3}^1(\mathbb{R})$ under the assumptions above; see, e.g., §6.8 in Chapter V of [Stein 1970]. Nevertheless, the following result holds.

Theorem 1.8. *The function $f_\sigma : \mathbb{R} \to \mathbb{R}$ from above does not have the* (1,1)-*N-property* (*with respect to $\mathcal{H}^1$-measure*).

This result is a direct consequence of the following two classical facts:

Theorem 1.9 [Saks 1937, Chapter IX, Theorem 7.7]. *If a continuous function $f:\mathbb{R}\to\mathbb{R}$ has the N-property, then it is differentiable on a set of positive measure.*

Theorem 1.10 [Zygmund 1959, Chapter V, §6, p. 206]. *The continuous function*

$$f(x)=\sum_{m=1}^{\infty} b^{-m}\varepsilon_m\cos(b^m x),$$

with $b>1$ and $\varepsilon_m\to 0$, $\sum_{m=1}^{\infty}\varepsilon_m^2=\infty$, is not differentiable almost everywhere.

Note that the functions f_σ, f from Theorems 1.8 and 1.10 are the typical examples of so-called lacunary Fourier series.

From Theorem 1.8 it follows that there exists a function $v\in W_3^2(\mathbb{R}^4)$ whose restriction to the real line coincides with f_σ; i.e., v does *not* have the (1,1)-N-property. The construction of the counterexample is finished.

1B. ***Fubini-type theorems for N-properties.*** The N-properties formulated above have an important application in the recent extension of the Morse–Sard theorem to Sobolev spaces (see [Ferone et al. 2017] and also Section 1C below). Here we need the following notion.

For a pair numbers $\tau,\sigma>0$ we will say that a continuous function $v:\mathbb{R}^n\to\mathbb{R}^d$ satisfies the (τ,σ)-N_*-property if for every $q\in[0,\sigma]$ and for any set $E\subset\mathbb{R}^n$ with $H^\tau(E)=0$ we have

$$\mathcal{H}^{\tau(1-\frac{q}{\sigma})}(E\cap v^{-1}(y))=0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y\in\mathbb{R}^d. \tag{1-2}$$

This implies, in particular, the usual (τ,σ)-N-property

$$\mathcal{H}^\sigma(v(E))=0 \quad \text{whenever } \mathcal{H}^\tau(E)=0.$$

(Indeed, it is sufficient to take $q=\sigma$ in (1-2).) In other words, the (τ,σ)-N_*-property is stronger than the usual (τ,σ) one.

The N_*-property can be considered as a Fubini-type theorem for the usual N-property. Now we can strengthen our previous results in the following way.

Theorem 1.11. *Let $\alpha>0$, $1<p<\infty$, $\alpha p>n$, and $v\in\mathscr{L}_p^\alpha(\mathbb{R}^n,\mathbb{R}^d)$. Suppose that $0<\tau\le n$. Then:*

(i) *If $\tau\ne\tau_*=n-(\alpha-1)p$, then v has the (τ,σ)-N_*-property, where the value $\sigma=\sigma(\tau)$ is defined in* (1-1).

(ii) *If $\alpha>1$ and $\tau=\tau_*$, then $\sigma(\tau)=\tau_*$ and the mapping v in general has no (τ_*,τ_*)-N-property; i.e., it could be that $\mathcal{H}^{\tau_*}(v(E))>0$ for some $E\subset\mathbb{R}^n$ with $\mathcal{H}^{\tau_*}(E)=0$.*

Remark 1.12. If $\alpha=1$ and $p>n$, then $\tau_*=n$ and $\mathscr{L}_p^\alpha(\mathbb{R}^n,\mathbb{R}^d)=W_p^1(\mathbb{R}^n,\mathbb{R}^d)$, and the validity of the (τ,σ)-N_*-property for all $\tau\in(0,n]$ and for all mappings of these spaces is a simple corollary of the classical Morrey inequality and Theorem 4.1 below.

Of course, Theorem 1.11 omits the limiting cases $\alpha p=n$ and $\tau=\tau_*$. Again, it is possible to cover these cases as well using the Lorentz norms.

Theorem 1.13. *Let $\alpha > 0$, $1 < p < \infty$, $\alpha p \geq n$, and $0 < \tau \leq n$. Suppose that $v \in \mathscr{L}^{\alpha}_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$. Then v is a continuous function satisfying the (τ, σ)-N_*-property, where again the value $\sigma = \sigma(\tau)$ is defined in* (1-1).

Remark 1.14. In the case $\alpha = k \in \mathbb{N}$, $kp = n$, $p \geq 1$, we have $\tau_* = p$ and the validity of the (τ, σ)-N_*-property for mappings of the corresponding Sobolev–Lorentz space $W^k_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$ was proved in [Bourgain et al. 2015; Hajłasz et al. 2017].

1C. ***Application to the Morse–Sard and Dubovitskiĭ–Federer theorems.*** The classical Morse–Sard theorem claims that for a mapping $v\colon \mathbb{R}^n \to \mathbb{R}^m$ of class C^k the measure of the set of critical values $v(Z_{v,m})$ is zero under the condition $k > \max(n-m, 0)$. Here the critical set, or m-critical set, is defined as $Z_{v,m} = \{x \in \mathbb{R}^n : \operatorname{rank} \nabla v(x) < m\}$. Further Dubovitskiĭ [1957; 1967] and Federer [1969, Theorem 3.4.3] independently found some elegant extensions of this theorem to the case of other (e.g., lower) smoothness assumptions. They also established the sharpness of their results within the C^k category.

Recently the following *bridge theorem*, which includes all the results above as particular cases, was found.

We say that a mapping $v : \mathbb{R}^n \to \mathbb{R}^d$ belongs to the class $C^{k,\alpha}$ for some integer positive k and $0 < \alpha \leq 1$ if there exists a constant $L \geq 0$ such that

$$|\nabla^k v(x) - \nabla^k v(y)| \leq L|x-y|^{\alpha} \quad \text{for all } x, y \in \mathbb{R}^n.$$

To simplify the notation, let us make the following agreement: for $\alpha = 0$ we identify $C^{k,\alpha}$ with usual spaces of C^k-smooth mappings. The following theorem was obtained in [Ferone et al. 2017].

Theorem 1.15. *Let $m \in \{1, \dots, n\}$, $k \geq 1$, $d \geq m$, $0 \leq \alpha \leq 1$, and $v \in C^{k,\alpha}(\mathbb{R}^n, \mathbb{R}^d)$. Then for any $q \in (m-1, \infty)$ the equality*

$$\mathcal{H}^{\mu_q}(Z_{v,m} \cap v^{-1}(y)) = 0 \quad \textit{for } \mathcal{H}^q\textit{-a.a. } y \in \mathbb{R}^d$$

holds, where

$$\mu_q = n - m + 1 - (k+\alpha)(q - m + 1),$$

and $Z_{v,m}$ denotes the set of m-critical points of v, that is, $Z_{v,m} = \{x \in \mathbb{R}^n : \operatorname{rank} \nabla v(x) \leq m-1\}$.

Here and in the following we interpret $\mathcal{H}^{\beta}$ as the counting measure when $\beta \leq 0$. Let us note that for the classical C^k-case, i.e., when $\alpha = 0$, the behaviour of the function μ_q is very natural:

$$\begin{aligned}
\mu_q &= 0 && \text{for } q = q_\circ = m - 1 + (n-m+1)/k && \text{(Dubovitskiĭ–Federer theorem, 1967)},\\
\mu_q &< 0 && \text{for } q > q_\circ && \text{(Dubovitskiĭ–Federer theorem, 1967)},\\
\mu_q &= n - m - k + 1 && \text{for } q = m && \text{(Dubovitskiĭ theorem, 1957)},\\
\mu_q &= n - m + 1 && \text{for } q = m - 1.
\end{aligned}$$

The last value cannot be improved in view of the trivial example of a linear mapping $L\colon \mathbb{R}^n \to \mathbb{R}^d$ of rank $m-1$.

Thus, Theorem 1.15 contains all the previous theorems (Morse–Sard, Dubovitskiĭ–Federer, and even the Bates theorem [1993] for $C^{k,1}$-Lipschitz functions) as particular cases.

Intuitively, the sense of this bridge theorem is very close to *Heisenberg's uncertainty principle* in theoretical physics: the more precise the information we receive on the measure of the image of the critical set, the less precisely the preimages are described, and vice versa.

The following analog of the bridge theorem, Theorem 1.15, was obtained for the Sobolev and fractional Sobolev cases (items (i)–(ii) and items (iii)–(iv), respectively).

Theorem 1.16 [Hajłasz et al. 2017; Ferone et al. 2017]. *Let* $m \in \{1, \dots, n\}$, $k \geq 1$, $d \geq m$, $0 \leq \alpha < 1$, $p \geq 1$ *and let* $v : \mathbb{R}^n \to \mathbb{R}^d$ *be a mapping for which one of the following cases holds*:

(i) $\alpha = 0$, $kp > n$, *and* $v \in W^k_p(\mathbb{R}^n, \mathbb{R}^d)$.

(ii) $\alpha = 0$, $kp = n$, *and* $v \in W^k_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$.

(iii) $0 < \alpha < 1$, $p > 1$, $(k+\alpha)p > n$, *and* $v \in \mathscr{L}^{k+\alpha}_p(\mathbb{R}^n, \mathbb{R}^d)$.

(iv) $0 < \alpha < 1$, $p > 1$, $(k+\alpha)p = n$, *and* $v \in \mathscr{L}^{k+\alpha}_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$.

Then for any $q \in (m-1, \infty)$ *the equality*

$$\mathcal{H}^{\mu_q}(Z_{v,m} \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d$$

holds, *where again*

$$\mu_q = n - m + 1 - (k+\alpha)(q - m + 1),$$

and $Z_{v,m}$ *denotes the set of* m*-critical points of* v, *that is*, $Z_{v,m} = \{x \in \mathbb{R}^n \setminus A_v : \operatorname{rank} \nabla v(x) \leq m-1\}$.

Here A_v means the set of nondifferentiability points for v. Recall, that by approximation results [Swanson 2002; Korobkov and Kristensen 2018] under the conditions of Theorem 1.16 the equalities

$$\begin{aligned} \mathcal{H}^\tau(A_v) = 0 &\quad \text{for all } \tau > \tau_* := n - (k+\alpha-1)p \quad \text{in cases (i), (iii)}, \\ \mathcal{H}^{\tau_*}(A_v) = \mathcal{H}^p(A_v) = 0 &\quad \tau_* := n - (k+\alpha-1)p = p \qquad \text{in cases (ii), (iv)} \end{aligned}$$

are valid (in particular, $A_v = \varnothing$ if $(k+\alpha-1)p > n$). Our purpose is to prove that the impact of the "bad" set A_v is negligible in the bridge Dubovitskiĭ–Federer theorem (Theorem 1.16), i.e., that the following statement holds:

Theorem 1.17. *Let the conditions of Theorem 1.16 be fulfilled for a function* $v : \mathbb{R}^n \to \mathbb{R}^d$. *Then*

$$\mathcal{H}^{\mu_q}(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d$$

for any $q > m-1$.

Remark 1.18. Since $\mu_q \leq 0$ for $q \geq q_\circ = m - 1 + \frac{n-m+1}{k+\alpha}$, the assertions of Theorems 1.16–1.17 are equivalent to the equality $0 = \mathcal{H}^q[v(A_v \cup Z_{v,m})]$ for $q \geq q_\circ$, so it is sufficient to check the assertions of Theorems 1.16–1.17 for $q \in (m-1, q_\circ]$ only.

Finally, let us comment briefly that the merge ideas for the proofs are from our previous papers [Bourgain et al. 2015; Korobkov and Kristensen 2014; 2018; Hajłasz et al. 2017]. In particular, the papers [Bourgain et al. 2013; 2015] by one of the authors with J. Bourgain and J. Kristensen contain many of the key ideas that allow us to consider nondifferentiable Sobolev mappings. For the implementation of these ideas one relies on estimates for the Hardy–Littlewood maximal function in terms of Choquet-type integrals with respect to Hausdorff capacity. In order to take full advantage of the Lorentz context we exploit the recent estimates from [Korobkov and Kristensen 2018] (recalled in Theorem 2.11 below); see also [Adams 1988] for the case $p = 1$.

2. Preliminaries

By an *n-dimensional interval* we mean a closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. If Q is an n-dimensional cubic interval then we write $\ell(Q)$ for its side-length.

For a subset S of $\mathbb{R}^n$ we write $\mathscr{L}^n(S)$ for its outer Lebesgue measure (sometimes we use the symbol $\operatorname{meas} S$ for the same object). The m-dimensional Hausdorff measure is denoted by $\mathcal{H}^m$ and the m-dimensional Hausdorff content by $\mathcal{H}^m_\infty$. Recall that for any subset S of $\mathbb{R}^n$ we have by definition

$$\mathcal{H}^m(S) = \lim_{t\searrow 0} \mathcal{H}^m_t(S) = \sup_{t>0} \mathcal{H}^m_t(S),$$

where for each $0 < t \le \infty$,

$$\mathcal{H}^m_t(S) = \inf\left\{ \sum_{i=1}^{\infty} (\operatorname{diam} S_i)^m : \operatorname{diam} S_i \le t,\ S \subset \bigcup_{i=1}^{\infty} S_i \right\}.$$

It is well known that $\mathcal{H}^n(S) = \mathcal{H}^n_\infty(S) \sim \mathscr{L}^n(S)$ for sets $S \subset \mathbb{R}^n$ ("$\sim$" means, here and in the following, that these values have upper and lower bounds with positive constants independent of the set S).

By $L_p(\mathbb{R}^n)$, $1 \le p \le \infty$, we will denote the usual Lebesgue space equipped with the norm $\|\cdot\|_{L_p}$. The notation $\|f\|_{L_p(E)}$ means $\|1_E \cdot f\|_{L_p}$, where 1_E is the indicator function of E.

Working with locally integrable functions, we always assume that the precise representatives are chosen. If $w \in L_{1,\mathrm{loc}}(\Omega)$, then the precise representative w^* is defined for *all* $x \in \Omega$ by

$$w^*(x) = \begin{cases} \lim_{r\searrow 0} ⨍_{B(x,r)} w(z)\,\mathrm{d}z & \text{if the limit exists and is finite,} \\ 0 & \text{otherwise,} \end{cases}$$

where the dashed integral as usual denotes the integral mean,

$$⨍_{B(x,r)} w(z)\,\mathrm{d}z = \frac{1}{\mathscr{L}^n(B(x,r))} \int_{B(x,r)} w(z)\,\mathrm{d}z,$$

and $B(x,r) = \{y : |y - x| < r\}$ is the open ball of radius r centred at x. Henceforth we omit special notation for the precise representative, writing simply $w^* = w$.

For $0 \le \beta < n$, the *fractional maximal function* of w of order β is given by

$$M_\beta w(x) = \sup_{r>0} r^\beta ⨍_{B(x,r)} |w(z)|\,\mathrm{d}z. \tag{2-1}$$

When $\beta = 0$, M_0 reduces to the usual Hardy-Littlewood maximal operator M.

The Sobolev space $\mathrm{W}^k_p(\mathbb{R}^n, \mathbb{R}^d)$ is as usual defined as consisting of those $\mathbb{R}^d$-valued functions $f \in \mathrm{L}_p(\mathbb{R}^n)$ whose distributional partial derivatives of orders $l \le k$ belong to $\mathrm{L}_p(\mathbb{R}^n)$; for detailed definitions and differentiability properties of such functions, see, e.g., [Evans and Gariepy 1992; Mazya 1985; Ziemer 1989; Dorronsoro 1989]. Denote by $\nabla^k f$ the vector-valued function consisting of all k-th order partial derivatives of f arranged in some fixed order. However, for the case of first order derivatives $k=1$ we shall often think of $\nabla f(x)$ as the Jacobi matrix of f at x, that is, the $d\times n$ matrix whose r-th row is the vector of partial derivatives of the r-th coordinate function.

We use the norm

$$\|f\|_{\mathrm{W}^k_p}=\|f\|_{\mathrm{L}_p}+\|\nabla f\|_{\mathrm{L}_p}+\cdots+\|\nabla^k f\|_{\mathrm{L}_p},$$

and unless otherwise specified all norms on the spaces $\mathbb{R}^s$ ($s\in\mathbb{N}$) will be the usual euclidean norms.

If $k<n$, then it is well known that functions from Sobolev spaces $\mathrm{W}^k_p(\mathbb{R}^n)$ are continuous for $p>n/k$ and can be discontinuous for $p\le p_\circ=n/k$ [Mazya 1985; Ziemer 1989]. The Sobolev–Lorentz space $\mathrm{W}^k_{p_\circ,1}(\mathbb{R}^n)\subset\mathrm{W}^k_{p_\circ}(\mathbb{R}^n)$ is a refinement of the corresponding Sobolev space. Among other things, functions that are locally in $\mathrm{W}^k_{p_\circ,1}$ on $\mathbb{R}^n$ are in particular continuous.

Here we only mentioned the Lorentz space $\mathrm{L}_{p,1}$, and in this case one may rewrite the norm as follows [Malý 2003, Proposition 3.6]:

$$\|f\|_{L_{p,1}}=\int_0^{+\infty}[\mathscr{L}^n(\{x\in\mathbb{R}^n:|f(x)|>t\})]^{\frac{1}{p}}\,\mathrm{d}t.$$

As for Lebesgue norm we set $\|f\|_{L_{p,1}(E)}:=\|1_E\cdot f\|_{L_{p,1}}$. Of course, we have the inequality

$$\|f\|_{L_p}\le\|f\|_{L_{p,1}}. \tag{2-2}$$

Moreover, recall that by properties of Lorentz spaces, the standard estimate

$$\|Mf\|_{L_{p,q}}\le C\|f\|_{L_{p,q}} \tag{2-3}$$

holds for $1<p<\infty$ [Malý 2003, Theorem 4.4].

Denote by $\mathrm{W}^k_{p,1}(\mathbb{R}^n)$ the space of all functions $v\in\mathrm{W}^k_p(\mathbb{R}^n)$ such that in addition the Lorentz norm $\|\nabla^k v\|_{\mathrm{L}_{p,1}}$ is finite.

2A. *On potential spaces $\mathscr{L}^\alpha_p$.* In this paper we deal with the (*Bessel*) *potential spaces* $\mathscr{L}^\alpha_p$ with $\alpha>0$. Recall that a function $v:\mathbb{R}^n\to\mathbb{R}^d$ belongs to the space $\mathscr{L}^\alpha_p$ if it is a convolution of the Bessel kernel K_α with a function $g\in L_p(\mathbb{R}^n)$:

$$v=\mathscr{G}_\alpha(g):=K_\alpha*g,$$

where $\widehat{K}_\alpha(\xi)=(1+4\pi^2\xi^2)^{-\alpha/2}$. In particular,

$$\|v\|_{\mathscr{L}^\alpha_p}:=\|g\|_{L_p}.$$

It is well known that

$$\mathscr{L}^\alpha_p(\mathbb{R}^n)=W^\alpha_p(\mathbb{R}^n)\quad\text{if }\alpha\in\mathbb{N}\text{ and }1<p<\infty, \tag{2-4}$$

and we use the agreement that $\mathscr{L}^\alpha_p(\mathbb{R}^n)=L_p(\mathbb{R}^n)$ when $\alpha=0$. Moreover, the following well-known result holds:

Theorem 2.1 [Stein 1970, Lemma 3, p. 136]. *Let $\alpha \geq 1$ and $1 < p < \infty$. Then $v \in \mathscr{L}_p^\alpha(\mathbb{R}^n)$ if and only if $v \in \mathscr{L}_p^{\alpha-1}(\mathbb{R}^n)$ and $\partial v/\partial x_j \in \mathscr{L}_p^{\alpha-1}(\mathbb{R}^n)$ for every $j = 1, \dots, n$.*

The following technical bounds will be used on several occasions (for convenience, we prove them in the Appendix).

Lemma 2.2. *Let $\alpha > 1$, $n + p > \alpha p > n$, and $p > 1$. Suppose that $v \in \mathscr{L}_p^\alpha(\mathbb{R}^n)$; i.e., $v = \mathscr{G}_\alpha(g)$ for some $g \in L_p(\mathbb{R}^n)$. Then for every n-dimensional cubic interval $Q \subset \mathbb{R}^n$ with $r = \ell(Q) \leq 1$ the estimate*

$$\operatorname{diam} v(Q) \leq C\left[\|Mg\|_{L_p(Q)} r^{\alpha-\frac{n}{p}} + \frac{1}{r^{n-1}} \int_Q I_{\alpha-1}|g|(y)\,\mathrm{d}y\right] \tag{2-5}$$

holds, where the constant C depends on n, p, d, α only, and

$$I_\beta f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|y-x|^{n-\beta}}\,\mathrm{d}y$$

is the Riesz potential of order β.

Sometimes it is not convenient to work with the Riesz potential, and we need also the following variant of the estimates above.

Lemma 2.3. *Let $\alpha > 0$, $n + p > \alpha p > n$, and $p > 1$. Suppose that $v \in \mathscr{L}_p^\alpha(\mathbb{R}^n)$; i.e., $v = \mathscr{G}_\alpha(g)$ for some $g \in L_p(\mathbb{R}^n)$. Fix arbitrary $\theta > 0$ such that $\alpha + \theta \geq 1$. Then for every n-dimensional cubic interval $Q \subset \mathbb{R}^n$ with $r = \ell(Q) \leq 1$ the estimate*

$$\operatorname{diam} v(Q) \leq C\left[\|Mg\|_{L_p(Q)} r^{\alpha-\frac{n}{p}} + \frac{1}{r^{n+\theta-1}} \int_Q M_{\alpha-1+\theta}\, g(y)\,\mathrm{d}y\right] \tag{2-6}$$

holds, where the constant C depends on n, p, d, α, θ only.

For reader's convenience, we prove Lemmas 2.2–2.3 in the Appendix.

2B. *On Lorentz potential spaces $\mathscr{L}_{p,1}^\alpha$.* To cover some other limiting cases, denote by $\mathscr{L}_{p,1}^\alpha(\mathbb{R}^n, \mathbb{R}^d)$ the space of functions which can be represented as a convolution of the Bessel potential K_α with a function g from the Lorentz space $L_{p,1}$; that is,

$$\|v\|_{\mathscr{L}_{p,1}^\alpha} := \|g\|_{L_{p,1}}.$$

Because of inequality (2-2), we have the evident inclusion

$$\mathscr{L}_{p,1}^\alpha(\mathbb{R}^n) \subset \mathscr{L}_p^\alpha(\mathbb{R}^n).$$

Since these spaces are not so common, let us discuss briefly some of their properties. We need some technical facts concerning the Lorentz spaces.

Lemma 2.4 [Rakotondratsimba 1998]. *Let $1 < p < \infty$. Then for any $j = 1, \dots, n$ the Riesz transform $\mathscr{R}_j$ is continuous from $L_{p,1}(\mathbb{R}^n)$ to $L_{p,1}(\mathbb{R}^n)$.*

Lemma 2.5 [Schep 1995]. *Let $1 < p < \infty$ and μ be a finite Borel measure on $\mathbb{R}^n$. Then the convolution transform $f \mapsto f * \mu$ is continuous in the space $L_{p,1}(\mathbb{R}^n)$ and in $\mathscr{L}_p^\alpha(\mathbb{R}^n)$ for all $\alpha > 0$.*

Using these facts and repeating almost word for word the arguments from [Stein 1970, §3.3–3.4], one can obtain the following very natural results.

Theorem 2.6 (cf. [Stein 1970, Lemma 3, p. 136]). *Let $\alpha \geq 1$ and $1 < p < \infty$. Then $f \in \mathscr{L}^{\alpha}_{p,1}(\mathbb{R}^n)$ if and only if $f \in \mathscr{L}^{\alpha-1}_{p,1}(\mathbb{R}^n)$ and $\partial f/\partial x_j \in \mathscr{L}^{\alpha-1}_{p,1}(\mathbb{R}^n)$ for every $j = 1, \dots, n$.*

Corollary 2.7. *Let $k \in \mathbb{N}$ and $1 < p < \infty$. Then $\mathscr{L}^{k}_{p,1}(\mathbb{R}^n) = W^{k}_{p,1}(\mathbb{R}^n)$, where $W^{k}_{p,1}(\mathbb{R}^n)$ is the space of functions such that all its distributional partial derivatives of order $\leq k$ belong to $L_{p,1}(\mathbb{R}^n)$.*

Note that the space $W^{k}_{p,1}(\mathbb{R}^n)$ admits an even simpler (but equivalent) description: it consists of functions f from the usual Sobolev space $W^{k}_{p}(\mathbb{R}^n)$ satisfying the additional condition $\nabla^k f \in L_{p,1}(\mathbb{R}^n)$ (i.e., this condition is on the highest derivatives only); see, e.g., [Malý 2003].

As before, we need some standard estimates.

Lemma 2.8. *Let $\alpha > 0$, $n + p \geq \alpha p \geq n$, and $p > 1$. Suppose that $v \in \mathscr{L}^{\alpha}_{p,1}(\mathbb{R}^n)$; i.e., $v = \mathscr{G}_{\alpha}(g)$ for some $g \in L_{p,1}(\mathbb{R}^n)$. Then the function v is continuous and for every n-dimensional cubic interval $Q \subset \mathbb{R}^n$ with $r = \ell(Q) \leq 1$ the estimate*

$$\operatorname{diam} v(Q) \leq C\left[\|Mg\|_{L_{p,1}(Q)} r^{\alpha-\frac{n}{p}} + \frac{1}{r^{n+\theta-1}} \int_Q M_{\alpha-1+\theta}\, g(y)\,\mathrm{d}y\right] \tag{2-7}$$

holds for arbitrary (fixed) parameter $\theta > 0$ such that $\alpha + \theta \geq 1$ (here the constant C again depends on n, p, d, α, θ only). Furthermore, if $\alpha > 1$, then

$$\operatorname{diam} v(Q) \leq C\left[\|Mg\|_{L_{p,1}(Q)} r^{\alpha-\frac{n}{p}} + \frac{1}{r^{n-1}} \int_Q I_{\alpha-1}|g|(y)\,\mathrm{d}y\right]. \tag{2-8}$$

For the reader's convenience, we prove Lemma 2.8 in the Appendix.

2C. ***On Choquet-type integrals.*** Let $\mathscr{M}^{\beta}$ be the space of all nonnegative Borel measures μ on $\mathbb{R}^n$ such that

$$|||\mu|||_{\beta} = \sup_{I \subset \mathbb{R}^n} \ell(I)^{-\beta} \mu(I) < \infty,$$

where the supremum is taken over all n-dimensional cubic intervals $I \subset \mathbb{R}^n$ and $\ell(I)$ denotes the side length of I.

Recall the following classical theorem proved by D. R. Adams.

Theorem 2.9 (see [Mazya 1985, §1.4.1] or [Adams 1973]). *Let $\alpha > 0$, $n - \alpha p > 0$, $s > p > 1$ and μ be a positive Borel measure on $\mathbb{R}^n$. Then for any $g \in L_p(\mathbb{R}^n)$ the estimate*

$$\int |I_{\alpha} g|^s \,\mathrm{d}\mu \leq C\, |||\mu|||_{\beta} \cdot \|g\|^s_{L_p} \tag{2-9}$$

holds with $\beta = \frac{s}{p}(n - \alpha p)$, where C depends on n, p, s, α only.

The estimate (2-9) fails for the limiting case $s = p$. Namely, there exist functions $g \in L_p(\mathbb{R}^n)$ such that $|I_{\alpha} g|(x) = +\infty$ on some set of positive $(n-\alpha p)$-Hausdorff measure. Nevertheless, there are two ways to cover this limiting case $s = p$. The first way is to use the maximal function M_{α} instead of the Riesz potential on the left-hand side of (2-9).

Theorem 2.10 [Adams 1998, Theorem 7, p. 28]. *Let $\alpha > 0$, $n-\alpha p > 0$, $s \geq p > 1$ and μ be a positive Borel measure on $\mathbb{R}^n$. Then for any $g \in L_p(\mathbb{R}^n)$ the estimate*

$$\int |M_\alpha g|^s \,\mathrm{d}\mu \leq C\, |||\mu|||_\beta \cdot \|g\|^s_{L_p} \tag{2-10}$$

holds with $\beta = (s/p)(n-\alpha p)$, where C depends on n, p, s, α only.

The second way is to use the Lorentz norm instead of the Lebesgue norm on the right-hand side of (2-9):

Theorem 2.11 [Korobkov and Kristensen 2018, Theorem 0.2]. *Let $\alpha > 0$, $n-\alpha p > 0$, and μ be a positive Borel measure on $\mathbb{R}^n$. Then for any $g \in L_p(\mathbb{R}^n)$ the estimate*

$$\int |I_\alpha g|^p \,\mathrm{d}\mu \leq C\, |||\mu|||_\beta \cdot \|g\|^p_{L_{p,1}}$$

holds with $\beta = n-\alpha p$, where C depends on n, p, α only.

2D. ***On Fubini-type theorems for N-properties.*** Recall that by the usual Fubini theorem if a set $E \subset \mathbb{R}^2$ has zero plane measure, then for $\mathcal{H}^1$-almost all straight lines L parallel to the coordinate axes we have $\mathcal{H}^1(L \cap E) = 0$. The next result can be considered as a Fubini-type theorem for the N-property.

Theorem 2.12 [Hajłasz et al. 2017, Theorem 5.3]. *Let $\mu \geq 0$, $q > 0$, and $v : \mathbb{R}^n \to \mathbb{R}^d$ be a continuous function. For a set $E \subset \mathbb{R}^n$ define the set function*

$$\Phi(E) = \inf_{E \subset \bigcup_j D_j} \sum_j (\operatorname{diam} D_j)^\mu [\operatorname{diam} v(D_j)]^q,$$

where the infimum is taken over all countable families of compact sets $\{D_j\}_{j\in\mathbb{N}}$ such that $E \subset \bigcup_j D_j$. Then $\Phi(\cdot)$ is countably subadditive and we have the implication

$$\Phi(E) = 0 \quad \Longrightarrow \quad \big[\mathcal{H}^\mu(E \cap v^{-1}(y)) = 0 \quad \textit{for } \mathcal{H}^q\textit{-a.a. } y \in \mathbb{R}^d\big].$$

2E. ***On local properties of considered potential spaces.*** Let $\mathcal{B}$ be some family of continuous functions defined on $\mathbb{R}^n$. For a set $\Omega \subset \mathbb{R}^n$ define the space $\mathcal{B}_{\mathrm{loc}}(\Omega)$ in the following standard way:

$$\mathcal{B}_{\mathrm{loc}}(\Omega) := \{f : \Omega \to \mathbb{R} : \text{for any compact set } E \subset \Omega, \text{there exists } g \in \mathcal{B} \text{ such that } f(x) = g(x) \text{ for all } x \in E\}.$$

For simplicity put $\mathcal{B}_{\mathrm{loc}} = \mathcal{B}_{\mathrm{loc}}(\mathbb{R}^n)$.

It is easy to see that for $\alpha > 0$ and $q > s > p > 1$ with $\alpha p \geq n$ the following inclusions hold:

$$\mathscr{L}^\alpha_{q,\mathrm{loc}} \subset \mathscr{L}^\alpha_{s,\mathrm{loc}} \subset \mathscr{L}^\alpha_{p,1,\mathrm{loc}}.$$

Since the N-properties have a local nature, this means that if we prove some N- (or N_*-) property for $\mathscr{L}^\alpha_p$, then the same N-property will be valid for the spaces $\mathscr{L}^\alpha_{p,1}$ and $\mathscr{L}^\alpha_q$ for all $q > p$. Similarly, if we prove some N- (or N_*-) property for $\mathscr{L}^\alpha_{p,1}$, then the same N-property will be valid for the spaces $\mathscr{L}^\alpha_q$ with $q > p$, etc.

3. Proofs of the N-properties (Theorems 1.3, 1.6)

In this section we will prove Theorems 1.3 and 1.6. For each theorem, we will consider different cases. The most interesting case is when $\alpha p < n + p$, which implies that $\tau_* > 0$: in such a situation we will consider the supercritical case $\tau > \tau_* > 0$ and the subcritical case $0 < \tau < \tau_*$ (see, respectively, Sections 3A and 3B below). The case $\alpha p \geq n + p$ is contained in Section 3C.

In the proofs we will consider a particular family of intervals to cover a given set, whose properties are more suitable for our aims. Below *a dyadic interval* means a closed cube in $\mathbb{R}^n$ of the form $[k_1/2^l, (k_1+1)/2^l] \times \cdots \times [k_n/2^l, (k_n+1)/2^l]$, where k_i, l are integers. Define

$$\Lambda^s(E) = \inf\left\{ \sum_{i=1}^{\infty} \ell(Q_i)^s : E \subset \bigcup_{i=1}^{\infty} Q_i,\ Q_i \text{ dyadic} \right\}.$$

It is well known that $\Lambda^s(E) \sim \mathcal{H}^s(E)$ for all subset $E \subset \mathbb{R}^n$; in particular, Λ^s and $\mathcal{H}^s$ have the same null sets.

Let $\{Q_j\}_{j\in\mathbb{N}}$ be a family of n-dimensional dyadic intervals. For a given parameter $\tau > 0$ we say that the family $\{Q_j\}$ is *regular* if $\sum \ell(Q_j)^\tau < \infty$ and for any n-dimensional dyadic interval Q the estimate

$$\ell(Q)^\tau \geq \sum_{j:Q_j\subset Q} \ell(Q_j)^\tau \tag{3-1}$$

holds. Since dyadic intervals are either nonoverlapping or contained in one another, (3-1) implies that any regular family $\{Q_j\}$ must in particular consist of nonoverlapping intervals. Moreover, the following result holds.

Lemma 3.1 [Bourgain et al. 2015, Lemma 2.3]. *Let $\{J_i\}$ be a family of n-dimensional dyadic intervals with $\sum_i \ell(J_i)^\tau < \infty$. Then there exists a regular family $\{Q_j\}$ of n-dimensional dyadic intervals such that $\bigcup_i J_i \subset \bigcup_j Q_j$ and*

$$\sum_j \ell(Q_j)^\tau \leq \sum_i \ell(J_i)^\tau.$$

3A. ***Proof of Theorem 1.3: the supercritical case $\tau > \tau_* > 0$.*** Fix the parameters $n \in \mathbb{N}$, $\alpha > 0$, $p > 1$ such that

$$\alpha p > n, \quad \tau_* = n - (\alpha - 1)p > 0, \tag{3-2}$$

and take

$$\tau \in (\tau_*, n]. \tag{3-3}$$

Fix also a mapping $v \in \mathscr{L}^\alpha_p(\mathbb{R}^n, \mathbb{R}^d)$. If $\alpha = 1$, then $v \in W^1_p(\mathbb{R}^n)$ with $p > n$ and $\tau = n$, and the result is well known. So we restrict our attention to the nontrivial case $\alpha > 1$, $\tau < n$.

Now let $\{Q_i\}_{i\in\mathbb{N}}$ be a regular family of n-dimensional dyadic intervals. Consider the corresponding measure μ defined as

$$\int f \, d\mu := \sum_i \frac{1}{\ell(Q_i)^{n-\tau}} \int_{Q_i} f(y) \, dy. \tag{3-4}$$

As usual, for a measurable set $E \subset \mathbb{R}^n$ put $\mu(E) = \int 1_E \,\mathrm{d}\mu$, where 1_E is an indicator function of E.

Lemma 3.2 [Korobkov and Kristensen 2014, Lemma 2.4]. *For any regular family $\{Q_i\}_{i\in\mathbb{N}}$ of n-dimensional dyadic intervals the corresponding measure μ defined by* (3-4) *satisfies*

$$\mu(Q) \le \ell(Q)^\tau$$

for any dyadic cube $Q \subset \mathbb{R}^n$.

From this fact and from the Adams Theorem 2.9, we immediately obtain:

Lemma 3.3. *Let $g \in L_p(\mathbb{R}^n)$. Then for any regular family $\{Q_i\}$ of n-dimensional dyadic intervals the estimate*

$$\sum_i \frac{1}{\ell(Q_i)^{n-\tau}} \int_{Q_i} (I_{\alpha-1}|g|)^s \,\mathrm{d}y \le C\|g\|^s_{L_p} \tag{3-5}$$

holds, where $s := (\tau/\tau_*)p > p$ and C does not depend on g.

Now we are ready to formulate the key step of the proof.

Lemma 3.4. *Under the assumptions above, for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, v) > 0$ such that for any regular family $\{Q_i\}$ of n-dimensional dyadic intervals if*

$$\sum_i \ell(Q_i)^\tau < \delta,$$

then

$$\sum_i [\operatorname{diam} v(Q_i)]^\tau < \varepsilon.$$

Proof. Since $v \in \mathscr{L}^\alpha_p(\mathbb{R}^n, \mathbb{R}^d)$, by the definition of this space, it is easy to see that for any $\tilde{\varepsilon} > 0$ there exists a representation

$$v = v_1 + v_2,$$

where $v_i \in \mathscr{L}^\alpha_p(\mathbb{R}^n, \mathbb{R}^d)$, $v_1 \in C^\infty(\mathbb{R}^n)$,

$$\|\nabla v_1\|_{L_\infty(\mathbb{R}^n)} < \infty,$$

and

$$v_2 = \mathscr{G}_\alpha(g) \quad \text{with } \|g\|_{L_p} < \tilde{\varepsilon}. \tag{3-6}$$

This means, in particular, that

$$|\nabla v_1(x)| < K \quad \text{for all } x \in \mathbb{R}^n, \tag{3-7}$$

for some $K = K(\tilde{\varepsilon}, v) \in \mathbb{R}$. Take any regular family $\{Q_i\}$ of n-dimensional dyadic intervals such that

$$\sum_i \ell(Q_i)^\tau < \delta \tag{3-8}$$

(the exact value of δ will be specified below). Put $r_i = \ell(Q_i)$. Then by Lemma 2.2

$$\sum_i [\operatorname{diam} v(Q_i)]^\tau \le C(S_1 + S_2 + S_3),$$

where

$$S_1=\sum_i [\operatorname{diam} v_1(Q_i)]^\tau \overset{(3\text{-}7)-(3\text{-}8)}{\le} n^{\tau/2}K^\tau\delta,$$

$$S_2=\sum_i \|Mg\|^\tau_{L_p(Q_i)} r_i^{\tau(\alpha-\frac{n}{p})},$$

$$S_3=\sum_i\biggl(\frac{1}{r_i^{n-1}}\int_{Q_i} I_{\alpha-1}|g|(y)\,\mathrm{d}y\biggr)^\tau.$$

Let us estimate S_2. Since $\alpha-\frac{n}{p}<1$ by (3-2), we can apply the Hölder inequality to obtain

$$S_2\le\biggl(\sum_i \|Mg\|^{\tau\frac{p}{n-p(\alpha-1)}}_{L_p(Q_i)}\biggr)^{\frac{n}{p}-\alpha+1}\cdot\biggl(\sum_i r_i^\tau\biggr)^{\alpha-\frac{n}{p}} \overset{(3\text{-}8)}{\le} \biggl(\sum_i \|Mg\|^{\tau\frac{p}{n-p(\alpha-1)}}_{L_p(Q_i)}\biggr)^{\frac{n}{p}-\alpha+1}\cdot\delta^{\alpha-\frac{n}{p}}=$$

$$\overset{(3\text{-}2)}{=}\biggl(\sum_i \|Mg\|^{p\frac{\tau}{\tau_*}}_{L_p(Q_i)}\biggr)^{\frac{\tau_*}{p}}\cdot\delta^{\alpha-\frac{n}{p}} \overset{(3\text{-}3)}{\le} \|Mg\|^\tau_{L_p(\cup_i Q_i)}\cdot\delta^{\alpha-\frac{n}{p}} \overset{(3\text{-}6)}{\le} C\,\tilde\varepsilon^\tau\cdot\delta^{\alpha-\frac{n}{p}};$$

here C is the constant from the the Hardy–Littlewood maximal inequality. Similarly, taking $s=(\tau/\tau_*)p$ and applying twice the Hölder inequality in S_3 (the first time for the integrals, and the second time for sums), we obtain

$$S_3\le\sum_i\biggl(\int_{Q_i}(I_{\alpha-1}|g|)^s\,\mathrm{d}y\biggr)^{\frac{\tau_*}{p}}\cdot r_i^{n(\tau-\frac{\tau_*}{p})}\cdot r_i^{(1-n)\tau}=\sum_i\biggl(\frac{1}{r_i^{n-\tau}}\int_{Q_i}(I_{\alpha-1}|g|)^s\,\mathrm{d}y\biggr)^{\frac{\tau_*}{p}}\cdot r_i^{(1-\frac{\tau_*}{p})\tau}$$

$$\overset{\text{Hölder}}{\le}\biggl(\sum_i\frac{1}{r_i^{n-\tau}}\int_{Q_i}(I_{\alpha-1}|g|)^s\,\mathrm{d}y\biggr)^{\frac{\tau_*}{p}}\cdot\biggl(\sum_i r_i^\tau\biggr)^{1-\frac{\tau_*}{p}} \overset{(3\text{-}5),\,(3\text{-}6),\,(3\text{-}8)}{\le} C\,\tilde\varepsilon^\tau\cdot\delta^{1-\frac{\tau_*}{p}}.$$

So taking δ sufficiently small so that $K^\tau\delta<\frac12\varepsilon$ is small, we have $S_1+S_2+S_3<\varepsilon$ as required, and Lemma 3.4 is proved. □

Finally, if E is a set such that $\mathcal{H}^\tau(E)=0$, then also $\Lambda^\tau(E)=0$, and this lemma together with Lemma 3.1 implies the validity of the assertion Theorem 1.3(i) for the supercritical case $\tau>\tau_*>0$.

3B. *Proof of Theorem 1.3: the subcritical case* $\mathbf{0<\tau<\tau_*}$. Now fix the parameters $n\in\mathbb{N}$, $\alpha>0$, $p>1$ such that

$$\alpha p>n,\quad \tau_*=n-(\alpha-1)p>0, \tag{3-9}$$

and take

$$\tau\in(0,\tau_*),\quad \sigma=\frac{p\tau}{\alpha p-n+\tau}.$$

Evidently, by this definition

$$\sigma>\tau. \tag{3-10}$$

Fix also a mapping $v\in\mathscr{L}^\alpha_p(\mathbb{R}^n,\mathbb{R}^d)$. Take an additional parameter θ such that

$$(\alpha-1+\theta)>0\quad\text{and}\quad n-(\alpha-1+\theta)p>0.$$

From Lemma 3.2 and the Adams theorem 2.10, taking $s = p$, we immediately obtain:

Lemma 3.5. *Let $g \in L_p(\mathbb{R}^n)$. Then for any τ-regular family $\{Q_i\}$ of n-dimensional dyadic intervals the estimate*

$$\sum_i \frac{1}{\ell(Q_i)^{n-\tau_\theta}} \int_{Q_i} (M_{\alpha-1+\theta}|g|)^p \, dy \le C\|g\|_{L_p}^p \tag{3-11}$$

holds, where $\tau_\theta = n - (\alpha - 1 + \theta)p$ and C does not depend on g.

As in the previous case, the proof of Theorem 1.3 in the case $0 < \tau < \tau^*$ will be complete once we establish the following result.

Lemma 3.6. *Under above assumptions, for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, v) > 0$ such that for any regular family $\{Q_i\}$ of n-dimensional dyadic intervals if*

$$\sum_i \ell(Q_i)^\tau < \delta,$$

then

$$\sum_i [\operatorname{diam} v(Q_i)]^\sigma < \varepsilon.$$

Proof. Again, since $v \in \mathscr{L}_p^\alpha(\mathbb{R}^n, \mathbb{R}^d)$, by the definition of this space, for any $\tilde{\varepsilon} > 0$ there exists a representation

$$v = v_1 + v_2,$$

where $v_i \in \mathscr{L}_p^\alpha(\mathbb{R}^n, \mathbb{R}^d)$, $v_1 \in C^\infty(\mathbb{R}^n)$,

$$\|\nabla v_1\|_{L_\infty(\mathbb{R}^n)} < \infty,$$

and

$$v_2 = \mathscr{G}_\alpha(g) \quad \text{with } \|g\|_{L_p} < \tilde{\varepsilon}. \tag{3-12}$$

This means, in particular, that

$$|\nabla v_1(x)| < K \quad \text{for all } x \in \mathbb{R}^n, \tag{3-13}$$

for some $K = K(\tilde{\varepsilon}, v) \in \mathbb{R}$. Take any regular family $\{Q_i\}$ of n-dimensional dyadic intervals such that

$$\sum_i \ell(Q_i)^\tau < \delta < 1 \tag{3-14}$$

(the exact value of δ will be specified below). Put $r_i = \ell(Q_i)$. Then by Lemma 2.3

$$\sum_i [\operatorname{diam} v(Q_i)]^\sigma \le C(S_1 + S_2 + S_3),$$

where

$$S_1 = \sum_i [\operatorname{diam} v_1(Q_i)]^\sigma \overset{(3\text{-}10),(3\text{-}13)-(3\text{-}14)}{\leq} CK^\sigma \delta,$$

$$S_2 = \sum_i \|Mg\|^\sigma_{L_p(Q_i)} r_i^{\sigma(\alpha - \frac{n}{p})},$$

$$S_3 = \sum_i \left(\frac{1}{r_i^{n-1+\theta}} \int_{Q_i} M_{\alpha-1+\theta} g(y)\,\mathrm{d}y \right)^\sigma.$$

Let us estimate S_2. Since by assumptions (3-9) the inequality $\sigma < p$ holds and

$$\frac{p-\sigma}{p} = \frac{\alpha p - n}{\alpha p - n + \tau}, \quad \sigma \frac{p}{p-\sigma} = \frac{\tau}{\alpha - (n/p)} \tag{3-15}$$

we can apply the Hölder inequality to obtain

$$S_2 \leq \left(\sum_i \|Mg\|^p_{L_p(Q_i)} \right)^{\frac{\sigma}{p}} \cdot \left(\sum_i r_i^{\sigma(\alpha - \frac{n}{p}) \frac{p}{p-\sigma}} \right)^{\frac{p-\sigma}{p}}$$
$$= (\|Mg\|^p_{L_p(\cup_i Q_i)})^{\frac{\sigma}{p}} \cdot \left(\sum_i r_i^\tau \right)^{\frac{p-\sigma}{p}} \overset{(3\text{-}14),\,(3\text{-}12)}{\leq} C\,\tilde{\varepsilon}^\sigma \delta^{1-\frac{\sigma}{p}}.$$

Similarly, applying twice the Hölder inequality in S_3 (the first time for the integrals, and the second time for sums), we obtain

$$S_3 \leq \sum_i \left(\int_{Q_i} (M_{\alpha-1+\theta} g)^p \,\mathrm{d}y \right)^{\frac{\sigma}{p}} \cdot r_i^{n \frac{p-1}{p} \sigma} \cdot r_i^{(1-n-\theta)\sigma} = \sum_i \left(\frac{1}{r_i^{n-\tau\theta}} \int_{Q_i} (M_{\alpha-1+\theta}|g|)^p \,\mathrm{d}y \right)^{\frac{\sigma}{p}} \cdot r_i^{(\alpha - \frac{n}{p})\sigma}$$
$$\overset{\text{Hölder}}{\leq} \left(\sum_i \frac{1}{r_i^{n-\tau\theta}} \int_{Q_i} (M_{\alpha-1+\theta}|g|)^p \,\mathrm{d}y \right)^{\frac{\sigma}{p}} \cdot \left(\sum_i r_i^{(\alpha - \frac{n}{p})\sigma \frac{p}{p-\sigma}} \right)^{1-\frac{\sigma}{p}}$$
$$\overset{(3\text{-}15)}{=} \left(\sum_i \frac{1}{r_i^{n-\tau\theta}} \int_{Q_i} (M_{\alpha-1+\theta}|g|)^p \,\mathrm{d}y \right)^{\frac{\sigma}{p}} \cdot \left(\sum_i r_i^\tau \right)^{1-\frac{\sigma}{p}} \overset{(3\text{-}11),\,(3\text{-}12),\,(3\text{-}14)}{\leq} C\,\tilde{\varepsilon}^\sigma \cdot \delta^{1-\frac{\sigma}{p}}.$$

So taking δ sufficiently small so that $K^\tau \delta < \frac{1}{2}\varepsilon$ is small, we have $S_1 + S_2 + S_3 < \varepsilon$ as required, and the lemma is proved. □

Finally, we conclude exactly as in the previous case.

3C. ***Proof of Theorem 1.3: the supercritical case $\tau_* \leq 0 < \tau$.*** Consider now the case $\alpha p > n$ and $\tau_* = n - (\alpha - 1)p \leq 0$. If $(\alpha - 1)p > n$, then every function $v \in \mathscr{L}^\alpha_p(\mathbb{R}^n, \mathbb{R}^d)$ is locally Lipschitz (even C^1) and the result is trivial. Suppose now $(\alpha - 1)p = n$. Under these assumptions, let $\tau > 0$ and $v \in \mathscr{L}^\alpha_p(\mathbb{R}^n, \mathbb{R}^d)$. Take a number $1 < \tilde{p} < p$ such that $\alpha \tilde{p} > n$ and $\tau > \tau^*_* = n - (\alpha - 1)\tilde{p} > 0$. Then we have $v \in \mathscr{L}^\alpha_{\tilde{p},\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^d)$ (see Section 2E). Therefore, by the previous case $\tau > \tilde{\tau}_* > 0$, the mapping v has the (τ, τ)-N-property. □

3D. ***Proof of Theorem 1.6.*** The proof of Theorem 1.6 is very similar to that of Theorem 1.3: the main differences concern the limiting cases $\alpha p = n$ or $\tau = \tau^*$.

Case I: $\alpha p > n$ and $\tau \neq \tau^*$. The required assertion follows immediately from Theorem 1.3 and from the inclusion $\mathscr{L}^\alpha_{p,1}(\mathbb{R}^n) \subset \mathscr{L}^\alpha_p(\mathbb{R}^n)$ (this inclusion follows from the definitions of these space and from the relation $L_{p,1}(\mathbb{R}^n) \subset L_p(\mathbb{R}^n)$).

Case II: $\alpha p = n$ and $\tau > \tau_* > 0$. The required assertion can be proved by repeating almost word for word the same arguments as in the supercritical case in Theorem 1.3 with the following evident modifications: now one has to apply the estimate (2-8) (which covers the case $\alpha p = n$) instead of previous estimate (2-5), and, in addition, one needs the following analog of the additivity property for the Lorentz norms:

$$\sum_i \|f\|^p_{\mathrm{L}_{p,1}(Q_i)} \leq \|f\|^p_{\mathrm{L}_{p,1}(\cup_i Q_i)}$$

for any family of disjoint cubes [Malý 2003, Lemma 3.10].

Case III: $\alpha p \geq n$ and $\tau = \tau^*$. The required assertion can be proved by repeating almost word for word the same arguments as in the supercritical case in Theorem 1.3 with the following evident modifications: now $\tau = \tau_*$ (this simplifies the calculations a little bit) and one has to apply Theorem 2.11 (which covers the case $s = p$) and the estimate (2-8) instead of Theorem 2.9 (where $s > p$) and the inequality (2-5), respectively.

Case IV: $\alpha p = n$ and $0 < \tau < \tau^*$. By a direct calculation, we get $\sigma(\tau) \equiv p$ for any $\tau \in (0, \tau_*]$, and the result follows from the above-considered critical case $\tau = \tau_*$.

Thus Theorems 1.3 and 1.6 are proved completely.

Remark 3.7. Really, we have proved that under the assumptions of Theorems 1.3 and 1.6, for every fixed function $v : \mathbb{R}^n \to \mathbb{R}^d$ from the considered potential spaces and for the corresponding pair (τ, σ) the following assertion holds: for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every τ-regular family of cubes $Q_i \subset \mathbb{R}^n$ if $\sum_i \ell(Q_i)^\tau < \delta$, then $\sum_i [\operatorname{diam} v(Q_i)]^\sigma < \varepsilon$.

4. Proof of "Fubini-type" N_*-properties

Here we have to prove Theorems 1.11 and 1.13. We need the following general fact.

Theorem 4.1. *Let $\tau \in (0, n]$, $\sigma > 0$, and let $v : \mathbb{R}^n \to \mathbb{R}^d$ be a continuous function. Suppose that for any $E \subset \mathbb{R}^n$ with $\mathcal{H}^\tau(E) = 0$ and for every $\varepsilon > 0$ there exists a family of compact sets $\{D_i\}_{i \in \mathbb{N}}$ such that*

$$E \subset \bigcup_i D_i \quad \text{and} \quad \sum_i [\operatorname{diam} D_i]^\tau < \varepsilon \quad \text{and} \quad \sum_i [\operatorname{diam} v(D_i)]^\sigma < \varepsilon. \tag{4-1}$$

Then v has the (τ, σ)-N_*-property; i.e., for every $q \in [0, \sigma]$ and for any set $E \subset \mathbb{R}^n$ with $H^\tau(E) = 0$ we have

$$\mathcal{H}^{\tau\left(1-\frac{q}{\sigma}\right)}(E \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d. \tag{4-2}$$

Proof. Let the assumptions of the theorem be fulfilled. Fix $q \in [0,\sigma]$. If $q=0$ or $q=\sigma$, then the required assertion (4-2) follows trivially from these assumptions. Suppose now that

$$0 < q < \sigma.$$

Fix an arbitrary $\varepsilon > 0$ and take the corresponding sequence of compact sets D_i satisfying (4-1). Put $\mu = \tau\bigl(1-\frac{q}{\sigma}\bigr) < \tau$. Then

$$\begin{aligned}\sum_i (\operatorname{diam} D_i)^{\mu}[\operatorname{diam} v(D_i)]^q &\overset{\text{Hölder}}{\le} \Bigl(\sum_i [\operatorname{diam} D_i]^{\mu\frac{\sigma}{\sigma-q}}\Bigr)^{1-\frac{q}{\sigma}} \cdot \Bigl(\sum_i [\operatorname{diam} v(D_i)]^{\sigma}\Bigr)^{\frac{q}{\sigma}} \\ &= \Bigl(\sum_i [\operatorname{diam} D_i]^{\tau}\Bigr)^{1-\frac{q}{\sigma}} \Bigl(\sum_i [\operatorname{diam} v(D_i)]^{\sigma}\Bigr)^{\frac{q}{\sigma}} \overset{(4\text{-}1)}{<} \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, now the required assertion follows immediately from Theorem 2.12. □

The theorem just proved and Remark 3.7 clearly imply the assertions of Theorems 1.11 and 1.13.

4A. ***Proof of Theorem 1.17.*** Fix a mapping $v : \mathbb{R}^n \to \mathbb{R}^d$ for which the assumptions of Theorem 1.16 are fulfilled. We have to prove that

$$\mathcal{H}^{\mu_q}(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d, \tag{4-3}$$

for any $q > m-1$, where $\mu_q = n-m+1-(k+\alpha)(q-m+1)$ and A_v is the set of nondifferentiability points of v. Recall that, by approximation results [Swanson 2002; Korobkov and Kristensen 2018], under the conditions of Theorem 1.16 the equalities

$$\mathcal{H}^{\tau}(A_v) = 0 \quad \text{for all } \tau > \tau_* := n-(k+\alpha-1)p \quad \text{in cases (i), (iii),} \tag{4-4}$$

$$\mathcal{H}^{\tau_*}(A_v) = \mathcal{H}^{p}(A_v) = 0 \quad \tau_* := n-(k+\alpha-1)p = p \quad \text{in cases (ii), (iv)} \tag{4-5}$$

are valid.

Because of Remark 1.18 we can assume without loss of generality that $q \in (m-1, q_\circ]$. Then for all cases (i)–(iv) we have

$$\begin{aligned}\Bigl(\frac{n}{k+\alpha} \le p\Bigr) \;&\Longrightarrow\; \Bigl(q-m+1 \le q_\circ - m+1 = \frac{n-m+1}{k+\alpha} \le p\Bigr) \\ &\Longrightarrow\; \mu_q = n-m+1-(k+\alpha)(q-m+1) \\ &\qquad\quad = n-(k+\alpha-1)(q-m+1)-q \ge n-(k+\alpha-1)p-q = \tau_*-q.\end{aligned}$$

In other words,

$$\mu_q \ge \tau_* - q, \tag{4-6}$$

where the equality holds if and only if

$$k=1, \quad \alpha=0, \quad \mu_q = n-q = \tau_*-q \tag{4-7}$$

or

$$m = 1, \quad (k+\alpha)p = n, \quad q = p = \tau_*, \quad \mu_q = 0. \tag{4-8}$$

Below for convenience we consider the cases (i)–(iv) of Theorem 1.16 separately.

Case I: $\alpha = 0,\ kp > n,\ p \geq 1,\ v \in W^k_p(\mathbb{R}^n, \mathbb{R}^d)$. This case splits into the following three subcases.

Case Ia: $k = 1,\ p > n,\ \tau_* = n,\ \mu_q = n - q$. Then the required assertion (4-3) follows immediately from the equality $\mathcal{H}^n(A_v) = 0$ and from Remark 1.12.

Case Ib: $\tau_* < 0$ or $\tau_* = 0,\ k = n+1,\ p = 1$. Then the set A_v is empty (since functions of the space $W^k_p(\mathbb{R}^n, \mathbb{R}^d)$ are C^1-smooth), and there is nothing to prove.

Case Ic: $\tau_* \geq 0,\ p > 1,\ k > 1,\ kp > n$. Then by (4-4) we have

$$\text{for all } \tau > \tau_*, \quad \mathcal{H}^\tau(A_v) = 0. \tag{4-9}$$

Further, by Theorem 1.11 the function v has the (τ,τ)-N_*-property for every $\tau > \tau_*$. This implies, in particular, by virtue of (4-9), that for every $\tau > \tau_*$ and for every $q \in [0, \tau]$ the equality

$$\mathcal{H}^{\tau - q}(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d \tag{4-10}$$

holds. Fix $q \in (m-1, q_\circ]$ and take $\tau = q + \mu_q$. Since by construction $\mu_q \geq 0$, we have $\tau \geq q$. Moreover, by (4-6)–(4-8) we have $\tau > \tau_*$. The last two inequalities together with (4-10) imply

$$\mathcal{H}^{\mu_q}(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d.$$

So the required assertion is proved for this case.

Case II: $\alpha = 0,\ kp = n,\ p \geq 1,\ v \in W^k_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$. In this case by definition

$$\tau_* := n - (k-1)p = p,$$

and, by (4-5) we have

$$\mathcal{H}^p(A_v) = 0. \tag{4-11}$$

Further, by [Hajłasz et al. 2017, Theorem 2.3] the function v has the (τ,τ)-N_*-property for every $\tau \geq p$. This implies, in particular, by virtue of (4-11), that for every $\tau \geq p$ and for every $q \in [0, \tau]$ the equality

$$\mathcal{H}^{\tau - q}(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d \tag{4-12}$$

holds. Fix $q \in (m-1, q_\circ]$ and take $\tau = q + \mu_q$. Since by construction $\mu_q \geq 0$, we have $\tau \geq q$. Moreover, by (4-6)–(4-8) we have $\tau \geq \tau_* = p$. The last two inequalities together with (4-12) imply

$$\mathcal{H}^{\mu_q}(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d.$$

So the required assertion is proved for this case.

Case III: $0 < \alpha < 1,\ (k+\alpha)p > n,\ p > 1,\ v \in \mathscr{L}^{k+\alpha}_p(\mathbb{R}^n, \mathbb{R}^d)$. If $\tau_* = n - (k+\alpha-1)p < 0$, then $A_v = \varnothing$ and there is nothing to prove. Suppose now that $\tau_* \geq 0$. We obtain from Theorem 1.11 that v has the (τ,τ)-N_*-property for every $\tau>\tau_*:=n-(\alpha-1)p$. This implies, in particular, by virtue of (4-4), that for every $\tau>\tau_*$ and for every $q\in[0,\tau]$ the equality

$$\mathcal{H}^{\tau-q}(A_v\cap v^{-1}(y))=0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y\in\mathbb{R}^d \tag{4-13}$$

holds. Fix $q\in(m-1,q_\circ]$ and take $\tau=q+\mu_q$. Since by construction $\mu_q\ge 0$, we have $\tau\ge q$. Moreover, by (4-6)–(4-8) we have $\tau>\tau_*$. The last two inequalities together with (4-13) imply

$$\mathcal{H}^{\mu_q}(A_v\cap v^{-1}(y))=0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y\in\mathbb{R}^d.$$

So the required assertion is proved for this case.

Case IV: $0<\alpha<1,\ (k+\alpha)p=n,\ p>1,\ v\in\mathscr{L}^{k+\alpha}_{p,1}(\mathbb{R}^n,\mathbb{R}^d)$. In this case by definition

$$\tau_*:=n-(k-1)p=p,$$

and, by (4-5) we have

$$\mathcal{H}^p(A_v)=0. \tag{4-14}$$

Further, by Theorem 1.13 the function v has the (τ,τ)-N_*-property for every $\tau\ge p$. This implies, in particular, by virtue of (4-14), that for every $\tau\ge p$ and for every $q\in[0,\tau]$ the equality

$$\mathcal{H}^{\tau-q}(A_v\cap v^{-1}(y))=0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y\in\mathbb{R}^d \tag{4-15}$$

holds. Fix $q\in(m-1,q_\circ]$ and take $\tau=q+\mu_q$. Since by construction $\mu_q\ge 0$, we have $\tau\ge q$. Moreover, by (4-6)–(4-8) we have $\tau\ge\tau_*=p$. The last two inequalities together with (4-15) imply

$$\mathcal{H}^{\mu_q}(A_v\cap v^{-1}(y))=0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y\in\mathbb{R}^d.$$

So the required assertion is proved for this case, which is the last one.

Thus Theorem 1.17 is proved completely. □

Appendix

We prove the technical estimates of Lemmas 2.2, 2.3 and 2.8. Fix a cube $Q\subset\mathbb{R}^n$ of size $r=\ell(Q)\le 1$. Recall that by $2Q$ we denote the double cube with the same centre as Q of size $\ell(2Q)=2\ell(Q)$. We need some general elementary estimates.

Lemma A.1. *For any measurable function $g:\mathbb{R}^n\to\mathbb{R}_+$ and for every $x\in Q$ the inequality*

$$\int_{2Q}\frac{g(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y\le C\int_{Q}\frac{Mg(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y \tag{A-1}$$

holds.

Here C denotes some universal constant that does not depend on g, Q, etc.

Proof. Fix $x \in Q$. Define $r_0 = \frac{7}{2}\sqrt{n}r$. In particular, $2Q \subset B\big(x, \frac{1}{2}r_0\big)$.

Now put $r_j = 2^{-j} r_0$ and $B_j = B(x, r_j) \setminus B(x, r_{j+1}),\ j \in \mathbb{N}$. Clearly,

$$2Q = \bigcup_{j \in \mathbb{N}} (2Q \cap B_j) \tag{A-2}$$

and

$$\operatorname{meas}(Q \cap B_j) \geq C r_j^n \quad \text{for all } j \in \mathbb{N} \tag{A-3}$$

(here and henceforth we denote by C general constants depending on the parameters n, p, d, α only).

Since $|x-y| \sim r_j$ for $y \in B_j$, by the definition of the maximal function, it is easy to see that the estimate

$$\int_{2Q \cap B_j} \frac{g(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y \leq C r_j^{\alpha} Mg(z) \quad \text{for all } z \in Q \cap B_j$$

holds for all $j \in \mathbb{N}$. Integrating this inequality with respect to $z \in Q \cap B_j$ and using (A-3), we have

$$\int_{2Q \cap B_j} \frac{g(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y \leq C r_j^{\alpha-n} \int_{Q \cap B_j} Mg(z)\,\mathrm{d}z. \tag{A-4}$$

Since $|x-z| \sim r_j$ for $z \in Q \cap B_j$, the last inequality implies

$$\int_{2Q \cap B_j} \frac{g(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y \leq C \int_{Q \cap B_j} \frac{Mg(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y. \tag{A-5}$$

Then summing these inequalities for all $j \in \mathbb{N}$ and taking into account (A-2), we obtain the required estimate (A-1). □

Henceforth, fix $p > 1$, $\alpha > 0$ with $n + p \geq \alpha p \geq n$ (in particular, $\alpha < n+1$), and a function $v(x) = \mathscr{G}_\alpha(x) = \int_{\mathbb{R}^n} g(y)\, K_\alpha(x-y)\,\mathrm{d}y$ with some $g \in L_p(\mathbb{R}^n)$.

Split our function v into a sum

$$v = v_1 + v_2, \tag{A-6}$$

where

$$v_1 := \int_{\mathbb{R}^n} g_1(y)\, K_\alpha(x-y)\,\mathrm{d}y, \qquad v_2 := \int_{\mathbb{R}^n} g_2(y)\, K_\alpha(x-y)\,\mathrm{d}y,$$

and

$$g_1 := g \cdot 1_{2Q}, \qquad g_2 := g \cdot 1_{\mathbb{R}^n \setminus 2Q}.$$

Lemma A.2. *If $n + p > \alpha p > n$, we have*

$$\operatorname{diam} v_1(Q) \leq C\, \|Mg\|_{L_p(Q)}\, r^{\alpha - \frac{n}{p}}. \tag{A-7}$$

Proof. If $0 < \alpha < n$, then $K_\alpha(x) < c_\alpha\, |x|^{\alpha-n}$ (see [Adams and Hedberg 1996, page 10], for example), and from Lemma A.1 we have

$$|v_1(x)| \leq C \int_Q \frac{Mg(y)}{|x-y|^{n-\alpha}}\,\mathrm{d}y \quad \text{for all } x \in Q,$$

so the required estimate (A-7) follows immediately from the Hölder inequality.

If $n \le \alpha < n+1$, then

$$\left|\nabla K_\alpha(x)\right| \le C\, |x|^{\alpha-n-1}$$

(see [Adams and Hedberg 1996, page 13], for example), and by Lemma A.1 we have

$$|\nabla v_1(x)| \le C \int_Q \frac{Mg(y)}{|x-y|^{n-\alpha+1}}\,\mathrm{d}y \quad \text{for all } x \in Q. \tag{A-8}$$

Then by the Hardy–Littlewood–Sobolev inequality for Riesz potentials we have

$$\|\nabla v_1\|_{L_q(Q)} \le C\, \|Mg\|_{L_p(Q)},$$

where

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha-1}{n}.$$

It is easy to see that $q > n$, then by the Morrey inequality

$$\operatorname{diam} v_1(Q) \le C\, \|\nabla v_1\|_{L_q(Q)}\, r^{1-\frac{n}{q}} \le C_1\, \|Mg\|_{L_p(Q)} r^{\alpha-\frac{n}{p}}$$

as required. □

We need a modification of lemma above to the case of Lorentz spaces.

Lemma A.3. *If $n+p \ge \alpha p \ge n$, we have*

$$\operatorname{diam} v_1(Q) \le C\, \|Mg\|_{L_{p,1}(Q)}\, r^{\alpha-\frac{n}{p}}.$$

Proof. We have to repeat the previous arguments using the following facts for Lorentz norms: the generalized Hölder inequality

$$\int_Q \frac{f(y)}{|y-x|^{n-\alpha}}\,\mathrm{d}y \le \|f\|_{L_{p,1}} \cdot \left\| \frac{1_Q}{|\cdot - x|^{n-\alpha}} \right\|_{L_{\frac{p}{p-1},\infty}} = C\, \|f\|_{L_{p,1}} r^{\alpha-\frac{n}{p}}$$

for $n > \alpha \ge \frac{n}{p}$ [Malý 2003, Theorem 3.7], and the generalized Hardy–Littlewood–Sobolev inequality for Riesz potentials

$$\|I_\beta f\|_{L_{q,1}(Q)} \le C\, \|f\|_{L_{p,1}(Q)} \qquad \text{with} \quad \frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$$

if $\beta p \le n$ [Bennett and Sharpley 1988, Theorem IV.4.18], and the generalized Morrey inequality

$$\operatorname{diam} v_1(Q) \le C\, \|\nabla v_1\|_{L_{q,1}(Q)}\, r^{1-\frac{n}{q}}$$

for $q \ge n$ (see, e.g., [Korobkov and Kristensen 2014, Lemma 1.3]). □

Now we have to estimate the term v_2.

Lemma A.4. *For an arbitrary positive parameter $\theta \ge 1-\alpha$ the inequality*

$$\operatorname{diam}\bigl[v_2(Q)\bigr] \le C\, r^{1-\theta-n} \int_Q M_{\alpha+\theta-1} g(y)\,\mathrm{d}y \tag{A-9}$$

holds, where we recall that $r = \ell(Q)$.

Proof. Without loss of generality suppose that Q is centred at the origin. Since

$$C_1|y| \le |y-x| \le C_2|y| \quad \text{for all } x \in Q, \text{ for all } y \in \mathbb{R}^n \setminus 2Q, \tag{A-10}$$

and $K'_\alpha(\rho) \le C\rho^{\alpha-1-n}$ for $0 < \alpha < n+1$, it is easy to deduce that

$$\begin{aligned} \operatorname{diam} v_2(Q) &\le \sup_{x_1,x_2 \in Q} \int_{\mathbb{R}^n \setminus 2Q} |g(y)|[K_\alpha(x_1-y) - K_\alpha(x_2-y)]\,\mathrm{d}y \\ &\le C\,r \int_{\mathbb{R}^n\setminus 2Q} \frac{|g(y)|}{|y|^{n-\alpha+1}}\,\mathrm{d}y. \end{aligned} \tag{A-11}$$

Fix $\theta > 0$ such that

$$\alpha + \theta - 1 \ge 0. \tag{A-12}$$

Put $r_0 = \frac{1}{2}r,\ r_j = 2^j r_0$, and consider a sequence of sets $B_j = B(0, r_{j+1}) \setminus B(0, r_j)$. By construction,

$$\mathbb{R}^n \setminus 2Q \subset \bigcup_{j \in \mathbb{N}} B_j \tag{A-13}$$

and

$$\int_{B_j} \frac{|g_2(y)|}{|y|^{n-\alpha+1}}\,\mathrm{d}y \le C r_j^{-\theta} r_j^{\alpha+\theta-1} \fint_{B_j} |g_2(y)|\,\mathrm{d}y \le C r_j^{-\theta} M_{\alpha+\theta-1} g_2(0), \tag{A-14}$$

where we recall that $g_2 := g \cdot 1_{\mathbb{R}^n \setminus 2Q}$. Therefore, by summing over j and using (A-13) and the elementary formula for geometric progressions, we obtain

$$\int_{\mathbb{R}^n\setminus 2Q} \frac{|g_2(y)|}{|y|^{n-\alpha+1}}\,\mathrm{d}y \le C M_{\alpha+\theta-1} g_2(0) \sum_{j=1}^{\infty} r_j^{-\theta} \le C r^{-\theta} M_{\alpha+\theta-1} g_2(0), \tag{A-15}$$

It is easy to check (using the assumption that $g_2 \equiv 0$ on $2Q$) that $M_{\alpha+\theta-1} g_2(0) \le C M_{\alpha+\theta-1} g_2(z)$ for every $z \in Q$. Therefore,

$$M_{\alpha+\theta-1} g_2(0) \le C \fint_Q M_{\alpha+\theta-1} g_2(z)\,\mathrm{d}z; \tag{A-16}$$

thus

$$\int_{\mathbb{R}^n\setminus 2Q} \frac{|g_2(y)|}{|y|^{n-\alpha+1}}\,\mathrm{d}y \le C r^{-\theta-n} \int_Q M_{\alpha+\theta-1} g_2(z)\,\mathrm{d}z. \tag{A-17}$$

Finally we obtain from (A-11) that

$$\operatorname{diam}[v(Q)] \le C r^{1-\theta-n} \int_Q M_{\alpha+\theta-1} g_2(z)\,\mathrm{d}z \tag{A-18}$$

as required. □

The next result is established using the same arguments, with some evident simplifications.

Lemma A.5. *If, in addition to the assumptions above, we have* $\alpha > 1$, *then the estimate*

$$\operatorname{diam} v_2(Q) \le C\,r^{1-n} \int_Q I_{\alpha-1}|g|(y)\,\mathrm{d}y \tag{A-19}$$

holds, where we recall that $I_{\alpha-1}|g|$ *is the corresponding Riesz potential of the function* $|g|$.

Lemmas A.2–A.5 clearly imply the assertions of Lemmas 2.2, 2.3 and 2.8.

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