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Cohomological invariants and Brauer groups of algebraic stacks in positive characteristic

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Abstract

We introduce a theory of cohomological invariants with mod p^r coefficients for algebraic stacks in characteristic p. Using these new tools, we complete the computation of the Brauer group and cohomological invariants of the stack of elliptic curves over any field.

1. Introduction

1.1 Background on cohomological invariants

Given an algebraic group G over a base field $\mathbf{k}$, Serre [GMS03] defined the group of cohomological invariants of G as the group of natural transformations from the functor

$$\mathrm{T}_G\colon (\text{field}/\mathbf{k}) \longrightarrow (\text{set})\,, \quad F \longmapsto \{G\text{-torsors}/F\}/\simeq$$

to Galois cohomology with coefficients in some torsion Galois module. Cohomological invariants were, and are, studied by Garibaldi, Merkurjev, Rost, Totaro and many others. A cohomological invariant of G can be thought of as an arithmetic equivalent to a characteristic class, functorially assigning to each torsor $E \to \mathrm{Spec}(F)$ an element in the cohomology of F.

The functor T_G of G-torsors modulo isomorphism can be seen as the functor of points of the classifying stack $\mathcal{B}G$. From this point of view, the group of cohomological invariants appears naturally as an invariant of the stack $\mathcal{B}G$ rather than the group G. Based on this idea, the second-named author [Pir18a] extended the concept to define cohomological invariants of algebraic stacks. When restricted to smooth schemes, this theory recovers the theory of unramified cohomology (see [Sal84, CO89]). Moreover, the authors showed in [DLP21a] that for a smooth quotient stack $\mathcal{X}$ and any positive integer ℓ coprime to the characteristic of the base field, cohomological invariants in degree 2 compute the ℓ-torsion $\mathrm{Br}'(\mathcal{X})_\ell$ of the cohomological Brauer group.

In [Pir17, Pir18b, DL21, DLP21b, DLP21a, DLP23] the authors investigated the cohomological invariants of the stacks of smooth hyperelliptic curves and their compactifications; using these computations, they obtained an explicit presentation of the Brauer group of the stacks of smooth hyperelliptic curves over any field of characteristic zero, and the prime-to-p part in characteristic p.

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1.2 In positive characteristic

Until recently most results on classical cohomological invariants used mod ℓ Galois cohomology, where ℓ is a positive integer coprime to the characteristic of the base field, though there are a few notable exceptions, such as [EKLV98, Appendix B]. The assumption is crucial to the theory, most notably providing homotopy invariance. This changed with some recent works of Blinstein–Merkurjev [BM13], Lourdeaux [Lou22] and Totaro [Tot22] which explored classical cohomological invariants with p-torsion coefficients in characteristic p. Totaro in particular developed a complete theory and provided the first examples of full computations of mod p cohomological invariants of a group G in characteristic p. The key idea is that the coefficients should be in the "motivic" groups $\mathrm{H}^n(F, \underline{\mathbb{Z}}/p^r(j))$, where $\underline{\mathbb{Z}}/p^r(j)$ is Voevodsky's complex in the derived category of étale sheaves on schemes smooth over $\mathrm{Spec}(\mathbf{k})$.

In this paper we extend Totaro's ideas in the following direction.

Theorem. *There is a theory of mod p^r cohomological invariants for algebraic stacks in characteristic p extending the classical theory, and moreover for a smooth quotient stack $\mathcal{X}$, we have*

$$\mathrm{Br}'(\mathcal{X})_{p^r} = \mathrm{Inv}\big(\mathcal{X}, \mathrm{H}^2(-, \underline{\mathbb{Z}}/p^r(1))\big)\,.$$

Developing the theory requires some non-trivial preliminary work, mostly to extend known results on étale motivic cohomology of fields and discrete valuation rings from mod p to mod p^r coefficients.

We then study mod p^r cohomological invariants of the moduli stack $\mathcal{M}_{1,1}$ of elliptic curves over a field $\mathbf{k}$, completing the classification from [DLP21a, Theorem 3.1]. The most interesting case is the following, which we state in the notation of Section 5:

$$\mathrm{H}^n_{p^r}(F) = \mathrm{H}^n(F, \underline{\mathbb{Z}}/p^r(n-1))\,, \quad \mathrm{K}^n_{p^r}(F) = \mathrm{H}^n(F, \underline{\mathbb{Z}}/p^r(n))\,.$$

Theorem. *Let $\mathbf{k}$ be a field of characteristic p. Then*

$$\begin{cases} \mathrm{Inv}^\bullet(\mathcal{M}_{1,1}, \mathrm{H}_{2^r}) = \mathrm{Inv}^\bullet\big(\mathbb{A}^1, \mathrm{H}_{2^r}\big) \oplus \mathrm{J}^{\bullet-1}_{2^r}(\mathbf{k}) & \text{if } p = 2\,, \\ \mathrm{Inv}^\bullet(\mathcal{M}_{1,1}, \mathrm{H}_{3^r}) = \mathrm{Inv}^\bullet\big(\mathbb{A}^1, \mathrm{H}_{3^r}\big) \oplus \mathrm{H}^{\bullet-1}_{3^r}(\mathbf{k}) & \text{if } p = 3\,, \\ \mathrm{Inv}^\bullet(\mathcal{M}_{1,1}, \mathrm{H}_{p^r}) = \mathrm{Inv}^\bullet\big(\mathbb{A}^1, \mathrm{H}_{p^r}\big) & \text{if } p > 3\,. \end{cases}$$

The group $\mathrm{J}^\bullet_{2^r}$ above is a fiber product

$$\mathrm{J}^\bullet_{2^r} = \mathrm{H}^\bullet_{2^r}(\mathbf{k}) \times_{\mathrm{H}^\bullet_{2^r}(\mathbf{k})} \mathrm{K}^{\bullet-1}_2(\mathbf{k})\,,$$

where the first map is multiplication by 4 and the second map comes from the natural morphism

$$\mathrm{K}^{n-1}_2(\mathbf{k}) = \Omega^{n-1}_{\log,\mathbf{k}} \longrightarrow \Omega^{n-1}_{\mathbf{k}} \longrightarrow \mathrm{H}^n_{2^r}(\mathbf{k})\,.$$

We remark that all of the generators in the groups above are described constructively, meaning that given a smooth irreducible scheme $S/\mathbf{k}$ and a family of elliptic curves $E\colon S \to \mathcal{M}_{1,1}$, their pullbacks to $\mathrm{Inv}^\bullet(S, \mathrm{H}_{p^r})$ can be obtained explicitly from a Weierstrass equation for E_ξ, where ξ is the generic point of S.

1.3 The Brauer group of $\mathcal{M}_{1,1}$

We apply our study of mod p^r cohomological invariants of $\mathcal{M}_{1,1}$ to the computation of its Brauer group.

While the Brauer group is a fundamental and highly studied invariant, computations of the Brauer group of moduli stacks have appeared only recently. The breakthrough result in this direction is Antieau and Meier's paper [AM20], where they compute the Brauer group of the stack $\mathcal{M}_{1,1}$ of elliptic curves over various bases, including $\mathbb{Q}$, all finite fields of characteristic different from 2 and, most notably, $\mathbb{Z}[1/2]$ and $\mathbb{Z}$. The authors contributed to the topic in [DLP21a, Corollary 3.2], though it should be noted that the corollary is implied by the stronger result in Meier's unpublished draft [Mei18], and finally Shin obtained a somewhat surprising result in [Shi19], proving that unlike in all other characteristics, the Brauer group of $\mathcal{M}_{1,1}$ is not trivial over an algebraically closed field of characteristic 2, and moreover computing it for all finite fields of characteristic 2.

Our computation of mod p^r cohomological invariants of $\mathcal{M}_{1,1}$ over fields of characteristic p, together with [DLP21a, Corollary 3.2] and some extra mod ℓ computations, gives us the following description of the Brauer group of $\mathcal{M}_{1,1}$ which holds over any field, with no assumptions on characteristic, perfection or algebraic closure.

Theorem. *Let $\mathcal{M}_{1,1}$ be the stack over $\mathrm{Spec}(\mathbf{k})$ parametrizing elliptic curves. If $\mathrm{char}(\mathbf{k}) = c$, the group $\mathrm{Br}(\mathcal{M}_{1,1})$ is*

$$\begin{cases} \mathrm{Br}\left(\mathbb{A}^1_{\mathbf{k}}\right) \oplus \mathrm{H}^1(\mathbf{k}, \mathbb{Z}/12\mathbb{Z}) & \text{if } c \neq 2\,, \\ \mathrm{Br}\left(\mathbb{A}^1_{\mathbf{k}}\right) \oplus \mathrm{H}^1(\mathbf{k}, \mathbb{Z}/3\mathbb{Z}) \oplus \mathrm{J} & \text{if } c = 2 \text{ and } x^2+x+1 \text{ irreducible over } \mathbf{k}\,, \\ \mathrm{Br}\left(\mathbb{A}^1_{\mathbf{k}}\right) \oplus \mathrm{H}^1(\mathbf{k}, \mathbb{Z}/12\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } c = 2 \text{ and } x^2+x+1 \text{ has a root in } \mathbf{k}\,, \end{cases}$$

where $\mathrm{H}^1(\mathbf{k}, \mathbb{Z}/4) \subset \mathrm{J} \subseteq \mathrm{H}^1(\mathbf{k}, \mathbb{Z}/8)$ sits in an exact sequence

$$0 \longrightarrow \mathrm{H}^1(\mathbf{k}, \mathbb{Z}/4) \longrightarrow \mathrm{J} \longrightarrow \mathbb{Z}/2 \longrightarrow 0\,.$$

As for cohomological invariants, our construction provides an explicit description of each generator.

1.4 Content of the paper

In Section 2 we recall some basic facts on étale motivic cohomology, in particular its relation to logarithmic differential forms.

In Section 3 we define mod p^r cohomological invariants for smooth algebraic stacks over a base field of characteristic p. After proving some general properties, we restrict to invariants with coefficients in the étale motivic cohomology and characterize them as the sheafification of unramified cohomology.

In Section 4 we show that cohomological invariants of degree 2 can be used to compute the cohomological Brauer group. We also give an interpretation of cohomological invariants of degree 1 in terms of first cohomology groups of the sheaves $\mathbb{Z}/p^r$ and μ_{p^r} (the latter regarded as a sheaf in the flat topology).

The next two sections are the technical core of the paper. In Section 5 we study the étale motivic cohomology groups of fields, in particular their explicit description in terms of log differential forms and symbols. We establish several useful facts, and we recall Izhboldin's results on the tamely ramified and wild subgroups.

In Section 6 we study cohomology groups of discrete valuation rings (DVRs). The results contained here are necessary for the definitions in Section 3.

Section 7 is devoted to the computation of the cohomological invariants of stacks of the form $X \times \mathrm{B}\mathbb{Z}/n$, which will be necessary later.

In Section 8 we compute the mod p^r cohomological invariants of $\mathcal{M}_{1,1}$, dividing our analysis into three parts, depending on the characteristic of the base field.

In Section 9 we take a little detour to compute the cohomological invariants of $\mathcal{M}_{1,1}$ with coefficients in cycle modules of ℓ-torsion, with ℓ coprime to the characteristic of the base field. These have already been computed in the previous works by the authors except when the characteristic of the base field is 2 or 3, which is what is done here.

Finally, in Section 10 we compute the Brauer group of $\mathcal{M}_{1,1}$ over any field, and we describe its generators.

1.5 Notation

Throughout the paper $\mathbf{k}$ will be a field characteristic $p > 0$ unless stated otherwise. All schemes and algebraic stacks will be assumed to be of finite type over $\mathbf{k}$ or localizations thereof unless stated otherwise. With the notation $\mathrm{H}^i(\mathcal{X}, A)$ we always mean étale cohomology with coefficients in A, or lisse-étale if $\mathcal{X}$ is not a Deligne–Mumford stack. If R is a $\mathbf{k}$-algebra, we will often write $\mathrm{H}^i(R, A)$ for $\mathrm{H}^i(\mathrm{Spec}(R), A)$. In general, for a functor F, when no confusion is possible, we will often write $\mathrm{F}(A)$ for $\mathrm{F}(\mathrm{Spec}(A))$.

2. Étale motivic cohomology

Étale cohomology with torsion coefficients behaves rather differently when the coefficients are of ℓ-torsion, with ℓ coprime to the characteristic of our base field, and when they are instead of $p = \mathrm{char}(\mathbf{k})$-primary torsion, with the latter case being significantly harder to deal with. The most basic example is the following: consider the map $\mathcal{P}\colon \mathbb{G}_{\mathrm{a}} \to \mathbb{G}_{\mathrm{a}}$ given by $a \mapsto a^p - a$. It is a surjection of étale sheaves with kernel the constant sheaf $\mathbb{Z}/p$. Consequently, we have an exact sequence

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{G}_{\mathrm{a}} \xrightarrow{\mathcal{P}} \mathbb{G}_{\mathrm{a}} \longrightarrow 0\,,$$

which induces a long exact sequence in cohomology

$$0 \longrightarrow \mathrm{H}^0(-,\mathbb{Z}/p) \longrightarrow \mathrm{H}^0(-,\mathbb{G}_{\mathrm{a}}) \xrightarrow{\mathcal{P}} \mathrm{H}^0(-,\mathbb{G}_{\mathrm{a}}) \longrightarrow \mathrm{H}^1(-,\mathbb{Z}/p) \longrightarrow \mathrm{H}^1(-,\mathbb{G}_{\mathrm{a}}) \longrightarrow \cdots\,.$$

If we apply it to $\mathbb{A}^1$, as it is an affine scheme, we get

$$0 \longrightarrow \mathrm{H}^0(\mathbb{A}^1,\mathbb{Z}/p) \longrightarrow \mathrm{H}^0\big(\mathbb{A}^1,\mathbb{G}_{\mathrm{a}}\big) \xrightarrow{\mathcal{P}} \mathrm{H}^0\big(\mathbb{A}^1,\mathbb{G}_{\mathrm{a}}\big) \longrightarrow \mathrm{H}^1\big(\mathbb{A}^1,\mathbb{Z}/p\big) \longrightarrow 0\,,$$

which implies that $\mathrm{H}^1\big(\mathbb{A}^1,\mathbb{Z}/p\big) = \mathbf{k}[t]/\mathcal{P}(\mathbf{k}[t])$, a group that is not finitely generated, even if the base field $\mathbf{k}$ is algebraically closed. This shows that mod p étale cohomology is not homotopy invariant, thus lacking a fundamental property of its mod ℓ counterpart.

Nonetheless, there are many tools at our disposal in this situation as well. At the beginning of the 1980s Kato [Kat82, Section 0, p. 219] used differential forms to define cohomology groups $\mathrm{H}^i(\mathbf{k}, \mathbb{Z}/p^r(j))$, which turned out to work impressively well: when $i = 1$, $j = 0$, we have $\mathrm{H}^1(\mathbf{k}, \mathbb{Z}/p^r(0)) = \mathrm{H}^1_{\text{ét}}(\mathbf{k}, \mathbb{Z}/p^r)$, and when $i = 2$, $j = 1$, we have $\mathrm{H}^2(\mathbf{k}, \mathbb{Z}/p^r(1)) = \mathrm{Br}(\mathbf{k})_{p^r}$. To explain why this happens, we need more sophisticated technology: let $\mathbb{Z}_{\mathrm{tr}}$ be Voevodsky's sheaf with transfers. Define [MVW06, Definition 3.1]

$$\underline{\mathbb{Z}}(j) = \mathrm{C}_*\mathbb{Z}_{\mathrm{tr}}\big(\mathbb{G}_{\mathrm{m}}^{\wedge j}\big)[-j]$$

in the derived category of étale sheaves over schemes smooth over $\mathbf{k}$. Then $\underline{\mathbb{Z}}(0)$ is quasi-isomorphic to $\mathbb{Z}$, and $\underline{\mathbb{Z}}(1)$ is quasi-isomorphic to $\mathbb{G}_{\mathrm{m}}[-1]$; see [MVW06, Theorem 4.1].

There exist a natural map $\underline{\mathbb{Z}}(j) \xrightarrow{\cdot n} \underline{\mathbb{Z}}(j)$ and an exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}}(j) \xrightarrow{\cdot n} \underline{\mathbb{Z}}(j) \longrightarrow \underline{\mathbb{Z}}/n(j) \longrightarrow 0\,,$$

which in particular for $j = 1$ gives us the familiar-looking

$$\begin{array}{ccccccccc}
\mathrm{H}^2(X, \underline{\mathbb{Z}}(1)) & \xrightarrow{\cdot n} & \mathrm{H}^2(X, \underline{\mathbb{Z}}(1)) & \longrightarrow & \mathrm{H}^2(X, \underline{\mathbb{Z}}/n(1)) & \longrightarrow & \mathrm{H}^3(X, \underline{\mathbb{Z}}(1))_n & \longrightarrow & 0 \\
\| & & \| & & \| & & \| & & \\
\mathrm{H}^1(X, \mathbb{G}_\mathrm{m}) & \xrightarrow{\cdot n} & \mathrm{H}^1(X, \mathbb{G}_\mathrm{m}) & \dashrightarrow & (?) & \dashrightarrow & \mathrm{H}^2(X, \mathbb{G}_\mathrm{m})_n\,. & &
\end{array}$$

When n is coprime to $\mathrm{char}(\mathbf{k})$, we know that $\mathrm{H}^2(X, \underline{\mathbb{Z}}/n(1)) = \mathrm{H}^2(X, \mathbb{Z}/n(1)) = \mathrm{H}^2(X, \mu_n)$, retrieving the usual Kummer exact sequence. In general if $\mathrm{char}(\mathbf{k})$ does not divide n, then $\underline{\mathbb{Z}}/n(j)$ is quasi-isomorphic to $\mathbb{Z}/n(j) = \mathbb{Z}/n \otimes \mu_n^{\otimes j}$ by [MVW06, Theorem 10.2]. It remains to understand what the groups $\mathrm{H}^n(X, \underline{\mathbb{Z}}/p^r(j))$ are when $p = \mathrm{char}(\mathbf{k})$.

Let $X/\mathbf{k}$ be a smooth scheme, and write Ω^n_X for the sheaf of differentials of X over $\mathrm{Spec}(\mathbb{Z})$. Inside this sheaf consider the subsheaf (of $\mathbb{Z}/p$-modules) $\Omega^n_{\log}$ of logarithmic differentials, that is, the subsheaf generated by elements of the form

$$\frac{\mathrm{d}b_1}{b_1} \wedge \cdots \wedge \frac{\mathrm{d}b_n}{b_n}$$

with $b_1, \ldots, b_n$ units. There are corresponding notions of de Rham–Witt differentials $\mathrm{W}_r\Omega^n$ and logarithmic de Rham–Witt differentials $\mathrm{W}_r\Omega^n_{\log}$. Geisser and Levine [GL00, Proposition 3.1, Theorems 8.3 and 8.5] proved that in the derived category of Zariski or étale sheaves on smooth schemes over a *perfect* field of characteristic p, there is an isomorphism

$$\underline{\mathbb{Z}}/p^r(j) = \mathrm{W}_r\Omega^q_{\log}[-j]\,,$$

so that $\mathrm{H}^i\big(X, \underline{\mathbb{Z}}/p^r(j)\big) = \mathrm{H}^{i-j}\big(X, \mathrm{W}_r\Omega^j_{\log}\big)$. Note that their Theorem 8.5 is stated for $r = 1$ but the proof works for general r.

Geisser and Levine's result also computes the étale motivic cohomology of non-perfect fields thanks to Quillen's method (see the proof of Proposition 6.2 or [GL00, Proposition 3.1]).

Fields of characteristic p have cohomological dimension 1, see [Ser02, Section II.2.2], implying that $\mathrm{H}^i(\mathbf{k}, \underline{\mathbb{Z}}/p^r(j)) = 0$ unless i is either j or $j+1$. When $i = j$, we have

$$\mathrm{H}^j(\mathbf{k}, \underline{\mathbb{Z}}/p^r(j)) = \mathrm{W}_r\Omega^j_{\mathbf{k},\log} = \mathrm{K}^j_{\mathrm{Mil}}(\mathbf{k})/p^r\,,$$

where $\mathrm{K}^\bullet_{\mathrm{Mil}}$ is Milnor's K-theory [GS06, Chapter 7] and the identification is originally by Bloch and Kato [BK86]. For $i = j+1$ the description by Geisser and Levine gives

$$\mathrm{H}^{j+1}(\mathbf{k}, \underline{\mathbb{Z}}/p^r(j)) = \mathrm{H}^1_{\mathrm{Gal}}\big(\mathbf{k}, \mathrm{W}_r\Omega^j_{\mathbf{k}^s,\log}\big)\,,$$

where $\mathbf{k}^s$ is the separable closure of $\mathbf{k}$. This description is not particularly enlightening, but it agrees with Kato's original definition (thus explaining why his groups worked so well), and when $r = 1$, the latter is rather simple.

The group $\Omega^n_{\mathbf{k}}$ is generated additively by elements of the form $a(\mathrm{d}b_1/b_1) \wedge \cdots \wedge (\mathrm{d}b_n/b_n)$. We have a well-defined additive homomorphism $\mathcal{P} = \Omega^n_{\mathbf{k}} \to \Omega^n_{\mathbf{k}}$ given by

$$\mathcal{P}\left(a\frac{\mathrm{d}b_1}{b_1} \wedge \cdots \wedge \frac{\mathrm{d}b_n}{b_n}\right) = \mathcal{P}(a)\frac{\mathrm{d}b_1}{b_1} \wedge \cdots \wedge \frac{\mathrm{d}b_n}{b_n} = (a^p - a)\frac{\mathrm{d}b_1}{b_1} \wedge \cdots \wedge \frac{\mathrm{d}b_n}{b_n}\,,$$

and there is an exact sequence [Izh91, Corollary 6.5]

$$0 \longrightarrow \Omega^n_{\mathbf{k},\log} \longrightarrow \Omega^n_{\mathbf{k}} \xrightarrow{\mathcal{P}} \Omega^n_{\mathbf{k}}/\mathrm{d}\Omega^{n-1}_{\mathbf{k}} \longrightarrow \mathrm{H}^{n+1}(\mathbf{k}, \underline{\mathbb{Z}}/p(n)) \longrightarrow 0\,. \tag{2.1}$$

In other words, we can see $\mathrm{H}^{n+1}(\mathbf{k}, \underline{\mathbb{Z}}/p(n))$ as $\Omega^n_{\mathbf{k}}/N$, where N is the subgroup generated by exact differentials and elements of the form $(a^p - a)(\mathrm{d}b_1/b_1) \wedge \cdots \wedge (\mathrm{d}b_n/b_n)$, with $b_1, \ldots, b_n \in \mathbf{k}^*$.

The last tool we need to recall from motivic cohomology is Gros and Suwa's resolution of the sheaf of logarithmic differentials on a scheme X smooth over a perfect field [GS88, Theorem 1.4]. Write

$$\mathcal{H}^n(X,\underline{\mathbb{Z}}/p^r(j)) = \mathrm{H}^0_{\mathrm{Zar}}(X,\mathrm{H}^n(-,\underline{\mathbb{Z}}/p^r(j)))\,.$$

Gros and Suwa construct a resolution [GS88, Corollary 1.6]

$$0 \longrightarrow \mathcal{H}^n(X,\underline{\mathbb{Z}}/p^r(n)) = \mathrm{W}_r\Omega_{X,\log} \longrightarrow \mathrm{W}_r\Omega^n_{\mathbf{k}(X),\log} \xrightarrow{\partial} \bigoplus_{x\in X^{(1)}} \mathrm{W}_r\Omega^{n-1}_{\mathbf{k}(x),\log} \xrightarrow{\partial} \dots\,. \tag{2.2}$$

Using Gros and Suwa's results together with Quillen's method, over any field of characteristic p, we get the following exact sequence, which appears in [BM13, Appendix A]:

$$0 \longrightarrow \mathcal{H}^n(X,\underline{\mathbb{Z}}/p^r(j)) \longrightarrow \mathrm{H}^n(k(X),\underline{\mathbb{Z}}/p^r(j)) \xrightarrow{\partial} \bigoplus_{x\in X^{(1)}} \mathrm{H}^{n+1}_{\text{ét},x}(X,\underline{\mathbb{Z}}/p^r(j))\,, \tag{2.3}$$

where the groups appearing at the end of the sequence above are defined as

$$\mathrm{H}^{\bullet}_{\text{ét},x}(X,\underline{\mathbb{Z}}/p^r(j)) \overset{\text{def}}{=\!=} \varinjlim_{x\in U} \mathrm{H}^{\bullet}_{\text{ét},U\cap\{\overline{x}\}}(X,\underline{\mathbb{Z}}/p^r(j))\,.$$

The colimit runs over all of the open subsets containing x, and the groups on the right are the cohomology groups with support on a closed subscheme.

The sequence, importantly, works at the level of local rings, giving us

$$0 \longrightarrow \mathcal{H}^n(\mathcal{O}_{X,s},\underline{\mathbb{Z}}/p^r(j)) \longrightarrow \mathrm{H}^n(k(X),\underline{\mathbb{Z}}/p^r(j)) \xrightarrow{\partial} \bigoplus_{x\in X^{(1)},\, s\in\overline{x}} \mathrm{H}^{n+1}_{\text{ét},x}(X,\underline{\mathbb{Z}}/p^r(j))\,. \tag{2.4}$$

3. Invariants in characteristic p

In this section we generalize the definition of cohomological invariants from [DLP21a, Definition 2.3] to include coefficients of p^r-torsion even when our base field $\mathbf{k}$ has characteristic p. While the functors we will be interested in for the rest of the paper are specifically those of the form $\mathrm{H}^n(-,\underline{\mathbb{Z}}/p(j))$ for $j\in\{n-1,n\}$, allowing for general functors still has value, at least in terms of exposition. The following is the minimum we can ask of our coefficients functor for our theory to make sense.

Definition 3.1. Let (F,v) be a discretely valued field (respectively, let (R,v) be a DVR), let $\mathcal{O}_F=\{a\in F \mid v(a)\geqslant 0\}$, and let $\mathbf{k}_v=\mathcal{O}_F/\{a\mid v(a)>0\}=R/\mathrm{m}_v$.

We say v is a geometric discrete valuation if

- F and $\mathbf{k}_v$ are finitely generated over $\mathbf{k}$;
- v is trivial on $\mathbf{k}$;
- the transcendence degree of F over $\mathbf{k}$ is 1 more than the transcendence degree of $\mathbf{k}_v$ over $\mathbf{k}$.

Equivalently, (R,v) is a geometric DVR if it is the localization of a regular variety over $\mathbf{k}$ at a point of codimension 1.

From now on, coherently with our notation choices, we will always assume that valuations and DVRs are geometric unless stated otherwise.

Write $(\mathcal{F}/\mathbf{k})$ for the category of finitely generated extensions of $\mathbf{k}$, and with $(\mathrm{gv}\mathcal{F}/\mathbf{k})$ for the category whose objects are (F,v), where F is a finitely generated extension of $\mathbf{k}$ and v is a geometric discrete valuation.

Finally, write (Ab) for the category of abelian groups and (gr-Ab) for graded abelian groups.

Definition 3.2. A v-functor is the data of two functors

$$\Theta\colon (\mathcal{F}/\mathbf{k}) \longrightarrow ((\text{gr-})\text{Ab})\,,\quad \Theta'\colon (\text{gv}\mathcal{F}/\mathbf{k}) \longrightarrow ((\text{gr-})\text{Ab})$$

with, for each object $(F,v) \in (\text{gv}\mathcal{F}/\mathbf{k})$, compatible maps

$$\mathrm{j}_v\colon \Theta(\mathbf{k}_v) \hookrightarrow \Theta'(F,v)\,,\quad \mathrm{p}_v\colon \Theta(F) \longrightarrow \Theta'(F,v)\,,$$

where j_v is injective. We will often write Θ for the v-functor (Θ,Θ',j,p).

Remark 3.3. This definition includes all of the functors we picked as coefficients in the mod-ℓ case, that is, cycle modules, as we can pick the functors Θ, Θ' to be, respectively, the cycle module M and the group M_v described in [Ros96, Remark 1.6].

More specifically, when using étale cohomology with coefficients in a locally constant ℓ-torsion Galois module D, we can define a v-functor using Gabber's theorem [Sta22, Tag 09ZI]. Define $\Theta(F) = \mathrm{H}^n(F,D)$ and for any DVR (R,v)

$$\Theta'(\mathbf{k}(R),v) = \mathrm{H}^n\big(\mathbf{k}\big(R^{\mathrm{h}}\big),D\big)\,,\quad \mathrm{p}_v = \pi_{\mathrm{h}}^*\,,\quad \mathrm{j}_v = (i')^* \circ (i^*)^{-1}\,,$$

where R^{h} is the Henselization, $\pi_{\mathrm{h}}\colon \operatorname{Spec}(R^{\mathrm{h}}) \to \operatorname{Spec}(R)$ is the projection, the map $i\colon \operatorname{Spec}(\mathbf{k}_v) \to \operatorname{Spec}(R)$ is the inclusion of the closed point, which induces an isomorphism on cohomology, and $i'\colon \operatorname{Spec}\big(\mathbf{k}\big(R^{\mathrm{h}}\big)\big) \to \operatorname{Spec}\big(R^{\mathrm{h}}\big)$ is the inclusion of the generic point.

Using the notion of a v-functor, we can give a broad definition of cohomological invariants. Given an algebraic stack $\mathcal{X}/\mathbf{k}$, let $\mathrm{Pt}_{\mathcal{X}}$ be the functor

$$\mathrm{Pt}_{\mathcal{X}}\colon (\text{fields}/\mathbf{k}) \longrightarrow (\text{sets})\,,\quad \mathrm{Pt}_{\mathcal{X}}(F) = \mathcal{X}(\operatorname{Spec}(F))/\simeq.$$

Definition 3.4. Let Θ be a v-functor. A cohomological invariant with coefficients in Θ of an algebraic stack $\mathcal{X}/\mathbf{k}$ is a natural transformation

$$\alpha\colon \mathrm{Pt}_{\mathcal{X}} \longrightarrow \Theta$$

satisfying the following continuity condition: for any DVR (R,v) with a map $\operatorname{Spec}(R) \to \mathcal{X}$, there exists a finite étale extension $\big(R',v'\big) \to (R,v)$ such that $\mathbf{k}_{v'} = \mathbf{k}_v$ and

$$\mathrm{p}_{v'}(\alpha(\mathbf{k}(R))) = \mathrm{j}_{v'}(\alpha(\mathbf{k}_{v'}))\,.$$

Remark 3.5. Using Remark 3.3 it is easy to see that this definition retrieves the general definition for mod ℓ coefficients given in [DLP21a, Definition 2.3].

The more practical way to see a cohomological invariant α is as a way to functorially assign to each point $x\colon \operatorname{Spec}(F) \to \mathcal{X}$ an element $\alpha(x) \in \Theta(F)$, with the continuity condition requiring that it behaves well with respect to specialization.

Cohomological invariants of $\mathcal{X}$ with coefficients in Θ clearly form a group, graded if Θ is, that is, if it maps fields (respectively, valued fields) to graded abelian groups. We call it $\operatorname{Inv}(\mathcal{X},\Theta)$. If Θ is graded, we write $\operatorname{Inv}^\bullet(\mathcal{X},\Theta)$ for the whole group and $\operatorname{Inv}^n(\mathcal{X},\Theta)$ for $\operatorname{Inv}^\bullet\big(\mathcal{X},\Theta^n\big)$.

There is an obvious pullback map: a morphism $f\colon \mathcal{Y} \to \mathcal{X}$ induces a natural transformation $f\colon \mathrm{Pt}_{\mathcal{Y}} \to \mathrm{Pt}_{\mathcal{X}}$, and given an element $\alpha \in \operatorname{Inv}^\bullet(\mathcal{X},\Theta)$, we compose to obtain $f^*\alpha \in \operatorname{Inv}^\bullet(\mathcal{Y},\Theta)$.

In other words, given $y\colon \operatorname{Spec}(F) \to \mathcal{Y}$ we define

$$(f^*\alpha)(y) = \alpha(f \circ y)\,.$$

Now recall the following definition [Pir18a, Definition 3.2]: a *smooth-Nisnevich* morphism $\mathcal{Y} \to \mathcal{X}$ is a smooth, representable map of algebraic stacks such that for any field F and any morphism $\operatorname{Spec}(F) \to \mathcal{X}$, we have a lifting

$$\begin{array}{ccc} & & \mathcal{Y} \\ & \nearrow & \downarrow f \\ \operatorname{Spec}(F) & \longrightarrow & \mathcal{X}\,. \end{array}$$

Typical examples are a quotient map $X \to [X/G]$, where G is a *special* smooth affine algebraic group, in the sense that G-torsors are always Zariski-locally trivial (for example GL_n, $\mathbb{G}_{\mathrm{a}}$), as well as vector or affine bundles.

Smooth-Nisnevich morphisms define a Grothendieck topology, which we refer to as the smooth-Nisnevich topology, on the category of representable smooth morphisms $\mathcal{Y} \to \mathcal{X}$. We call the resulting site the smooth-Nisnevich site of $\mathcal{X}$. If $\mathcal{X}$ is a scheme, then any smooth-Nisnevich morphism has a (étale) Nisnevich section, so on schemes the smooth-Nisnevich site is equivalent to the ordinary Nisnevich one [Pir18a, Proposition 3.3].

In [Pir18a, Proposition 3.6] the second-named author proves that an algebraic stack with affine stabilizers has a smooth-Nisnevich cover if the base field is infinite, and relies on a finer topology, the m-smooth Nisnevich topology, for finite fields. Using result by Aizenbud and Avni, we can easily show that smooth-Nisnevich covers of finite type exist over any base field.

Proposition 3.6. *Let $\mathcal{X}/\mathbf{k}$ be an algebraic stack with affine geometric stabilizers. Then there exist a scheme X and a smooth-Nisnevich cover $X \to \mathcal{X}$ of finite type.*

Proof. The proof of [Pir18a, Proposition 3.6] can be applied verbatim to show that there is a scheme X_1 with a smooth, finite-type map $X_1 \to \mathcal{X}$ lifting all points over infinite fields. On the other hand, [AA22, Theorem A] shows that there exists a scheme X_2 with a smooth, finite-type map $X_2 \to \mathcal{X}$ lifting all points over finite fields. Then $X_1 \sqcup X_2 \to \mathcal{X}$ is the smooth-Nisnevich cover we are looking for. □

Given an algebraic stack $\mathcal{X}$, we can consider the functor $\operatorname{Inv}^\bullet(-, \Theta)$ on the smooth-Nisnevich site of $\mathcal{X}$.

Theorem 3.7. *The functor $\operatorname{Inv}^\bullet(-, \Theta)$ is a smooth-Nisnevich sheaf.*

Proof. Let $\mathcal{Y} \to \mathcal{X}$ be a smooth-Nisnevich covering, and assume given an element $\alpha \in \operatorname{Inv}^\bullet(\mathcal{Y}, \Theta)$ that satisfies $\mathrm{pr}_1^* \alpha = \mathrm{pr}_2^* \alpha$. For each morphism $p\colon \operatorname{Spec}(F) \to \mathcal{X}$ define $\beta(p)$ to be $\alpha(p')$, where p' is any lifting of p. This is well defined as given a second lifting p'', there is a map $q\colon \operatorname{Spec}(F) \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ such that $\mathrm{pr}_1 \circ q = p'$ and $\mathrm{pr}_2 \circ q = p''$, so the values of α at the two points must be the same. Functoriality is immediate.

Finally, given a map from the spectrum of a DVR (R, v) to $\mathcal{X}$, we can always find a finite extension $\left(R', v'\right)$ with $\mathbf{k}_{v'} = \mathbf{k}_v$ which lifts to $\mathcal{Y}$. This is because the map from the Henselization $\left(R^{\mathrm{h}}, v\right)$ always lifts. This shows that we can check the continuity condition $\mathrm{p}_{v'}\beta\big(\mathbf{k}\big(R'\big)\big) = \mathrm{j}_{v'}\beta(j(\mathbf{k}_{v'}))$ on the lifting $\operatorname{Spec}\big(R'\big) \to \mathcal{Y}$, concluding our proof. □

A direct consequence of the continuity condition is that a cohomological invariant of an irreducible smooth stack is determined by its “generic” value.

Lemma 3.8. *Let $\mathcal{X}/\mathbf{k}$ be an algebraic stack, let (A, m) be a regular local ring with a map $\operatorname{Spec}(A) \to \mathcal{X}$, and let $\alpha \in \operatorname{Inv}^\bullet(\mathcal{X}, \Theta)$. Let ξ, x be, respectively, the images of the generic and closed points of $\operatorname{Spec}(A)$. Then if $\alpha(\xi) = 0$, we have $\alpha(x) = 0$.*

Proof. First assume that A is a DVR. We may pass to an extension A' such that $k_{v'} = k_v$ and $\mathrm{j}_{v'}$ is injective. Then the fact that $\alpha(\xi) = 0$ implies that $\alpha(\mathbf{k}(A')) = 0$, which in turn implies that $\mathrm{j}_{v'}(\alpha(x)) = \mathrm{p}_{v'}(0) = 0$. As $\mathrm{j}_{v'}$ is injective, we conclude that $\alpha(x) = 0$.

Now assume that the dimension d of A is greater than 1, and let $a_1, \dots, a_d$ be a regular sequence defining the maximal ideal m. By the 1-dimensional case, we know that the statement holds for the DVR $A_{(a_1)}$. This shows that if α is zero at ξ, it is zero at the generic point ξ' of $\operatorname{Spec}(A/(a_1))$. As $A/(a_1)$ is a regular local ring of dimension $d-1$, by induction we have that $\alpha(\xi') = 0$ implies $\alpha(x) = 0$. □

Proposition 3.9. *Let $\mathcal{X}/\mathbf{k}$ be an irreducible smooth algebraic stack, and let $\mathcal{U} \subset \mathcal{X}$ be a non-empty open substack. Then the following pullback is injective:*

$$\operatorname{Inv}^\bullet(\mathcal{X}, \Theta) \longrightarrow \operatorname{Inv}^\bullet(\mathcal{U}, \Theta)\,.$$

Proof. Consider a smooth-Nisnevich covering $X \to \mathcal{X}$, and let $U \to \mathcal{U}$ be the pullback to $\mathcal{U}$. Let α be a cohomological invariant of $\mathcal{X}$. Given $p\colon \operatorname{Spec}(F) \to \mathcal{X}$, take a lifting $p'\colon \operatorname{Spec}(F) \to X$, and let ξ be the generic point of the corresponding connected component of X. Then ξ belongs to the image of U as $U \to X$ is dominant on every component. Now if $\alpha_U = 0$, we must have $\alpha(\xi) = 0$, which implies $\alpha(p') = \alpha(p) = 0$. □

3.1 Invariants with coefficients in étale motivic cohomology

For the rest of this section, we will restrict our attention to $\Theta = \mathrm{H}^n(-, \underline{\mathbb{Z}}/p^r(j))$. For now we will not see these as graded functors. First we need to show that they are v-functors. This will require some facts that will be proven in Sections 5 and 6, which we postpone as they require some work.

We can define a v-functor structure for $\Theta = \mathrm{H}^n(-, \underline{\mathbb{Z}}/p^r(n))$ by following the idea in [Ros96, Remark 1.6]. Recall that for a field $F/\mathbf{k}$ we have

$$\mathrm{H}^n(F, \underline{\mathbb{Z}}/p^r(n)) = \mathrm{K}^n_{\mathrm{Mil}}(F)/p^r\,.$$

For a DVR (R, v) define $\mathrm{K}^\bullet_v = \mathrm{K}^\bullet_{\mathrm{Mil}}(F)/\{1 + m_v\}\mathrm{K}^\bullet_{\mathrm{Mil}}(F)$. We set $\Theta'(\mathbf{k}(R), v) = \mathrm{K}^n_v/p^r$. There are maps

$$\mathrm{p}_v\colon \mathrm{H}^n(\mathbf{k}(R), \underline{\mathbb{Z}}/p^r(n)) \longrightarrow \mathrm{K}^n_v/p^r, \quad \mathrm{j}_v\colon \mathrm{H}^n(\mathbf{k}_v, \underline{\mathbb{Z}}/p^r(n)) \longrightarrow \mathrm{K}^n_v/p^r$$

given, respectively, by the projection and

$$\mathrm{j}_v\{b_1, \dots, b_n\} \longrightarrow \mathrm{p}_v\{\tilde{b}_1, \dots \tilde{b}_n\}\,,$$

where $\tilde{b}_1, \dots, \tilde{b}_n$ are liftings of $b_1, \dots, b_n$. We have an exact sequence

$$0 \longrightarrow \mathrm{H}^n(\mathbf{k}_v, \underline{\mathbb{Z}}/p^r(n)) \xrightarrow{\mathrm{j}_v} \mathrm{K}^n_v/p^r \xrightarrow{\partial} \mathrm{H}^n(\mathbf{k}_v, \underline{\mathbb{Z}}/p^r(n)) \longrightarrow 0\,,$$

where ∂ is the ramification map defined in Section 5.3.

To obtain a structure of v-functor on $\Theta = \mathrm{H}^{n+1}(-, \underline{\mathbb{Z}}/p^r(n))$, we use Proposition 5.6 and the identification obtained in Section 6 of the unramified subgroup of $\mathrm{H}^{n+1}(\mathbf{k}(R), \underline{\mathbb{Z}}/p^r(n))$ with the cohomology group $\mathrm{H}^{n+1}(R, \underline{\mathbb{Z}}/p^r(n))$ (see Proposition 6.7).

Consider a DVR (R, v). If we take the Henselization (R^{h}, v), we get

$$\mathrm{H}^{n+1}(R^{\mathrm{h}}, \underline{\mathbb{Z}}/p^r(n)) = \mathrm{H}^{n+1}(\mathbf{k}_v, \underline{\mathbb{Z}}/p^r(n)) \xrightarrow{(i')^* \circ (i^*)^{-1}} \mathrm{H}^{n+1}\big(\mathbf{k}\big(R^{\mathrm{h}}\big), \underline{\mathbb{Z}}/p^r(n)\big)\,,$$

where i is the inclusion of the closed point of $\mathrm{Spec}(R^{\mathrm{h}})$ and i' is the inclusion of the generic point. This shows that we can set

$$\Theta'(\mathbf{k}(R), v) = \mathrm{H}^{n+1}\big(\mathbf{k}\big(R^{\mathrm{h}}\big), \underline{\mathbb{Z}}/p^r(n)\big)\,, \quad \mathrm{p}_v = \pi_{\mathrm{h}}^*, \quad \mathrm{j}_v = (i')^* \circ (i^*)^{-1}\,.$$

Note that we have a map

$$\mathrm{H}^n(\mathcal{X}, \underline{\mathbb{Z}}/p^r(j)) \longrightarrow \mathrm{Inv}(\mathcal{X}, \mathrm{H}^n(-, \underline{\mathbb{Z}}/p^r(j)))$$

given by restriction; that is, $h \in \mathrm{H}^n(\mathcal{X}, \underline{\mathbb{Z}}/p^r(j))$ maps to the element

$$\tilde{h} \in \mathrm{Inv}(\mathcal{X}, \mathrm{H}^n(-, \underline{\mathbb{Z}}/p^r(j)))$$

defined by setting, for a point $x\colon \mathrm{Spec}(F) \to \mathcal{X}$,

$$\tilde{h}(x) = x^* h \in \mathrm{H}^n(F, \underline{\mathbb{Z}}/p^r(j))\,.$$

It is immediate that $\tilde{h}$ is functorial and satisfies the continuity condition. The following lemma shows that thanks to the continuity condition, when we restrict to DVRs, this map is *Nisnevich-locally* surjective.

Lemma 3.10. *Let $j \in \{n-1, n\}$, and let (R, v) be a DVR. In both cases, we have*

$$\mathrm{p}_v^{-1}\big(\mathrm{j}_v\big(\mathrm{H}^n\big(\mathbf{k}_v, \underline{\mathbb{Z}}/p^r(j)\big)\big)\big) = (i')^*\big(\mathrm{H}^n(R, \underline{\mathbb{Z}}/p^r(j)\big)\,.$$

Proof. The case $j = n$ is a direct consequence of Proposition 6.2. The case $j = n-1$ is shown in Proposition 5.6, Corollary 5.12 and Proposition 6.7. □

Consider a smooth irreducible scheme $X/\mathbf{k}$, with generic point ξ. As a consequence of Proposition 3.9, we have an inclusion

$$\mathrm{Inv}\big(X, \mathrm{H}^n(-, \underline{\mathbb{Z}}/p^r(j))\big) \subseteq \mathrm{H}^n(\xi, \underline{\mathbb{Z}}/p^r(j))\,.$$

Recall from equation (2.3) that for any point $x \in X^{(1)}$ we have a ramification map

$$\partial_x\colon \mathrm{H}^n(\xi, \underline{\mathbb{Z}}/p^r(j)) \longrightarrow \mathrm{H}^{n+1}_x(X, \underline{\mathbb{Z}}/p^r(j))\,.$$

We claim that for any $\alpha \in \mathrm{Inv}\big(X, \mathrm{H}^n(-, \underline{\mathbb{Z}}/p^r(j))\big)$ the element $\alpha(\xi)$, which belongs to $\mathrm{H}^n(\xi, \underline{\mathbb{Z}}/p^r(j))$, must lie in the kernel of ∂_x.

Lemma 3.11. *Let $X/\mathbf{k}$ be a smooth irreducible scheme with generic point ξ. Then for any $\alpha \in \mathrm{Inv}\big(X, \mathrm{H}^n(-, \mathbb{Z}/p^r(j))\big)$ and any $x \in X^{(1)}$, we have $\partial_x(\alpha(\xi)) = 0$.*

Proof. By Lemma 3.10 the continuity condition implies that after we pass to a Nisnevich neighbourhood $\big(X', x'\big)$ of x, the element $\alpha(\xi')$ belongs to $\mathrm{H}^n\big(\mathcal{O}^{\mathrm{h}}_{X',x'}, \underline{\mathbb{Z}}/p^r(j)\big)$. But, for example, [Mil16, Lemma 1.16, Proposition 1.27] implies that

$$\mathrm{H}^{\bullet}_x\big(X, \underline{\mathbb{Z}}/p^r(j)\big) = \mathrm{H}^{\bullet}_{x'}\big(X', \underline{\mathbb{Z}}/p^r(j)\big) = \mathrm{H}^{\bullet}_x\big(\mathcal{O}^{\mathrm{h}}_{X,x}, \underline{\mathbb{Z}}/p^r(j)\big)\,.$$

Thus we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^n(\mathbf{k}(X), \mathbb{Z}/p^r(j)) & \xrightarrow{\partial_x} & \mathrm{H}^{n+1}_x(X, \mathbb{Z}/p^r(j)) \\ \downarrow & & \downarrow \simeq \\ \mathrm{H}^n\big(\mathbf{k}(X'), \underline{\mathbb{Z}}/p^r(j)\big) & \xrightarrow{\partial_{x'}} & \mathrm{H}^{n+1}_{x'}\big(X', \underline{\mathbb{Z}}/p^r(j)\big)\,, \end{array}$$

and applying the local exact sequence (2.4) at $\big(X', x'\big)$, we conclude that $\partial_{x'}(\alpha(\xi')) = 0$, proving our claim. □

We are ready to extend the description in [Pir18a, Theorem 4.4] to invariants with coefficients in $\mathrm{H}^n(-,\mathbb{Z}/p^r(j))$; we will do it in two steps. First we describe the cohomological invariants of a smooth, irreducible scheme.

PROPOSITION 3.12. *Let $X/\mathbf{k}$ be a smooth, irreducible scheme. Then*

$$\mathrm{Inv}\big(X, \mathrm{H}^n(-,\underline{\mathbb{Z}}/p^r(j))\big) = \mathrm{H}^0_{\mathrm{Zar}}\big(X, \mathrm{H}^n(-,\underline{\mathbb{Z}}/p^r(j))\big)\,.$$

Proof. There is an obvious map

$$h\colon \mathrm{H}^0_{\mathrm{Zar}}\big(X, \mathrm{H}^n(-,\underline{\mathbb{Z}}/p^r(j))\big) \longrightarrow \mathrm{Inv}\big(X, \mathrm{H}^n(-,\underline{\mathbb{Z}}/p^r(j))\big)$$

as the map $\mathrm{H}^i\big(X,\underline{\mathbb{Z}}/p^r(j)\big) \to \mathrm{Inv}^i\big(X, \mathrm{H}^n(-,\underline{\mathbb{Z}}/p^r(j))\big)$ factors through the smooth-Nisnevich sheafification, which in turn factors through the Zariski sheafification. Now note that the map $\mathrm{Inv}\big(X, \mathrm{H}^n(-,\underline{\mathbb{Z}}/p^r(j))\big) \to \mathrm{H}^n(\mathbf{k}(X),\underline{\mathbb{Z}}/p^r(j))$ is injective by Lemma 3.8 and factors through the subgroup of unramified elements by Lemma 3.11. Now we have maps

$$\begin{array}{ccc} & & \mathrm{Inv}\big(X, \mathrm{H}^n(-,\underline{\mathbb{Z}}/p^r(j))\big) \\ & \overset{h}{\nearrow} & \big\downarrow f \\ \mathrm{H}^0_{\mathrm{Zar}}\big(X, \mathrm{H}^n(-,\underline{\mathbb{Z}}/p^r(j))\big) & \xrightarrow{\;g\;} & \mathrm{Ker}(\partial) \subseteq \mathrm{H}^n(\mathbf{k}(X),\underline{\mathbb{Z}}/p^r(j))\,, \end{array}$$

where f is the value at the generic point and g is the pullback to the generic point, an isomorphism thanks to equation (2.3). Now $f \circ h = g$ is an isomorphism and f is injective, so we conclude that h and f must both be isomorphisms as well. □

As an immediate consequence we obtain the following.

THEOREM 3.13. *Let $\mathcal{X}/\mathbf{k}$ be a smooth algebraic stack. Then the functor*

$$\mathrm{Inv}\big(-, \mathrm{H}^n(-,\underline{\mathbb{Z}}/p^r(j))\big)\colon (\mathrm{Sm}/\mathcal{X}) \longrightarrow (\textit{abelian groups})$$

is the smooth-Nisnevich sheafification of $\mathrm{H}^n(-,\underline{\mathbb{Z}}/p^r(j))$.

Proof. This is an immediate consequence of the fact that $\mathrm{Inv}\big(-, \mathrm{H}^n(-,\underline{\mathbb{Z}}/p^r(j))\big)$ is a Nisnevich sheaf (Theorem 3.7), the existence of a map $\mathrm{H}^n(-,\underline{\mathbb{Z}}/p^r(j)) \to \mathrm{Inv}\big(-, \mathrm{H}^n(-,\underline{\mathbb{Z}}/p^r(j))\big)$ and Proposition 3.12. □

The cohomological invariants of a stack $\mathcal{X}$ provide a lower bound for its *essential dimension* [BR97, BF03], here conjugated in the sense of [BRV11], which, roughly speaking, measures the minimum number of independent variables necessary to define all the objects parametrized by it.

Recall (see for example [Mil16, Lemma V.1.12]) that if $f\colon Y \to X$ is a finite étale map of constant degree d and A is an étale sheaf on X, there is a transfer morphism $f_*\colon \mathrm{H}^n(Y, A_{|Y}) \to \mathrm{H}^n(X, A)$ and $f_* f^* \alpha = d\alpha$.

PROPOSITION 3.14. *Let $\mathcal{X}$ be a smooth algebraic stack and $\alpha \in \mathrm{Inv}\big(\mathcal{X}, \mathrm{H}^n(-,\underline{\mathbb{Z}}/p(j))\big)$. Assume that α is not zero when pulled back to $\mathcal{X}_{\overline{\mathbf{k}}}$. Then we have*

$$\mathrm{ed}(\mathcal{X}) \geqslant \mathrm{ed}_p(\mathcal{X}) \geqslant n\,.$$

Proof. We may assume $\mathbf{k} = \overline{\mathbf{k}}$ as passing to $\overline{\mathbf{k}}$ cannot increase the essential dimension and p-essential dimension.

We will follow the idea in [Tot19, Lemma 3.1]. The two relevant cases are $j = n$ and $j = n-1$. We know, respectively by [GL00, Theorem 8.3] and the fact that, in Geisser and Levine's notation, we have $\nu^n(X) = 0$ for $n > \dim(X)$ for $j = n$, and by [KK86, Section 3, Corollary 2] for $j = n-1$, that in both cases the fact that $\alpha(x) \neq 0$, where $x\colon \operatorname{Spec}(F) \to \mathcal{X}$, implies that the transcendence degree of F over k is at least n. This proves that $\operatorname{ed}(\mathcal{X}) \geqslant n$.

To show the inequality for $\operatorname{ed}_p(\mathcal{X})$, observe that if $d = [L : F]$ is coprime to p and $f\colon \operatorname{Spec}(L) \to \operatorname{Spec}(F)$, then for an element $\alpha \in \mathrm{H}^n(F, \underline{\mathbb{Z}}/p(j))$ we must have $f_* f^* \alpha = d\alpha$, which shows that if α is non-zero, it must stay so after we pull back to L. So if we have an object $x'\colon \operatorname{Spec}(E) \to \mathcal{X}$ such that x, x' are equal after passing to an extension F'/F that is finite of degree coprime to p, we must have that $\alpha(x') \neq 0$, which shows that the transcendence degree of E over $\mathbf{k}$ must be at least n. □

4. Invariants in low degrees

The purpose of this section is to prove that, in analogy to their mod ℓ counterparts, when $\mathcal{X}/\mathbf{k}$ is a smooth quotient stack, its cohomological invariants compute the ordinary étale cohomology groups $\mathrm{H}^1(\mathcal{X}, \mathbb{Z}/p^r)$, the flat cohomology group $\mathrm{H}^1_{\mathrm{fl}}(\mathcal{X}, \mu_p)$ in degree 1 and the p-primary torsion in the cohomological Brauer group in degree 2. The ideas are the same as for the mod ℓ invariants, with some updates to make up for the lack of homotopy invariance. We begin with the easier case of degree 1 invariants. In this case we get a more complete result than what we have in [DLP21a, Lemma 2.18] as we have no requirement on the group action. First we show that for smooth schemes the Nisnevich cohomology is trivial when the coefficients are constant.

Lemma 4.1. *Let A be a finite abelian group, and let $X/\mathbf{k}$ be a smooth connected scheme. Then $\mathrm{H}^i_{\mathrm{Nis}}(X, A) = 0$ for $i > 0$.*

Proof. Let $\xi\colon \operatorname{Spec}(\mathbf{k}(X)) \to X$ be the generic point of X. Then as X is smooth, we have $\xi_*\xi^*\mathbb{Z}/p^r = \mathbb{Z}/p^r$. The Leray–Cartan spectral sequence applied to the map ξ reads

$$\mathrm{H}^i_{\mathrm{Nis}}\big(X, \mathrm{R}^j\xi_*\mathbb{Z}/p^r\big) \Longrightarrow \mathrm{H}^{i+j}_{\mathrm{Nis}}(\operatorname{Spec}(\mathbf{k}(X)), \mathbb{Z}/p^r)\,,$$

but the Nisnevich site of the spectrum of a field is trivial, so all of the terms in the abutment except the first and all of the R^i for $i > 0$ vanish, which shows that $\mathrm{H}^i_{\mathrm{Nis}}(X, \mathbb{Z}/p^r) = 0$ for $i > 0$. □

Thanks to results by Illusie, Gros and Suwa, on smooth schemes we can compare the flat cohomology group $\mathrm{H}^1_{\mathrm{fl}}(X, \mu_p)$ with $\operatorname{Inv}\big(X, \mathrm{H}^1(-, \underline{\mathbb{Z}}/p(1)\big)$.

Lemma 4.2. *Let $X/\mathbf{k}$ be a smooth scheme. Then*

$$\mathrm{H}^1_{\mathrm{fl}}(X, \mu_{p^r}) = \mathrm{H}^1(X, \underline{\mathbb{Z}}/p^r(1))\,.$$

As a consequence, if $\mathcal{X}$ is a smooth algebraic stack, an element $\alpha \in \mathrm{H}^1_{\mathrm{fl}}(\mathcal{X}, \mu_{p^r})$ induces a cohomological invariant by pullback.

Proof. The first statement is [MVW06, Remark 4.10]. Now given $\alpha \in \mathrm{H}^1_{\mathrm{fl}}(\mathcal{X}, \mu_{p^r})$ all we have to check is that the association

$$(x\colon \operatorname{Spec}(F) \longrightarrow \mathcal{X}) \longmapsto x^*\alpha \in \mathrm{H}^1_{\mathrm{fl}}(\operatorname{Spec}(F), \mu_{p^r}) = \mathrm{H}^1(F, \underline{\mathbb{Z}}(p^r(1))$$

satisfies the continuity condition, but this is clear as given any Henselian DVR (R, v) with a map $\operatorname{Spec}(R) \to \mathcal{X}$, we can see the pullback of α to $\operatorname{Spec}(R)$ as an element of $\mathrm{H}^1(R, \underline{\mathbb{Z}}/p^r(1))$. □

Recall [Tot99, EG98] that, given a scheme $X/\mathbf{k}$ with an action by an affine smooth group scheme $G/\mathbf{k}$, an *equivariant approximation* X' of $[X/G]$ is obtained by taking a representation V of G such that G acts freely on an open subset $U \subset V$ and $\operatorname{codim}(V \smallsetminus U, V) > 1$. Such a representation can always be found under these hypotheses [EG98, Lemma 9].

Lemma 4.3. *Let $X/\mathbf{k}$ be a smooth scheme with an action of a smooth algebraic group $G/\mathbf{k}$, and write $\mathcal{X} = [X/G]$. Then*

- $\mathrm{H}^i_{\text{sm-Nis}}(\mathcal{X}, \mathbb{Z}/p^r) = 0$ *for* $i > 0$;
- $\mathrm{H}^i_{\text{sm-Nis}}(\mathcal{X}, \mu_{p^r}) = 0$ *for* $i > 0$.

Proof. First observe that μ_{p^r} is a constant sheaf on the Nisnevich site of any smooth scheme. We write A for either $\mathbb{Z}/p^r$ or μ_{p^r}.

Pick an equivariant approximation $X' = (X \times U)/G \xrightarrow{\pi} \mathcal{X}$, and consider the Leray spectral sequence associated with the morphism $f\colon X' \to \mathcal{X}$:

$$\mathrm{H}^i_{\text{sm-Nis}}\big(\mathcal{X}, \mathrm{R}^j f_* A\big) \Longrightarrow \mathrm{H}^{i+j}_{\text{sm-Nis}}\big(X', A\big)\,.$$

We claim that the sequence collapses on the first row and all of the terms in the abutment are zero except for the first. To see this, first note that on schemes the smooth-Nisnevich and Nisnevich cohomology are equal [Pir18a, Proposition 3.3]. Then by Lemma 4.1 all of the terms in the abutment except for the first are zero. Moreover, the fibers of $X' \to \mathcal{X}$ are smooth and connected, which means that all of the R^q are zero for $q > 0$. Finally, $\mathrm{R}^0 f_* A = A$, concluding our proof. □

With this, all that is left is to apply the appropriate spectral sequence.

Proposition 4.4. *Let $X/\mathbf{k}$ be a smooth scheme with an action of a smooth algebraic group $G/\mathbf{k}$, and write $\mathcal{X} = [X/G]$. We have*

$$\operatorname{Inv}\big(\mathcal{X}, \mathrm{H}^1(-, \underline{\mathbb{Z}/p^r}(0))\big) = \mathrm{H}^1(\mathcal{X}, \mathbb{Z}/p^r)\,,$$
$$\operatorname{Inv}\big(\mathcal{X}, \mathrm{H}^1(-, \underline{\mathbb{Z}/p^r}(1))\big) = \mathrm{H}^1_{\text{fl}}(\mathcal{X}, \mu_{p^r})\,.$$

Proof. We begin by proving the first equality. Consider the category $\mathrm{Sm}/\mathcal{X}$ whose objects are representable smooth morphisms $\mathcal{Y} \to \mathcal{X}$, with morphisms the commutative squares. On this category we can consider both the smooth and smooth-Nisnevich topology, and we write $(\mathrm{Sm}/\mathcal{X})_{\mathrm{Sm}}$ and $(\mathrm{Sm}/\mathcal{X})_{\text{sm-Nis}}$ for the respective Grothendieck sites. The identity $(\mathrm{Sm}/\mathcal{X}) \to (\mathrm{Sm}/\mathcal{X})$ is continuous from the smooth-Nisnevich to the smooth sites, and by [Sta22, TAG 00X6] it induces a morphism of sites $\pi\colon (\mathrm{Sm}/\mathcal{X})_{\mathrm{sm}} \to (\mathrm{Sm}/\mathcal{X})_{\text{sm-Nis}}$. Consequently, there is a Leray spectral sequence

$$\mathrm{H}^i_{\text{sm-Nis}}\big(\mathcal{X}, \mathrm{R}^j \pi_* \mathbb{Z}/p^r\big) \Longrightarrow \mathrm{H}^{i+j}_{\mathrm{sm}}(\mathcal{X}, \mathbb{Z}/p^r)\,.$$

By Lemma 4.3 the first row of the page is zero except for the first group, showing that

$$\mathrm{H}^0_{\text{sm-Nis}}\big(\mathcal{X}, \mathrm{R}^1 \pi_* \mathbb{Z}/p^r\big) = \mathrm{H}^1_{\mathrm{sm}}(\mathcal{X}, \mathbb{Z}/p^r)\,.$$

Now, smooth cohomology with coefficients in $\mathbb{Z}/p^r$ is equal to étale cohomology, and $\underline{\mathbb{Z}/p^r}(0)$ is quasi-isomorphic to $\mathbb{Z}/p^r$, so the left term is the same as $\mathrm{H}^0_{\text{sm-Nis}}\big(\mathcal{X}, \mathrm{H}^1(-, \underline{\mathbb{Z}/p^r}(0))\big)$, and the right term is equal to $\mathrm{H}^1(\mathcal{X}, \mathbb{Z}/p^r)$, allowing us to conclude.

The second equality is proven exactly in the same way: using the Leray spectral sequence for the inclusion of the smooth-Nisnevich site into the flat site, we show that

$$\mathrm{H}^1_{\text{fl}}(\mathcal{X}, \mu_{p^r}) = \mathrm{H}^0_{\text{sm-Nis}}\big(\mathcal{X}, \mathrm{R}^1 \pi_* \mu_{p^r}\big)\,,$$

which shows that $\mathrm{H}^1_{\mathrm{fl}}(\mathcal{X}, \mu_{p^r})$ is a smooth-Nisnevich sheaf for smooth quotient stacks. But by Lemma 4.2 we know that it is equal to cohomological invariants on smooth schemes, showing that the two are equal on all smooth quotient stacks. □

Now we deal with degree 2. First we show that Br', and in fact the whole group $\mathrm{H}^2(-, \mathbb{G}_{\mathrm{m}})$, is a smooth-Nisnevich sheaf on smooth stacks.

Lemma 4.5. *Let $X/\mathbf{k}$ be a smooth scheme, let $G/\mathbf{k}$ be an affine smooth algebraic group acting on it, and set $\mathcal{X} = [X/G]$. Then*

$$\mathrm{H}^2(\mathcal{X}, \mathbb{G}_{\mathrm{m}}) = \mathrm{H}^0_{\text{sm-Nis}}\big(\mathcal{X}, \mathrm{H}^2(-, \mathbb{G}_{\mathrm{m}})\big)\,.$$

In other words, $\mathrm{H}^2(-, \mathbb{G}_{\mathrm{m}})$ is a smooth-Nisnevich sheaf over $\mathcal{X}$.

Proof. Consider again the morphism of sites $\pi\colon (\mathrm{Sm}/\mathcal{X})_{\mathrm{sm}} \to (\mathrm{Sm}/\mathcal{X})_{\text{sm-Nis}}$. It induces a Leray–Grothendieck spectral sequence

$$\mathrm{H}^i_{\text{sm-Nis}}\big(\mathcal{X}, R^j\pi_*\mathbb{G}_{\mathrm{m}}\big) \Longrightarrow \mathrm{H}^{i+j}_{\mathrm{sm}}(\mathcal{X}, \mathbb{G}_{\mathrm{m}})\,.$$

Note that $R^1 i\pi_*\mathbb{G}_{\mathrm{m}}$ is the smooth-Nisnevich sheafification of the Picard group, which is zero on a smooth stack because the Picard group of a local regular ring is zero. Then if we prove that $\mathrm{H}^i_{\text{sm-Nis}}(\mathcal{X}, \mathbb{G}_{\mathrm{m}}) = 0$ for $i > 1$, we have proven our claim as the smooth and étale cohomology of $\mathbb{G}_{\mathrm{m}}$ coincide on $\mathcal{X}$.

Now we use the fact that $\mathcal{X} = [X/G]$. Let $X' = (X \times U)/G$ be an equivariant approximation of $\mathcal{X}$, and consider the Leray–Grothendieck spectral sequence induced by the projection $X' \xrightarrow{f} \mathcal{X}$,

$$\mathrm{H}^i_{\text{sm-Nis}}\big(\mathcal{X}, R^j f_*\mathbb{G}_{\mathrm{m}}\big) \Longrightarrow \mathrm{H}^{i+j}_{\text{sm-Nis}}\big(X', \mathbb{G}_{\mathrm{m}}\big)\,.$$

For any smooth quasi-separated algebraic space Y, we have $H^n_{\text{sm-Nis}}(Y, \mathbb{G}_{\mathrm{m}}) = H^n_{\mathrm{Nis}}(Y, \mathbb{G}_{\mathrm{m}}) = 0$ when $n > 1$; see [Pir18a, Proposition 7.5]. This necessarily implies that $\mathrm{H}^i_{\text{sm-Nis}}\big(X', \mathbb{G}_{\mathrm{m}}\big) = 0$ for $i > 1$ and $R^q f_*\mathbb{G}_{\mathrm{m}} = 0$ for $q > 1$. Now observe that $R^1 f_*\mathbb{G}_{\mathrm{m}} = 0$ for the same reason as above, and $R^0 f_*\mathbb{G}_{\mathrm{m}} = \mathbb{G}_{\mathrm{m}}$ as a vector bundle induces an isomorphism on $\mathcal{O}^*$. Thus the spectral sequence collapses, and we can conclude. □

Now that we know that the cohomological Brauer group is a sheaf in the right topology, all we need to conclude is a morphism from the correct cohomology group onto it.

Theorem 4.6. *Let $\mathcal{X} = [X/G]$ be as above. Then we have*

$$\mathrm{Br}'(\mathcal{X})_{p^r} = \mathrm{Inv}^2\big(\mathcal{X}, \mathrm{H}^n(-, \underline{\mathbb{Z}}/p^r(1))\big)\,.$$

Proof. Observe that thanks to Lemma 4.5, we have a morphism

$$\mathrm{Inv}^2\big(\mathcal{X}, \mathrm{H}^n(-, \underline{\mathbb{Z}}/p^r(1))\big) = \mathrm{H}^0_{\text{sm-Nis}}\big(\mathcal{X}, \mathrm{H}^n(-, \underline{\mathbb{Z}}/p^r(1))\big) \longrightarrow \mathrm{Br}'(\mathcal{X})_{p^r}\,.$$

As this is a map of sheaves, it suffices to prove our claim when X is a smooth scheme. Due to the identification $\underline{\mathbb{Z}}(1) = \mathbb{G}_{\mathrm{m}}[-1]$ in the étale derived category of X and the exact sequence $0 \to \underline{\mathbb{Z}}(1) \xrightarrow{\cdot p^r} \underline{\mathbb{Z}}(1) \to \underline{\mathbb{Z}}/p^r \to 0$, we get the functorial exact sequence

$$\begin{aligned}\mathrm{H}^2(X, \underline{\mathbb{Z}}(1)) = \mathrm{Pic}(X) \xrightarrow{\cdot p^r} \mathrm{Pic}(X) \longrightarrow \mathrm{H}^2(X, \underline{\mathbb{Z}}(1)/p^r)& \\ \longrightarrow \mathrm{H}^2(X, \mathbb{G}_{\mathrm{m}}) \xrightarrow{\cdot p^r} \mathrm{H}^2(X, \mathbb{G}_{\mathrm{m}}) = \mathrm{H}^3(X, \underline{\mathbb{Z}}(1))&\,.\end{aligned}$$

Passing to the Zariski sheafification allows us to conclude immediately as the Picard group of a smooth scheme is Zariski-locally trivial. □

5. Étale motivic cohomology of fields

This section is devoted to the study of étale motivic cohomology of fields. In particular, we are interested in the explicit description of these groups via (log) differential forms and symbols. We also recall the definition of ramified, tamely ramified and unramified elements in these cohomology groups, and we discuss some related results.

5.1 Motivic cohomology, log differentials and symbols

As explained earlier, the functor $F \mapsto \mathrm{H}^n(F, \mathbb{Z}/p^r(j))$ is non-zero only for $j = n, n-1$. When $r = 1$, we have an explicit description of the functor, but in general the descriptions we gave for $r > 1$ are not quite simple enough for computations.

We will introduce another description of the functors $\mathrm{H}^n(F, \mathbb{Z}/p^r(j))$ for $j = n, n+1$ in terms of symbols that allows us to do (somewhat) explicit computations. We begin with the simpler case $j = n$ and $r = 1$. In this case we have $\mathrm{H}^n(F, \mathbb{Z}/p(n)) \simeq \Omega^n_{F,\log} \subset \Omega^n_F$, that is, the subsheaf of forms on F that can be written as

$$\frac{\mathrm{d}b_1}{b_1} \wedge \frac{\mathrm{d}b_2}{b_2} \wedge \cdots \wedge \frac{\mathrm{d}b_n}{b_n}$$

with $b_1, \ldots, b_n \in F^*$.

Let $\{b_1, \ldots, b_n\}$ denote the element $b_1 \otimes \cdots \otimes b_n \in \mathrm{K}^n_{\mathrm{Mil}}$. Define $\mathrm{K}^n_{p^r} = \mathrm{K}^n_{\mathrm{Mil}}/p^r\mathrm{K}^n_{\mathrm{Mil}}$. Bloch and Kato [BK86, Corollary 2.8] proved that there is an isomorphism

$$\mathrm{K}^n_{p^r}(F) \longrightarrow W_r\Omega^n_{F,\log} = \mathrm{H}^n(F, \mathbb{Z}/p^r(n))\,.$$

When $r = 1$, we can explicitly see the map as

$$\{b_1, \ldots, b_n\} \longmapsto \frac{\mathrm{d}b_1}{b_1} \wedge \frac{\mathrm{d}b_2}{b_2} \wedge \cdots \wedge \frac{\mathrm{d}b_n}{b_n}\,.$$

Note that the operation on the left is multiplication, while on the right it is addition. We will sometimes shorten the notation $\{b_1, \ldots, b_n\}$ to $\{\underline{b}\}$ when n is clear from context.

We have the following description by Izhboldin for $\mathrm{H}^{n+1}(F, \mathbb{Z}/p(n))$: there is an exact sequence

$$\Omega^n_{F,\log} \longrightarrow \Omega^n_F \xrightarrow{\mathcal{P}} \Omega^n_F/\mathrm{d}\Omega^{n-1}_F \longrightarrow \mathrm{H}^{n+1}(F, \mathbb{Z}/p(n)) \longrightarrow 0\,,$$

where

$$\mathcal{P}\left(a\frac{\mathrm{d}b_1}{b_1} \wedge \frac{\mathrm{d}b_2}{b_2} \wedge \cdots \wedge \frac{\mathrm{d}b_n}{b_n}\right) = (a^p - a)\frac{\mathrm{d}b_1}{b_1} \wedge \frac{\mathrm{d}b_2}{b_2} \wedge \cdots \wedge \frac{\mathrm{d}b_n}{b_n}\,.$$

The group of forms Ω^n_F has a description in symbols given by

$$\Omega^n_F = [a, b_1, \ldots, b_n\}\,, \quad a \in F\,,\ b_i \in F^*\,,$$

given by the map

$$[a, b_1, \ldots, b_n\} \longmapsto a\frac{\mathrm{d}b_1}{b_1} \wedge \frac{\mathrm{d}b_2}{b_2} \wedge \cdots \wedge \frac{\mathrm{d}b_n}{b_n}$$

subject to the relations that $[a, b_1, \ldots, b_i\} = 0$ if $b_i = b_j$ for some $i \neq j$ and

$$[u+v, u+v, \ldots, b_n\} = [u, u, \ldots, b_n\} + [v, v, \ldots, b_n\}$$

for $u, v, u+v \in F^*$. With the extra relations coming from the exact sequence above, we get the following description of $\mathrm{H}^{n+1}(F, \mathbb{Z}/p(n))$:

$$\mathrm{H}^{n+1}(F, \mathbb{Z}/p(n)) = F \otimes (F^*)^{\otimes n}/J\,,$$

where J is the subgroup generated by elements of the form

- $[a, b_1, \dots, b_n\}$, where $b_i = b_j$ for some $i \neq j$;
- $[a^p - a, b_1, \dots, b_n\}$;
- $[a, a, b_2, \dots, b_n\}$, where $a \in F^*$.

Following Izhboldin's notation we will write H_p^{n+1} for this functor. There are natural pairings $\mathrm{H}_p^{j+1}(F) \otimes \mathrm{K}_p^i(F) \to \mathrm{H}_p^{i+j+1}(F)$ given by

$$[a, b'_1, \dots, b'_j\} \otimes \{b_1, \dots, b_i\} \longmapsto [a, b'_1, \dots, b'_j, b_1, \dots, b_i\}.$$

A description of the cohomology groups and $\mathrm{H}^{n+1}(F, \mathbb{Z}/p^r(n))$ for $r > 1$ requires some additional work. Given an $\mathbb{F}_p$-algebra A, the group of Witt vectors [Ill79, Section 1] of length r on A, denoted by $W_r(A)$, is a group whose underlying set is A^r and whose group structure is defined in terms of a sequence of *universal polynomials* depending on the characteristic. Keep the following in mind:

- $W_1(A) = A$.
- The maps $(a_1, \dots, a_r) \mapsto (0, \dots, 0, a_1, \dots, a_r)$, $(a_1, \dots, a_r) \mapsto (a_1, \dots, a_{r-i})$ are group homomorphisms.
- We have $p(a_1, \dots, a_r) = \big(0, a_1^p, \dots, a_{r-1}^p\big)$.

Using Witt vectors, Izhboldin gives the following description of the groups $\mathrm{H}^n(F, \mathbb{Z}/p^r(n))$ and $\mathrm{H}^{n+1}(F, \underline{\mathbb{Z}}/p^r(n))$ (see [Izh91, Corollary 6.5]): write $[a_1, \dots, a_r, b_1, \dots, b_n\}$ for the element $(a_1, \dots a_r) \otimes b_1 \otimes \dots \otimes b_n$ in $W_r(F) \otimes \big(F^*\big)^{\otimes n}$, and define

$$Q^n(F, r) = W_r(F) \otimes \big(F^*\big)^{\otimes n} / I \,,$$

where I is the ideal generated by elements of the form

- $[a_1, \dots a_r, b_1, \dots, b_n\}$, where $b_i = b_j$ for some $i \neq j$;
- $[0, \dots, 0, a, 0, \dots, 0, a, b_2, \dots, b_n\}$, where $a \in F^*$.

By [Izh91, Theorem C] we have an isomorphism

$$\mathrm{K}_{p^r}^n(F) \simeq \ker\big(Q^n(F, r) \xrightarrow{\mathcal{P}} Q^n(F, r)\big)\,,$$

where

$$\mathcal{P}([a_1, \dots, a_r, b_1, \dots, b_n\}) \stackrel{\text{def}}{=\!=} [a_1^p, \dots, a_r^p, b_1, \dots, b_n\} - [a_1, \dots, a_r, b_1, \dots, b_n\}\,.$$

As $\mathrm{K}_{p^r}^n(F) \simeq \mathrm{H}^n(F, \mathbb{Z}/p^r(n))$, we can regard the latter as a subgroup of $Q^n(F, r)$. The group $\mathrm{H}^{n+1}(F, \mathbb{Z}/p^r(n))$ can be described as the cokernel of this map, that is,

$$\mathrm{H}^{n+1}(F, \underline{\mathbb{Z}}/p^r(n)) \simeq W_r(F) \otimes \big(F^*\big)^{\otimes n} / J\,,$$

where J is the subgroup generated by elements of the form

- $[a_1, \dots, a_r, b_1, \dots, b_n\}$, where $b_i = b_j$ for some $i \neq j$;
- $[a_1^p, \dots, a_r^p, b_1, \dots, b_n\} - [a_1, \dots, a_r, b_1, \dots, b_n\}$;
- $[0, \dots, 0, a, 0, \dots, 0, a, b_2, \dots, b_n\}$, where $a \in F^*$.

We will call these functors $\mathrm{H}_{p^r}^{n+1}$. We will sometimes shorten the notation for a Witt vector $[a_1, \dots, a_r]$ to $[\underline{a}]$ and write $[\underline{a}, \underline{b}\}$ for $[a_1, \dots, a_r, b_1, \dots, b_n\}$.

5.2 Torsion in $\mathrm{H}^{n+1}_{p^r}(F)$

We prove here some technical results on $\mathrm{H}^{n+1}_{p^r}(F)$ and its torsion subgroups. The main ingredient will be the following exact sequence due to Izhboldin.

Lemma 5.1 ([Izh96, Lemma 6.2]). *For $1 \leqslant s < r$ there is a short exact sequence*

$$0 \longrightarrow \mathrm{H}^{n+1}_{p^s}(F) \xrightarrow{\iota_s} \mathrm{H}^{n+1}_{p^r}(F) \xrightarrow{\pi_s} \mathrm{H}^{n+1}_{p^{r-s}}(F) \longrightarrow 0\,,$$

where the homomorphisms are defined as

$$\iota_s[a_1,\dots,a_s,b_1,\dots,b_n\} = [0,\dots,0,a_1,\dots,a_s,b_1,\dots,b_n\}\,,$$
$$\pi_s[a_1,\dots,a_r,b_1,\dots,b_n\} = [a_1,\dots,a_{r-s},b_1,\dots,b_n\}\,.$$

The image of ι_s is clearly of p^s-torsion, and in fact as the following proposition and corollary show, it is exactly equal to the p^s-torsion subgroup in $\mathrm{H}^{n+1}_{p^r}(F)$.

Proposition 5.2. *Let F be a field of characteristic $p > 0$. Then the subgroup of elements of p-torsion in $\mathrm{H}^{n+1}_{p^r}(F)$ is isomorphic to $\mathrm{H}^{n+1}_{p}(F)$, embedded via the homomorphism $[a,b_1,\dots,b_n\} \mapsto [0,\dots,0,a,b_1,\dots,b_n\}$.*

Proof. We argue by induction on r. For $r = 1$ the statement is obvious. For $r = 2$ define

$$\alpha = \sum_i [a_i, a_{2,i}, \underline{b_i}\}\,.$$

Then $p\alpha = \sum_i [0, a_i^p, \underline{b_i}\}$. Now note that taking the inverse image in $\mathrm{H}^{n+1}_{p}(F)$, we obtain

$$\sum_i \bigl([a_i^p, \underline{b_i}\} - [a_i, \underline{b_i}\}\bigr) = 0 \implies \sum_i [a_i, \underline{b_i}\} = 0 \iff \sum_i [a_i^p, \underline{b_i}\} = 0\,,$$

which proves the statement. Now let r be greater than 2, and let $\alpha = \sum_i [\underline{a_i}, \underline{b_i}\}$ be p-torsion. Then $\pi_{r-1}\alpha$ is p-torsion.

By the inductive hypothesis we know that

$$\pi_{r-1}\alpha = \sum_i [0,\dots,0,a_i',\underline{b_i'}\} \iff \alpha = \sum_i [0,\dots,a_i',0,\underline{b_i'}\} + \iota_1\beta$$

for some $\beta \in \mathrm{H}^n_p(F)$.

This shows that α belongs to $\iota_2(\mathrm{H}^{n+1}_{p^2}(F))$, so we can conclude by the case $r = 2$. □

Corollary 5.3. *The subgroup of elements of p^s-torsion in $\mathrm{H}^{n+1}_{p^r}(F)$ is isomorphic to $\mathrm{H}^{n+1}_{p^s}(F)$, embedded via the homomorphism*

$$[a_1,\dots,a_s,b_1,\dots,b_n\} \longmapsto [0,\dots,0,a_1,\dots,a_s,b_1,\dots,b_n\}\,.$$

Proof. We proceed by induction on s. An element $\alpha \in \mathrm{H}^{n+1}_{p^r}(F)$ is of p^s-torsion if and only if $p^{s-1}\alpha$ is of p-torsion, which by Proposition 5.2 means that $\pi_{r-1}(\alpha)$ is of p^{s-1}-torsion because the p-torsion of $\mathrm{H}^{n+1}_{p^r}(F)$ maps to zero in $\mathrm{H}^{n}_{p^{r-1}}(F)$. By the inductive hypothesis this means that we can write $\alpha = \sum_i [0,\dots,0,\underline{a}_i',0,\underline{b_i}\} + \iota_1\beta$ with $\underline{a}_i' \in F^{s-1}$, which allows us to conclude immediately. □

5.3 Unramified, tamely ramified and wildly ramified elements

The functor K_{p^r} has a natural notion of ramification at a point $x \in X^{(1)}$. If π is a uniformizer for the DVR $\mathcal{O}_{X,x}$, define

$$\partial_x \colon \mathrm{K}^n_{p^r}(\mathbf{k}(X)) \longrightarrow \mathrm{K}^n_{p^r}(\mathbf{k}(x))\,, \quad \partial_x\{\pi, b_2,\dots,b_n\} = \{\overline{b_2},\dots,\overline{b_n}\}\,,$$

where $\overline{b_i}$ is the image of b_i in $\mathbf{k}(x)$, and $\partial_x\{b_1,\dots,b_n\} = 0$ if $b_i \in \mathcal{O}^*_{X,x}$ for all i. We have the following description of $\mathrm{K}^\bullet_{p^r}(\mathbf{k}(t))$, from [GS06, Chapter 7], which shows that Milnor's K-theory mod p^r behaves the way we expect based on the mod ℓ case:

$$0 \longrightarrow \mathrm{K}^\bullet_{p^r}(\mathbf{k}) \longrightarrow \mathrm{K}^\bullet_{p^r}(\mathbf{k}(t)) \longrightarrow \bigoplus_{x\in(\mathbb{A}^1)^{(1)}} \mathrm{K}^{\bullet-1}_{p^r}(\mathbf{k}(x)) \longrightarrow 0\,. \tag{5.1}$$

The ramification map ∂_v turns out to be equal to the one in [GS88, Corollary 1.6] over perfect fields, and we will see later (Section 6) that we can use it to compute ramification in equation (2.3).

To get a map into $\mathrm{H}^n_{p^r}(\mathbf{k}(x))$ suiting our needs, we will have to restrict to an appropriate subgroup, the subgroup of *tamely ramified* elements.

First let us recall the notion of tamely ramified field extensions. Let (R,v) be a DVR, with fraction field F, residue field $\mathbf{k}_v$ and maximal ideal $\mathfrak{m}_v$. Let $F \subset L$ be a finite and separable extension, and denote by $\mathfrak{m}_i$ the maximal ideals of the integral closure $\overline{R}$ of R in L (there are only finitely many of these maximal ideals). Then $F \subset L$ is *tamely ramified* at v if either $\mathbf{k}_v$ has characteristic zero, or if $\mathbf{k}_v$ has characteristic p, the extensions $\mathbf{k}_v \subset \overline{R}/\mathfrak{m}_i$ are separable and the ramification indices of $\mathfrak{m}_v$ in $\overline{R}$ are prime to p.

Definition 5.4. Let F be a field and v a valuation on it. Define F^{tm} as the subfield of F^{sep} generated by extensions that are tamely ramified at v. An element $\alpha \in \mathrm{H}^\bullet_{p^r}(F)$ is tamely ramified at v if and only if $\alpha_{F^{\mathrm{tm}}} = 0$. We say that α is wildly ramified otherwise.

Given (F,v) as above, we write $\mathrm{H}^\bullet_{p^r}(F)_{\mathrm{tm}}$ for the subgroup of tamely ramified elements and $\widetilde{\mathrm{H}}^\bullet_{p^r}(F)$ for the wild quotient $\mathrm{H}^\bullet_{p^r}(F)/\mathrm{H}^\bullet_{p^r}(F)_{\mathrm{tm}}$.

Note that by [End72, Theorem 17.19 and Section 22], given a DVR (R,v) the quotient field of the strict Henselization $\mathbf{k}\big(R^{\mathrm{sh}}\big)$ is contained in $\mathbf{k}(R)^{\mathrm{tm}}$.

For $r = 1$ Izhboldin shows that the subgroup of tamely ramified elements is generated by symbols $[a, b_1,\dots,b_n\}$, where $v(a) \geqslant 0$, that is, $a \in \mathcal{O}_v$. The following proposition extends this result to general r.

Proposition 5.5. *Let (F,v) be as above. An element $\alpha \in \mathrm{H}^{n+1}_{p^r}(F)$ is tamely ramified at v if and only if α can be written as a sum of elements of the form $[a_1,\dots,a_r,b_1,\dots,b_n\}$ with $v(a_i) \geqslant 0$ for all i.*

Proof. We observe that the elements as above form a subgroup of $\mathrm{H}^{n+1}_{p^r}(F)$ corresponding to the image of $W_r(\mathcal{O}_v)\otimes \mathrm{K}^n_{p_r}(F)$. The case $r = 1$ is true by [Izh96, Corollary 2.6]. We proceed by induction. Assume that the statement is true for $r-1$.

First let $\alpha = [a_1,\dots,a_r,b_1,\dots,b_n\}$, and assume $v(a_i) \geqslant 0$ for all i. We want to show that α is tamely ramified. The extension F'/F obtained by adding a root of $x^p - x - a_1$ to F is unramified, and $\alpha_{F'} = [0, a'_2\dots,a'_r,b_1,\dots,b_n\}$ for some $a'_2,\dots,a'_r \in \mathcal{O}_{v'}$. Repeating the process up to $r-1$ more times, we see that α becomes zero after we pass to a finite unramified extension.

Now let $\alpha = \sum_i[\underline{a}_i, \underline{b}_i\}$ be a tamely ramified element, and let the morphisms π_s and ι_s be defined as in Lemma 5.1. Then $\pi_{r-1}(\alpha)$ is tamely ramified, which means that there is an element $\alpha' = \sum_j[a'_j, \underline{b}'_j\} \in \mathrm{H}^{n+1}_p(F)$ which is equal to $\pi_1(\alpha)$ and such that $v(a'_1) \geqslant 0$. But then $\alpha = \sum_j[a'_j,0,\dots,0,\underline{b}'_j\} + \alpha'$, where α' is a tamely ramified element coming from $\mathrm{H}^{n+1}_{p^{r-1}}(F)$, and we can conclude by the inductive hypothesis. □

Izhboldin [Izh96, Corollary 2.6 and Proposition 6.6] completely described the tame part of $\mathrm{H}^{n+1}_{p^r}(F)$ for a complete discrete valuation field (F, v). Totaro [Tot22, Theorem 4.3] showed that for $r = 1$ the same description extends to Henselian fields. Here we show that the description of the *tame* part of $\mathrm{H}^{n+1}_{p^r}(F)$ (trivially) works for Henselian fields for any r.

Proposition 5.6. *Let (R, v) be an Henselian DVR. We have a split exact sequence*

$$0 \longrightarrow \mathrm{H}^{n+1}_{p^r}(\mathbf{k}_v) \xrightarrow{j^*} \mathrm{H}^{n+1}_{p^r}(\mathbf{k}(R))_{\mathrm{tm}} \xrightarrow{\partial_v} \mathrm{H}^{n}_{p^r}(\mathbf{k}_v) \longrightarrow 0\,.$$

We can see j^* as the composition $\mathrm{H}^{n+1}_{p^r}(\mathbf{k}_v) = \mathrm{H}^{n+1}(R, \mathbb{Z}/p^r(n)) \to \mathrm{H}^{n+1}_{p^r}(\mathbf{k}(R))$. It is defined by $[a_1, \ldots, a_r, b_1, \ldots, b_n\} \mapsto [\tilde{a}_1, \ldots, \tilde{a}_r, \tilde{b}_1, \ldots, \tilde{b}_n\}$, where $\tilde{\varphi}$ is any lifting of φ to R.

The residue map ∂_v is $[a_1, \ldots, a_r, \pi, b_2, \ldots, b_n\} \mapsto [\overline{a}_1, \ldots, \overline{a}_r, \overline{b}_2, \ldots, \overline{b}_n\}$, where $\overline{\varphi}$ is the image of φ in $\mathbf{k}_v$.

Proof. The case $r = 1$ is proven in [Tot22, Theorem 4.3], and the general case can be proven by induction exactly as in [Izh96, Proposition 6.6], by considering the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{A}^{n+1}_{r-1} & \longrightarrow & \mathrm{A}^{n+1}_{r} & \longrightarrow & \mathrm{A}^{n+1}_{1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^{n+1}_{p^{r-1}}(\mathbf{k}(R)) & \xrightarrow{\iota_{r-1}} & \mathrm{H}^{n+1}_{p^r}(\mathbf{k}(R)) & \xrightarrow{\pi_1} & \mathrm{H}^{n+1}_{p}(\mathbf{k}(R)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{\mathrm{H}}^{n}_{p^{r-1}}(\mathbf{k}(R)) & \longrightarrow & \widetilde{\mathrm{H}}^{n}_{p^r}(\mathbf{k}(R)) & \longrightarrow & \widetilde{\mathrm{H}}^{n}_{p^r}(\mathbf{k}(R)) & \longrightarrow & 0\,,
\end{array}$$

where $A^{n+1}_i = \mathrm{H}^{n+1}_{p^i}(\mathbf{k}_v) \oplus \mathrm{H}^{n}_{p^i}(\mathbf{k}_v)$. The left and right columns are exact by the inductive hypothesis, which implies the same for the central one. The map

$$[a_1, \ldots, a_r, b_1, \ldots, b_n\} \longmapsto [\tilde{a}_1, \ldots, \tilde{a}_r, \tilde{b}_1, \ldots, \tilde{b}_n\}$$

can be shown to be well defined exactly as in [Tot22, Theorem 4.3] because if $\mathfrak{m} \subset R$ is the maximal ideal, any element $a \in \mathfrak{m}$ can be written as $u^p - u$ for some $u \in \mathbf{k}(R)$. It is an isomorphism by Proposition 6.7. $\square$

5.4 Izhboldin's description of wild ramification.

Coming back to the wild part of motivic cohomology, Izhboldin and Totaro give a nice description of $\widetilde{\mathrm{H}}^{n+1}_{p}(F)$, which we are going to recall. As before, we denote by (F, v) a discrete valuation field, with associated DVR R and residue field $\mathbf{k}_v$.

There is a filtration

$$0 \subset \mathrm{U}_0 \subset \mathrm{U}_1 \subset \cdots \subset \mathrm{U}_{i-1} \subset \mathrm{U}_i \subset \cdots \subset \mathrm{H}^{n+1}_{p}(F)\,,$$

where

$$\mathrm{U}_i = \langle [a, b_1, \ldots, b_n\} \mid v(i) \geqslant -i \rangle\,.$$

The graded pieces $\mathrm{U}_i/\mathrm{U}_{i-1}$ of this filtration admit a description in terms of forms *over the residue field.*

Indeed, for $p \nmid i$ there is a morphism

$$\psi_i \colon \Omega^n_{\mathbf{k}_v} \longrightarrow \mathrm{U}_i/\mathrm{U}_{i-1}$$

defined as follows: given an element $\alpha = a(\mathrm{d}b_1/b_1)\wedge\cdots\wedge(\mathrm{d}b_n/b_n)$ in $\Omega^n_{\mathbf{k}_v}$, pick liftings $a', b'_1, \ldots, b'_n$ in R of $a, b_1, \ldots, b_n$; then we define $\psi_i(\alpha)$ as $[t^{-i}a', b'_1, \ldots, b'_n\}$ in $\mathrm{U}_i/\mathrm{U}_{i-1}$.

This map is well defined because if we pick a second lifting, the associated element in $\mathrm{U}_i/\mathrm{U}_{i-1}$ differs from the previous one by a form in U_{i-1}, which is of course zero in the quotient.

For $p \mid i$ let $Z^n \subset \Omega^n_{\mathbf{k}_v}$ be the subgroup of closed forms. We define

$$\begin{aligned}\psi_i \colon \Omega^n_{\mathbf{k}_v}/Z^n \oplus \Omega^{n-1}_{\mathbf{k}_v}/Z^{n-1} &\longrightarrow \mathrm{U}_i/\mathrm{U}_{i-1}\,,\\ \left(a\frac{\mathrm{d}b_1}{b_1}\wedge\cdots\wedge\frac{\mathrm{d}b_n}{b_n}, 0\right) &\longmapsto [t^{-i}a', b'_1, \ldots, b'_n\}\,,\\ \left(0, a\frac{\mathrm{d}b_2}{b_2}\wedge\cdots\wedge\frac{\mathrm{d}b_n}{b_n}\right) &\longmapsto [t^{-i}a', t, b'_2, \ldots, b'_n\}\,.\end{aligned}$$

As before, the $a', b'_1, \ldots, b'_n$ denote liftings of $a, b_1, \ldots, b_n$ to R. Again, picking a different lifting changes the associated element in $\mathrm{U}_i/\mathrm{U}_{i-1}$ by a form in U_{i-1}, so the map is well defined.

Theorem 5.7 ([Izh96, Theorem 2.5] and [Tot22, Theorem 4.3]). *Let F be a field with a discrete valuation v, let $\mathbf{k}_v$ be the residue field of the corresponding DVR, and let $0 \subset \mathrm{U}_0 \subset \mathrm{U}_1 \subset \cdots \subset \mathrm{H}^{n+1}_p(F)$ be the filtration defined above.*

Then the subgroup U_0 is the tame subgroup $\mathrm{H}^{n+1}_p(F)_{\mathrm{tm}}$, and the maps ψ_i that we defined before induce isomorphisms

$$\mathrm{U}_i/\mathrm{U}_{i-1} \simeq \begin{cases}\Omega^n_{\mathbf{k}_v} & \text{if } p \nmid i\,,\\ \Omega^n_{\mathbf{k}_v}/Z_n \oplus \Omega^{n-1}_{\mathbf{k}_v}/Z_{n-1} & \text{if } p \mid i\,,\end{cases}$$

where $Z_n \subset \Omega^n_{\mathbf{k}_v}$ is the subgroup of closed forms.

Therefore, the group $\widetilde{\mathrm{H}}^{n+1}_p(F)$ is isomorphic to $\mathrm{H}^{n+1}_p(F)/\mathrm{U}_0$, and it has a filtration whose associated graded pieces $\mathrm{U}_i/\mathrm{U}_{i-1}$ are described by Theorem 5.7.

Nonetheless, given $\alpha \in \mathrm{U}_i$, understanding its image in $\mathrm{U}_i/\mathrm{U}_{i-1}$ and extrapolating information on its class in $\widetilde{\mathrm{H}}^{n+1}_p(F)$ may be highly non-trivial.

Luckily, when α is one of the standard generators of $\mathrm{U}_i/\mathrm{U}_{i-1}$, the situation gets much easier, as shown in the following corollary.

Corollary 5.8. *Let (F, v) be as above, and let t be a uniformizer for v. Assume moreover that $\mathbf{k}_v = \mathbf{k}$.*

1. *If $p \nmid i$, then for every $\varphi \in \Omega^n_{\mathbf{k}}$ the equality $t^{-i}\varphi = 0$ in $\mathrm{U}_i/\mathrm{U}_{i-1}$ implies $t^{-i}\varphi = 0$ in $\mathrm{H}^{n+1}_p(F)$.*
2. *If $p \mid i$, then for every $\varphi \in \Omega^n_{\mathbf{k}}$ the equality $t^{-i}\varphi = 0$ in $\mathrm{U}_i/\mathrm{U}_{i-1}$ implies $t^{-i}\varphi = t^{-i/p}\varphi'$ in $\mathrm{H}^{n+1}_p(F)$ for some $\varphi' \in \Omega^n_{\mathbf{k}}$.*
3. *If $p \mid i$, then for every $\psi \in \Omega^{n-1}_{\mathbf{k}}$ the equality $t^{-i}(\mathrm{d}t/t)\wedge\varphi = 0$ in $\mathrm{U}_i/\mathrm{U}_{i-1}$ implies $t^{-i}(\mathrm{d}t/t)\wedge\varphi = t^{-i/p}(\mathrm{d}t/t)\wedge\varphi'$ for some $\varphi' \in \Omega^{\mathbf{k}}_{n-1}$. If we further assume $p^2 \nmid i$, then we can replace $(\mathrm{d}t/t)\wedge\varphi'$ in the expression above with $\varphi'' \in \Omega^n_{\mathbf{k}}$.*

Remark 5.9. The hypothesis that $\mathbf{k}_v = \mathbf{k}$ in Corollary 5.8 is of key importance because it provides a canonical way to lift elements from $\Omega^n_{\mathbf{k}_v}$ to Ω^n_F using the embedding $\mathbf{k}_v = \mathbf{k} \hookrightarrow F$. Without this hypothesis, we would not be able to define an element in Ω^n_F because the maps that induce isomorphisms of Theorem 5.7 only lift to U_i up to elements of U_{i-1}.

Proof of Corollary 5.8. Point (1) follows from the fact that $\psi_i\colon \Omega^n_{\mathbf{k}_v} \to \mathrm{U}_i/\mathrm{U}_{i-1}$ is an isomorphism; hence if $\psi_i(\varphi) = 0$, we must have $\varphi = 0$.

If $p \mid i$, then the equality $t^{-i}\varphi = 0 \in \mathrm{U}_i/\mathrm{U}_{i-1}$, together with Theorem 5.7, implies that φ is closed. By Cartier's theorem [Izh96, Lemma 1.5.1], this implies that φ is a sum of exact forms and forms that can be written as $a^p(\mathrm{d}b_1/b_1) \wedge \cdots \wedge (\mathrm{d}b_n/b_n)$. If $\tau = \mathrm{d}\tau' \in \Omega_{\mathbf{k}}$ is exact, then $t^{-i}\tau = \mathrm{d}(t^{-i}\tau')$, so $t^{-i}\tau = 0 \in \mathrm{H}^{n+1}_p(F)$. On the other hand,

$$\big(t^{-i}a^p - t^{-i/p}a\big)\frac{\mathrm{d}b_1}{b_1} \wedge \cdots \wedge \frac{\mathrm{d}b_n}{b_n} = 0 \in \mathrm{H}^{n+1}_p(F)\,,$$

which immediately implies

$$\big(t^{-i}a^p\big)\frac{\mathrm{d}b_1}{b_1} \wedge \cdots \wedge \frac{\mathrm{d}b_n}{b_n} = t^{-i/p}a\frac{\mathrm{d}b_1}{b_1} \wedge \cdots \wedge \frac{\mathrm{d}b_n}{b_n} \in \mathrm{U}_{i/p}\,.$$

This proves point (2).

To prove point (3), proceeding as in point (3), we easily get that $t^{-i}(\mathrm{d}t/t)\wedge\varphi = t^{-i/p}(\mathrm{d}t/t)\wedge\varphi'$. Indeed, if $\varphi = \mathrm{d}\tau$, then

$$\mathrm{d}\big(t^{-i-1}\mathrm{d}t \wedge \tau\big) = t^{-i}\frac{\mathrm{d}t}{t} \wedge \mathrm{d}\tau + (i-1)t^{-i-2}\mathrm{d}t \wedge \mathrm{d}t \wedge \tau\,,$$

and both the left-hand side of the equality and the last term of the sum on the right-hand side of the equality are zero in $\mathrm{H}^{n+1}_p(F)$. If $\varphi = a^p(\mathrm{d}b_1/b_1) \wedge \cdots \wedge (\mathrm{d}b_n/b_n)$, then the same argument used to prove point (2) shows that $t^{-i}(\mathrm{d}t/t)\varphi = t^{-i/p}(\mathrm{d}t/t) \wedge \varphi'$. If we further assume $p^2 \nmid i$, then we have that

$$\mathrm{d}\left(at^{-i/p} \wedge \frac{\mathrm{d}b_1}{b_1} \wedge \cdots \wedge \frac{\mathrm{d}b_n}{b_n}\right) = -\frac{i}{p}at^{-i/p}\frac{\mathrm{d}t}{t} \wedge \frac{\mathrm{d}b_1}{b_1} \wedge \ldots \wedge \frac{\mathrm{d}b_n}{b_n} + t^{-i/p}\frac{\mathrm{d}a}{a} \wedge \frac{\mathrm{d}b_1}{b_1} \wedge \cdots \wedge \frac{\mathrm{d}b_n}{b_n},$$

which implies that in $\mathrm{H}^{n+1}_p(F)$ we have

$$t^{-i/p}\frac{\mathrm{d}t}{t} \wedge \varphi' = (-i/p)^{-1}t^{-i/p}\frac{\mathrm{d}a}{a} \wedge \frac{\mathrm{d}b_1}{b_1} \wedge \cdots \wedge \frac{\mathrm{d}b_n}{b_n}\,,$$

where i/p is invertible by hypothesis. This proves point (3). □

Another consequence of Theorem 5.7 is that every element in $\mathrm{H}^{n+1}_p(\mathbf{k}(t))$ can, up to subtracting a tamely ramified element, be written in a standard form.

LEMMA 5.10. *Let (F, v) be a field with a discrete valuation such that $\mathbf{k}_v = \mathbf{k}$, and let β be an element of $\mathrm{H}^{n+1}_p(F)$. Let t be a uniformizer for v. Then we have*

$$\beta = \sum_{i=0}^{m} t^{-i}\varphi_i + t^{-i}\frac{\mathrm{d}t}{t} \wedge \varphi'_i + \beta_{\mathrm{tm}} = \sum_{i=0}^{m}[c_i/t^i, \underline{f_i}\} + [c'_i/t^i, t, \underline{f'_i}\} + \beta_{\mathrm{tm}}\,,$$

where $\varphi_i \in \Omega^n_{\mathbf{k}}$, $\varphi'_i \in \Omega^{n-1}_{\mathbf{k}}$ and β_{tm} is tamely ramified. Moreover, we can pick $\varphi'_i = 0$ if $p \nmid i$ and φ, φ' not closed if $p \mid i$.

Proof. Assume $\beta \in \mathrm{U}_m$. By Theorem 5.7 there are φ_m, φ'_m in, respectively, $\Omega^n_{\mathbf{k}}$ and $\Omega^{n-1}_{\mathbf{k}}$ such that

$$\beta = t^{-m}\varphi + t^{-m}\frac{\mathrm{d}t}{t} \wedge \phi' \in \mathrm{U}_m/\mathrm{U}_{m-1}\,,$$

so that the element $\beta - t^{-m}\varphi - t^{-m}(\mathrm{d}t/t) \wedge \phi'$ belongs to U_{m-1}. If $p \mid m$, we can pick φ, φ' not closed, while if $p \nmid m$, we can pick $\varphi' = 0$.

Repeating this process up to $m-1$ times, we find that

$$\beta - \sum_{i\leqslant m,\, p|i}\left(t^{-i}\varphi + t^{-i}\frac{\mathrm{d}t}{t}\wedge\phi'\right) - \sum_{i\leqslant m,\, p\nmid i} t^{-i}\varphi$$

belongs to U_0, proving our claim. □

The following proposition describes a set of generators for $\widetilde{\mathrm{H}}^n_{p^r}(F)$ and shows that to check if an element is wildly ramified, we can reduce to the case $r=1$.

Proposition 5.11. *Let (F,v) be a field with a discrete valuation. The group*

$$\widetilde{\mathrm{H}}^{n+1}_{p^r}(F) = \mathrm{H}^{n+1}_{p^r}(F)/\mathrm{H}^{n+1}_{p^r}(F)_{\mathrm{tm}}$$

is generated by elements of the form

$$[0,\ldots,0,a_i,\ldots,a_r,b_1,\ldots,b_n\}\,,$$

where $[a_i,b_1,\ldots,b_n\}\in\mathrm{H}^n_p(F)$ is wildly ramified.

Moreover, given a wildly ramified element α, we can always write

$$\alpha = \iota_i\alpha' + \alpha_{\mathrm{tm}}\,,$$

where α_{tm} is tamely ramified and $\pi_{i-1}\alpha'\in\mathrm{H}^{n+1}_p(F)$ is wildly ramified.

Proof. To prove our first claim, we proceed by induction on r. The case $r=1$ is trivial. Now assume the claim for $r-1$. Write $\alpha=[a_1,\ldots,a_r,b_1,\ldots,b_n\}$. If $[a_1,b_1,\ldots,b_n\}$ is wildly ramified, there is nothing to prove.

If $[a_1,b_1,\ldots,b_n\}$ is tamely ramified, we must have

$$[a_1,b_1,\ldots,b_n\} = \sum_s [a_{1,s},b_{1,s},\ldots,b_{n,s}\}$$

with $v(a_{1,s})\geqslant 0$. Now let

$$\tilde\alpha = \sum_s [a_{1,s},0,\ldots,0,b_{1,s},\ldots,b_{n,s}\}\in\mathrm{H}^{n+1}_{p^r}(F)\,;$$

then $\alpha-\tilde\alpha$ belongs to the kernel of π_1, that is, the image of $\mathrm{H}^{n+1}_{p^{r-1}}(F)$, and $\tilde\alpha$ is clearly tamely ramified, so we can conclude by the inductive hypothesis.

Now take a wildly ramified element α, and consider the minimum s such that $\alpha=\iota_s\alpha'+\alpha_{\mathrm{tm}}$ with α_{tm} a tamely ramified element. If $\pi_1\alpha'$ is tamely ramified, then as above there is a tamely ramified element $\tilde\alpha'$ such that $\alpha'-\tilde\alpha'$ is in the image of $\mathrm{H}^{n+1}_{p^{s-1}}(F)$, contradicting the minimality of s. □

An immediate consequence is that checking if an element is tamely ramified or unramified can be done on the Henselization or even completion of (F,v).

Corollary 5.12. *Let (F,v) be a discretely valued field, and let α be an element of $\mathrm{H}^{n+1}_{p^r}(F)$. Then α is tamely ramified (respectively, unramified) if and only if the same is true for the pullback of α to the Henselization or completion of (F,v).*

Proof. Let F' be either the Henselization or completion or F at v. Proposition 5.11 shows that α is wildly ramified if a certain element $\alpha'\in\mathrm{H}^{n+1}_p(F)$ is, and Theorem 5.7 shows that the pullback $\widetilde{\mathrm{H}}^{n+1}_p(F)\to\widetilde{\mathrm{H}}^{n+1}_p(F')$ is an isomorphism, so α is wildly ramified on F if and only if it is on F'.

Now assume that α is tamely ramified. The differential ∂_v clearly commutes with passing to F' and as $\mathbf{k}_v=\mathbf{k}_{v'}$, we have the equivalence $\partial_v(\alpha)=0\Leftrightarrow\partial_{v'}(\alpha)=0$. □

6. Étale motivic cohomology of DVRs

Recall that by the work of Kato [Kat82] and Izhboldin [Izh91, Corollary 6.5], for a field F of characteristic $p > 0$, we have isomorphisms

$$\mathrm{H}^n(F, \underline{\mathbb{Z}}/p^r(n)) \simeq \mathrm{K}^n_{p^r}(F)\,, \quad \mathrm{H}^{n+1}(F, \underline{\mathbb{Z}}/p^r(n)) \simeq \mathrm{H}^{n+1}_{p^r}(F)\,.$$

With this identification in mind, given a DVR (R, v) with fraction field $\mathbf{k}(R)$ and residue field $\mathbf{k}_v$, from equation (5.1) and Proposition 5.6, we can derive residue homomorphisms

$$\partial_v \colon \mathrm{H}^n(\mathbf{k}(R), \underline{\mathbb{Z}}/p^r(n)) \longrightarrow \mathrm{H}^{n-1}(\mathbf{k}_v, \underline{\mathbb{Z}}/p^r(n-1))\,,$$
$$\partial_v \colon \mathrm{H}^{n+1}(\mathbf{k}(R), \underline{\mathbb{Z}}/p^r(n))_{\mathrm{tm}} \longrightarrow \mathrm{H}^n(\mathbf{k}_v, \underline{\mathbb{Z}}/p^r(n-1))\,.$$

The main goal of this section is to show that for i and j in this range, the image of

$$\mathrm{H}^i(R, \underline{\mathbb{Z}}/p^r(j)) \longrightarrow \mathrm{H}^i(\mathbf{k}(R), \underline{\mathbb{Z}}/p^r(j))$$

coincides with the kernel of the residue map ∂_v.

6.1 The case $i = j = n$

We begin with the easier case of $\mathrm{H}^n(R, \underline{\mathbb{Z}}/p^r(n))$. Define

$$\mathcal{K}^\bullet(R) = \mathrm{Ker}(\partial_v) \colon \mathrm{K}^\bullet_{\mathrm{Mil}}(\mathbf{k}(R)) \longrightarrow \mathrm{K}^\bullet_{\mathrm{Mil}}(\mathbf{k}_v)\,,$$
$$\mathcal{K}^\bullet_{p^r}(R) = \mathrm{Ker}(\partial_v) \colon \mathrm{K}^\bullet_{\mathrm{Mil}}(\mathbf{k}(R))/p^r \longrightarrow \mathrm{K}^\bullet_{\mathrm{Mil}}(\mathbf{k}_v)/p^r\,,$$

where ∂_v is the usual residue in Milnor's K-theory.

Now let D be a Dedekind domain which is the semi-localization of a regular finite-type algebra over $\mathbf{k}$. We can extend the functor $\mathcal{K}^\bullet_{p^r}$ to D by defining

$$\mathcal{K}^\bullet(D) = \bigcap_v \mathrm{Ker}(\partial_v) \colon \mathrm{K}^\bullet_{\mathrm{Mil}}(\mathbf{k}(D)) \longrightarrow \mathrm{K}^\bullet_{\mathrm{Mil}}(\mathbf{k}_v)$$

and correspondingly

$$\mathcal{K}^\bullet_{p^r}(D) = \bigcap_v \mathrm{Ker}(\partial_v) \colon \mathrm{K}^\bullet_{\mathrm{Mil}}(\mathbf{k}(D))/p^r \longrightarrow \mathrm{K}^\bullet_{\mathrm{Mil}}(\mathbf{k}_v)/p^r\,,$$

where v runs among all valuations $\mathbf{k}(D)$ that define a maximal ideal of D, that is, the closed points of $\mathrm{Spec}(D)$.

First, we show that $\mathcal{K}^\bullet_{p^r}$ behaves well when we change the exponent r.

Corollary 6.1. *Let (R, v) be a DVR. There is a natural isomorphism*

$$\mathcal{K}^n(R)/p^r \simeq \mathcal{K}^n_{p^r}(R)\,,$$

and the sequence

$$0 \longrightarrow \mathcal{K}^\bullet_{p^{r-i}}(R) \xrightarrow{i_{r-i}} \mathcal{K}^\bullet_{p^r}(R) \longrightarrow \mathcal{K}^\bullet_{p^i}(R) \longrightarrow 0\,,$$

where the first map is given by $\{b_1, \ldots, b_n\} \mapsto p^i\{b_1, \ldots, b_n\}$, is exact.

Proof. There is an obvious map $\mathcal{K}^n(R)/p^r \to \mathcal{K}^n_{p^r}(R)$. It suffices to prove that it is surjective. If $\beta \in \mathrm{K}^n_{\mathrm{Mil}}(\mathbf{k}(R))$ represents an element $\beta \in \mathcal{K}^n_{p^r}(R)$, then $\partial_v \beta = p^r \beta'$ for some $\beta' \in \mathrm{K}^{n-1}_{\mathrm{Mil}}(\mathbf{k}_v)$. Pick an element $\gamma \in (R^*)^{\otimes n-1}$ which restricts to β' on the residue field; now the element $\beta - p^r\{\pi\} \cdot \beta'$ belongs to $\mathcal{K}^n(R)$ and maps to $\beta \in \mathcal{K}^n_{p^r}(R)$.

This shows moreover that the last map in the exact sequence is surjective. The only non-trivial thing left to show for the sequence to be exact is that the map i_{r-1} is injective. If $i_{r-1}\alpha = 0$, then in $\mathrm{K}^{\bullet}_{\mathrm{Mil}}(\mathbf{k}(R))$ we must have $p^i\alpha' = p^r\beta$ for some lifting α' of α and some β.

Therefore, we must have $p^i(\alpha - p^{r-i}\beta) = 0$. It then follows that ι_{r-1} is injective if and only if $\alpha - p^{r-i}\beta = 0$ in this case, that is, if there is no p-torsion in $\mathrm{K}^{\bullet}_{\mathrm{Mil}}(\mathbf{k}(R))$. This is true by [Izh91, Theorem A]. ☐

What follows shows that the functor we defined is equal to $\mathrm{H}^n(D, \underline{\mathbb{Z}}/p^r(n))$ for all such D, and in particular for a DVR (R, v).

Proposition 6.2. *Let D be as above. There is a natural isomorphism*

$$\mathrm{H}^n(D, \underline{\mathbb{Z}}/p^r(n)) \simeq \mathcal{K}^n_{p^r}(D)\,.$$

Proof. The case $r = 1$ is proven by Geisser and Levine in [GL00, Proposition 3.1].

In general, if the base field is perfect, the proposition is immediate by [GL00, Theorem 8.3]. To get around this hypothesis, we apply Quillen's method: there exists a subfield $\mathbf{k}_0$ of $\mathbf{k}$ that is finitely generated over $\mathbb{F}_p$ and such that $R = R' \otimes_{\mathbf{k}_0} \mathbf{k}$. As all functors in the statement commute with direct limits, we just have to prove the proposition for the fields $\mathbf{k}_0 \subseteq \mathbf{k}' \subseteq \mathbf{k}$ which are finitely generated over $\mathbb{F}_p$. For such a field we can always see $D_{\mathbf{k}'}$ as a partial localization around a subset of codimension 1 of a smooth variety over $\mathbb{F}_p$. Thus we have reduced to the case of $\mathbf{k}$ being perfect. ☐

6.2 The case $i = j + 1 = n + 1$

Now we move on to the groups $\mathrm{H}^{n+1}(R, \underline{\mathbb{Z}}/p^r(n))$. Note that we always have

$$\mathrm{H}^{n+1}(R, \underline{\mathbb{Z}}/p^r(n)) \subseteq \mathrm{H}^{n+1}(\mathbf{k}(R), \underline{\mathbb{Z}}/p^r(n))_{\mathrm{tm}}$$

as $\mathbf{k}\big(R^{\mathrm{sh}}\big) \subset \mathbf{k}(R)^{\mathrm{tm}}$, where R^{sh} is the strict Henselization of R. The following lemma and proposition deal with the case $r = 1$. Proposition 6.4 is stated without proof in [Tot22, Section 4]; Burt Totaro kindly explained a way to derive it to the authors.

Lemma 6.3. *Let (R, v) be a DVR. Then the following sequence is exact in the small étale site of R:*

$$0 \longrightarrow \mathrm{H}^n(R, \underline{\mathbb{Z}}/p(n)) \longrightarrow \Omega^n \xrightarrow{1-\Phi} \Omega^n/\mathrm{d}\Omega^{n-1} \longrightarrow 0\,,$$

where Φ is the inverse Cartier operator defined by

$$\Phi(a\,\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_1) = \big(a^p f_1^{p-1} \cdots f_n^{p-1}\big)\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_1\,.$$

Proof. This statement is a direct consequence of [GL00, Proposition 3.1] and [Mor19, Corollaries 4.1(ii) and 4.2(iii)] ☐

Proposition 6.4. *The image of $\mathrm{H}^{n+1}(R, \underline{\mathbb{Z}}/p(n))$ in $\mathrm{H}(\mathbf{k}(R), \underline{\mathbb{Z}}/p(n))_{\mathrm{tm}}$ is equal to the kernel of the residue map ∂_v.*

Proof. As part of the proof of [Tot22, Theorem 4.3], Totaro shows that the kernel of ∂_v is equal to the subgroup generated by the differentials $a(\mathrm{d}b_1/b_1) \wedge \cdots \wedge (\mathrm{d}b_n)/b_n$, where $a \in R$ and $b_1, \ldots, b_n \in R^*$. Following his notation, we will refer to this subgroup as $\mathrm{H}^{n+1}(\mathbf{k}(R))_{\mathrm{nr}}$.

We want to show that the image of $\mathrm{H}^{n+1}(R, \underline{\mathbb{Z}}/p(n))$ is exactly $\mathrm{H}^{n+1}(\mathbf{k}(R))_{\mathrm{nr}}$. For this, observe that the short exact sequence in Lemma 6.3 induces a long exact sequence in cohomology, which by [GL00, Theorem 8.3] and, as usual, Quillen's method, gives us

$$\Omega^n_R \xrightarrow{1-\Phi} \Omega^n_R/\mathrm{d}\Omega^{n-1}_R \longrightarrow \mathrm{H}^1\big(R, \Omega^n_{\log,R}\big) \longrightarrow \mathrm{H}^1\big(R, \Omega^n_R\big) = 0\,,$$

where the last term vanishes because Ω^n_R is an R-module. This gives us a presentation of the group $\mathrm{H}^{n+1}(R, \underline{\mathbb{Z}}/p(n))$ as a group of differentials which is compatible with the presentation of $\mathrm{H}^{n+1}(\mathbf{k}(R), \underline{\mathbb{Z}}/p(n))$. By looking at the composition

$$\Omega^n_R/d\Omega^{n-1}_R \longrightarrow \mathrm{H}^{n+1}(R, \underline{\mathbb{Z}}/p(n)) \longrightarrow \mathrm{H}^{n+1}(\mathbf{k}(R), \underline{\mathbb{Z}}/p(n))_{\mathrm{tm}}\,,$$

we deduce that the image of $\mathrm{H}^{n+1}(R, \underline{\mathbb{Z}}/p(n))$ is generated, as above, by differentials of one of the two types

$$a\,\mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_n \quad \text{or} \quad a\,\mathrm{d}t \wedge \mathrm{d}b_2 \wedge \cdots \wedge \mathrm{d}b_n\,.$$

Differentials of the first type are clearly unramified and surject to $\mathrm{H}^{n+1}(\mathbf{k}(R))_{\mathrm{nr}}$. Now observe that symbols of the second form can be rewritten as

$$a\,\mathrm{d}t \wedge \mathrm{d}b_2 \wedge \cdots \wedge \mathrm{d}b_n = a\,b_2 \cdots b_n\,t \frac{\mathrm{d}t}{t} \wedge \frac{\mathrm{d}b_2}{b_2} \wedge \cdots \wedge \frac{\mathrm{d}b_n}{b_n}\,,$$

whose ramification is by definition equal to

$$\overline{a\,b_2 \cdots b_n} \cdot \bar{t} \cdot \frac{\mathrm{d}\overline{b_2}}{\overline{b_2}} \wedge \cdots \wedge \frac{\mathrm{d}\overline{b_n}}{\overline{b_n}} = 0 \cdot \left(\frac{\mathrm{d}\overline{b_2}}{\overline{b_2}} \wedge \cdots \wedge \frac{\mathrm{d}\overline{b_n}}{\overline{b_n}} \right) = 0\,. \qquad \square$$

To generalize this result to $r > 1$, we start by proving that the conclusion of Lemma 5.1 works in this setting as well.

Proposition 6.5. *Let (R, v) be a DVR. We have an exact sequence*

$$0 \longrightarrow \mathrm{H}^{n+1}\big(R, \underline{\mathbb{Z}}/p^{r-1}(n)\big) \longrightarrow \mathrm{H}^{n+1}(R, \underline{\mathbb{Z}}/p^r(n)) \longrightarrow \mathrm{H}^{n+1}(R, \underline{\mathbb{Z}}/p(n)) \longrightarrow 0\,,$$

which is compatible with the exact sequence on quotient fields

$$0 \longrightarrow \mathrm{H}^{n+1}_{p^{r-1}}(\mathbf{k}(R)) \longrightarrow \mathrm{H}^{n+1}_{p^r}(\mathbf{k}(R)) \longrightarrow \mathrm{H}^{n+1}_{p}(\mathbf{k}(R)) \longrightarrow 0\,.$$

Proof. We begin by applying Quillen's method to reduce to $\mathbf{k}$ perfect (see the beginning of the proof of Proposition 6.2). Now, the exact sequence from Corollary 6.1 implies that we have an exact sequence of the sheafifications of $\mathcal{K}^\bullet_{p^r}$ on the site of $\mathrm{Spec}(R)$. By [GL00, Theorem 8.3] the sheafification of $\mathcal{K}^n_{p^r}$ is equal to the sheaf $W_r\Omega^n_{R,\log}$, and the cohomology of the latter is equal to $\mathrm{H}^i(R, \underline{\mathbb{Z}}/p^r(n))$, which we know to be zero except when $i = n, n+1$. Therefore, we obtain the following long exact sequence in cohomology:

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^n\big(R, \underline{\mathbb{Z}}/p^{r-1}(n)\big) \longrightarrow \mathrm{H}^n\big(R, \underline{\mathbb{Z}}/p^r(n)\big) \longrightarrow \mathrm{H}^n\big(R, \underline{\mathbb{Z}}/p(n)\big) \longrightarrow \mathrm{H}^{n+1}\big(R, \underline{\mathbb{Z}}/p^{r-1}(n)\big) \\ \longrightarrow \mathrm{H}^{n+1}\big(R, \underline{\mathbb{Z}}/p^r(n)\big) \longrightarrow \mathrm{H}^{n+1}\big(R, \underline{\mathbb{Z}}/p^1(n)\big) \longrightarrow \mathrm{H}^{n+2}\big(R, \underline{\mathbb{Z}}/p^{r-1}(n)\big)\,. \end{aligned}$$

By equation (2.4) we know that the last term is zero as it injects into

$$\mathrm{H}^{n+2}\big(\mathbf{k}(R), \underline{\mathbb{Z}}/p^{r-1}(n)\big) = 0\,.$$

Moreover, we know by Corollary 6.1 that the sequence splits into two five-terms exact sequences. So we have the exact sequence

$$0 \longrightarrow \mathrm{H}^{n+1}\big(R, \underline{\mathbb{Z}}/p^{r-1}(n)\big) \longrightarrow \mathrm{H}^{n+1}\big(R, \underline{\mathbb{Z}}/p^r(n)\big) \longrightarrow \mathrm{H}^{n+1}\big(R, \underline{\mathbb{Z}}/p(n)\big) \longrightarrow 0\,.$$

The compatibility with the exact sequence on Izhboldin's presentation of the étale cohomology of $\mathbf{k}(R)$ is immediate from [Izh91, Corollary 6.5]. $\square$

To understand ramification in relation to the inclusion $\mathrm{H}^{n+1}(R, \underline{\mathbb{Z}}/p^r(n)) \to \mathrm{H}^{n+1}_{p^r}(\mathbf{k}(R))$, we want to describe its image in terms of symbols. We start from the exact sequence of étale sheaves

$$0 \longrightarrow \mathbb{Z}/p^r \longrightarrow W_r(\mathbb{G}_{\mathrm{a}}) \xrightarrow{\phi - 1} W_r(\mathbb{G}_{\mathrm{a}}) \longrightarrow 0\,,$$

where $\phi[a_1,\dots,a_r] = \left[a_1^p,\dots,a_r^p\right]$. This induces a map

$$W_r(R)/(\phi-1) \longrightarrow \mathrm{H}^1(R,\mathbb{Z}/p^r)\,.$$

Combining this with the isomorphism $\mathcal{K}^\bullet_{p^r}(R) = \mathrm{H}^n(R,\underline{\mathbb{Z}}/p^r(n))$ and taking the cup product, we get a map

$$W_r(R)\otimes\mathcal{K}^n_{p^r}(R) \longrightarrow \mathrm{H}^{n+1}(R,\underline{\mathbb{Z}}/p^r(n))\,.$$

Lemma 6.6. *The following map is surjective for all r, n:*

$$W_r(R)\otimes\mathcal{K}^n_{p^r}(R) \longrightarrow \mathrm{H}^{n+1}(R,\underline{\mathbb{Z}}/p^r(n))\,.$$

Proof. The case $r = 1$ is a consequence of Proposition 6.4. Now note that due to the equality $R = W_1(R) = W_r(R)/W_{r-1}(R)$, we have a natural exact sequence

$$W_{r-1}(R)\otimes\mathcal{K}_{p^{r-1}}(R) \longrightarrow W_r(R)\otimes\mathcal{K}_{p^r}(R) \longrightarrow R\otimes\mathcal{K}_p(R) \longrightarrow 0\,,$$

which commutes with the morphisms above. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \mathrm{H}^{n+1}(R,\underline{\mathbb{Z}}/p^{r-1}(n)) & \to & \mathrm{H}^{n+1}(R,\underline{\mathbb{Z}}/p^r(n)) & \to & \mathrm{H}^{n+1}(R,\underline{\mathbb{Z}}/p(n)) & \to 0 \\
& \uparrow & & \uparrow & & \uparrow & \\
& W_{r-1}(R)\otimes\mathcal{K}_{p^{r-1}}(R) & \to & W_r(R)\otimes\mathcal{K}_{p^r}(R) & \longrightarrow & R\otimes\mathcal{K}_p(R) & \longrightarrow 0\,.
\end{array}$$

We can assume by induction on r that the left and right vertical maps are surjective. Then a standard diagram chase proves the surjectivity of the central vertical map. □

Proposition 6.7. *Let (R,v) be a DVR. We have an exact sequence*

$$0 \longrightarrow \mathrm{H}^{n+1}(R,\underline{\mathbb{Z}}/p^r(n)) \longrightarrow \mathrm{H}^{n+1}_{p^r}(\mathbf{k}(R))_{\mathrm{tm}} \xrightarrow{\partial_v} \mathrm{H}^n_{p^r}(\mathbf{k}_v) \longrightarrow 0\,.$$

Proof. The statement is true for $r = 1$ by Proposition 6.4. Now consider the commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{H}^n_{p^{r-1}}(\mathbf{k}_v) & \longrightarrow & \mathrm{H}^n_{p^r}(\mathbf{k}_v) & \longrightarrow & \mathrm{H}^n_{p^r}(\mathbf{k}_v) & \longrightarrow 0 \\
& \partial_v\uparrow & & \partial_v\uparrow & & \partial_v\uparrow & \\
0 \longrightarrow & \mathrm{H}^{n+1}_{p^{r-1}}(\mathbf{k}(R))_{\mathrm{tm}} & \xrightarrow{\iota_{r-1}} & \mathrm{H}^{n+1}_{p^r}(\mathbf{k}(R))_{\mathrm{tm}} & \xrightarrow{\pi_1} & \mathrm{H}^{n+1}_{p}(\mathbf{k}(R))_{\mathrm{tm}} & \\
& \uparrow & & \uparrow & & \uparrow & \\
0 \to & \mathrm{H}^{n+1}(R,\underline{\mathbb{Z}}/p^{r-1}(n)) & \to & \mathrm{H}^{n+1}(R,\underline{\mathbb{Z}}/p^r(n)) & \to & \mathrm{H}^{n+1}(R,\underline{\mathbb{Z}}/p(n)) & \to 0\,.
\end{array}$$

Each row is exact, the lower vertical arrows are injective, and the upper vertical arrows are surjective. We want to show that all columns are exact as well.

First note that by Lemma 6.6 each element of $\mathrm{H}^{n+1}_{p^r}(\mathbf{k}(R))$ coming from $\mathrm{H}^{n+1}(R,\underline{\mathbb{Z}}/p^r(n))$ is of the form $[a_1,\dots,a_r,b_1,\dots,b_n\}$, where $a_1,\dots,a_r\in R$ and $\{b_1,\dots,b_n\}\in\mathcal{K}^n_{p^r}(R)$. These elements are unramified by construction, so all we need to prove is the exactness of

$$\mathrm{H}^{n+1}(R,\underline{\mathbb{Z}}/p^r(n)) \longrightarrow \mathrm{Ker}(\partial_v) \longrightarrow 0\,.$$

By induction we can assume that the left and right columns are exact, and we have to show that the central one is. Pick a $\gamma\in\mathrm{H}^{n+1}_{p^r}(\mathbf{k}(R))_{\mathrm{tm}}$ such that $\partial_v(\gamma) = 0$. As $\partial_v(\pi_1(\gamma)) = 0$, there is a $\gamma'\in\mathrm{H}^{n+1}(R,\underline{\mathbb{Z}}/p^r(n))$ such that $\pi_1(\gamma-\gamma') = 0$. Then $\gamma-\gamma' = \iota_{r-1}(\beta)$ with $\partial_v(\beta) = 0$, which implies that β is in the image of $\mathrm{H}^{n+1}(R,\underline{\mathbb{Z}}/p^{r-1}(n))$, allowing us to conclude by the commutativity of the diagram. □

With this it is easy to check that the two notions of ramification coming from equation (2.3) and Izhboldin's description are equivalent.

Proposition 6.8. *Let (R, v) be a DVR with fraction field F, and let y be the closed point of* $\mathrm{Spec}(R)$. *An element $\alpha \in \mathrm{H}^{n+1}_{p^r}(F)$ is unramified at y in the sense of Lemma 3.11 if and only if it is unramified at v_y as above.*

Proof. Comparing Proposition 5.6 and equation (2.4), we immediately see that both conditions amount to the element coming from the cohomology of $\mathrm{Spec}(R)$. □

Corollary 6.9. *Let (R, v) be a DVR with a map $\mathrm{Spec}(R) \to \mathcal{X}$. Then given any invariant $\alpha \in \mathrm{Inv}^\bullet(\mathcal{X}, \mathrm{H}_{p^r})$, the element $\alpha(\mathbf{k}(R))$ is tamely ramified and unramified at v.*

Given a scheme X smooth over $\mathbf{k}$, the map $\alpha \mapsto \alpha(\mathbf{k}(X))$ is an isomorphism between the cohomological invariants of X and the subgroup of $\mathrm{H}^\bullet_{p^r}(\mathbf{k}(X))_{\mathrm{tm}}$ of elements unramified at all $x \in X^{(1)}$.

7. Some computations of cohomological invariants

In this section we come back to cohomological invariants and present some explicit computations that will be useful for our study of the invariants of $\mathcal{M}_{1,1}$.

To start, we recall Izhboldin's description of $\mathrm{H}^{n+1}_{p^r}(\mathbf{k}(t))$, which we state here using Totaro's notation.

Theorem 7.1 ([Izh96, Theorems 4.5 and 6.10]). *Let S be the set of closed points of $\mathbb{P}^1_{\mathbf{k}}$. Write v_y for the valuation corresponding to the point y, and $\mathbf{k}(t)_{v_y}$ for the completion of $\mathbf{k}(t)$ at v_y. We have an exact sequence*

$$\mathrm{H}^{n+1}_{p^r}(\mathbf{k}(t)) \longrightarrow \bigoplus_{y \in S} \widetilde{\mathrm{H}}^\bullet_{p^r}(\mathbf{k}(t)_{v_y}) \longrightarrow 0\,.$$

The kernel of this homomorphism, which we call $\mathrm{H}^{n+1}_{p^r}(\mathbf{k}(t))_{\mathrm{tm}/\mathbb{P}^1}$, fits in the following exact sequence:

$$0 \longrightarrow \mathrm{H}^{n+1}_{p^r}(\mathbf{k}) \longrightarrow \mathrm{H}^{n+1}_{p^r}(\mathbf{k}(t))_{\mathrm{tm}/\mathbb{P}^1} \longrightarrow \bigoplus_{S \setminus \infty} \mathrm{H}^{n+1}_{p^r}(\mathbf{k}(y)) \longrightarrow 0\,.$$

Remark 7.2. Note that by [Tot22, Theorem 4.3], given a discrete valuation field (F, v), we have $\widetilde{\mathrm{H}}^\bullet_p(F) = \widetilde{\mathrm{H}}^\bullet_p(F_v)$, where F_v is the completion of F at v, simplifying the formula.

As an immediate corollary we get the cohomological invariants of $\mathbb{A}^1$ and $\mathbb{A}^1 \setminus \{0\}$.

Corollary 7.3. *We have*

$$0 \longrightarrow \mathrm{H}^\bullet_{p^r}(\mathbf{k}) \longrightarrow \mathrm{Inv}^\bullet\big(\mathbb{A}^1, \mathrm{H}_{p^r}\big) \longrightarrow \widetilde{\mathrm{H}}^\bullet_{p^r}(\mathbf{k}((1/t))) \longrightarrow 0\,,$$

$$0 \longrightarrow \mathrm{H}^\bullet_{p^r}(\mathbf{k}) \oplus \mathrm{H}^{\bullet-1}_{p^r}(\mathbf{k}) \longrightarrow \mathrm{Inv}^\bullet\big(\mathbb{A}^1 \setminus \{0\}, \mathrm{H}_{p^r}\big) \longrightarrow \widetilde{\mathrm{H}}^\bullet_{p^r}((\mathbf{k}(t))) \oplus \widetilde{\mathrm{H}}^\bullet_{p^r}(\mathbf{k}((1/t))) \longrightarrow 0\,.$$

Here the left-hand map sends a pair $[a, b_1, \dots, b_n\}, [a', b'_2, \dots, b'_n\}$ of degree $n+1$ to the element $[a, b_1, \dots, b_n\} + [a', t, b'_2, \dots, b'_n\}$, and its image is contained in $\mathrm{H}^\bullet_{p^r}(k(t))_{\mathrm{tm}/\mathbb{P}^1}$. The component on the right comes from the wild part of $\mathrm{H}^\bullet_{p^r}(k(t))$.

The cohomological invariants of $\mathrm{B}\mathbb{Z}/p$ with coefficients in H_p were computed by Totaro [Tot22, Proposition 8.1]. Here we give a different proof which we think is of independent interest, and then we will extend the description to coefficients in H_{p^r}.

Theorem 7.4 (Totaro). *We have*

$$\mathrm{Inv}^\bullet(\mathrm{B}\mathbb{Z}/p, \mathrm{H}_p) = \mathrm{H}_p^\bullet(\mathbf{k}) \oplus \mathrm{K}_p^{\bullet-1}(\mathbf{k}) \cdot \alpha\,,$$

where $\alpha \in \mathrm{H}^1(\mathrm{B}\mathbb{Z}/p, \mathbb{Z}/p)$ *is the identity element; that is, given a map* $\mathrm{Spec}(F) \to \mathrm{B}\mathbb{Z}/p$ *induced by a torsor* $\pi\colon E \to \mathrm{Spec}(F)$*, we have* $\alpha(F) = \pi \in H^1(F, \mathbb{Z}/p)$

Proof. Consider the map $\mathbb{G}_\mathrm{a} \to \mathbb{G}_\mathrm{a}$ given by $t \mapsto t^p - t$. It induces an additive action of $\mathbb{G}_\mathrm{a}$ on $\mathbb{A}^1$ which is transitive and has stabilizers equal to $\mathbb{Z}/p$, so $\left[\mathbb{A}^1/\mathbb{G}_\mathrm{a}\right] = \mathrm{B}\mathbb{Z}/p$. We have

$$\mathbb{A}^1 \times_{\mathrm{B}\mathbb{Z}/p} \mathbb{A}^1 = \mathbb{A}^1 \times \mathbb{G}_\mathrm{a} = \mathrm{Spec}(\mathbf{k}[t,s])\,,$$

where the two projections are the first projection pr_1 and the multiplication map m.

The map $\mathbb{A}^1 \to \mathrm{B}\mathbb{Z}/p$ is smooth-Nisnevich, so we can conclude that the cohomological invariants of $\mathrm{B}\mathbb{Z}/p$ are the invariants of $\mathbb{A}^1$ such that $\mathrm{pr}_1^*\alpha = \mathrm{m}^*\alpha$.

We claim that the subgroup of elements that descend to invariants of $\mathrm{B}\mathbb{Z}/p$ is generated by elements coming from the base field and elements of the form $t(\mathrm{d}b_1/b_1) \wedge \cdots \wedge (\mathrm{d}b_n/b_n)$. The elements coming from the base field form a direct summand in $\mathrm{Inv}^\bullet\left(\mathbb{A}^1, \mathrm{H}_p\right)$, and they trivially glue, so let α be an element of $\mathrm{Inv}^n\left(\mathbb{A}^1, \mathrm{H}_p\right)$ that restricts to zero at $t = 0$. These elements form a subgroup isomorphic to $\mathrm{Inv}^n\left(\mathbb{A}^1, \mathrm{H}_p\right)/\mathrm{H}_p^{n+1}(\mathbf{k})$.

By Corollary 7.3 and Lemma 5.10, we can write such an element as

$$\alpha = \sum_i t^i\varphi_i + t^i\frac{\mathrm{d}t}{t} \wedge \varphi_i'\,, \quad \varphi_i \in \Omega_k^n,\ \varphi_i' \in \Omega_\mathbf{k}^{n-1}\,, \tag{7.1}$$

where $\varphi_i' = 0$ if $p \nmid i$. We have

$$\mathrm{pr}_1^*\big(t^i\varphi_i\big) = t^i\varphi_i\,, \quad \mathrm{pr}_1^*\left(t^i\frac{\mathrm{d}t}{t} \wedge \varphi_i'\right) = t^i\frac{\mathrm{d}t}{t} \wedge \varphi_i'\,,$$
$$\mathrm{m}^*\big(t^i\varphi_i\big) = (t+s^p-s)^i\varphi_i\,, \quad \mathrm{m}^*\left(t^i\frac{\mathrm{d}t}{t} \wedge \varphi_i'\right) = (t+s^p-s)^i\frac{\mathrm{d}(t+s^p-s)}{t+s^p-s} \wedge \varphi_i'\,.$$

We first show that for an element to descend, we must have $i = 1$. Observe that if α glues, then the pullback of $\mathrm{pr}_1^*\alpha - \mathrm{m}^*\alpha$ to the copy of $\mathbb{G}_\mathrm{a}$ given by $t = 0$ must be zero; that is,

$$\alpha' = \sum_i (s^p-s)^i(\varphi_i) - s(s^p-s)^{i-1}\frac{\mathrm{d}s}{s} \wedge \varphi_i' = 0\,.$$

Pick the highest i such that $\alpha \in \mathrm{U}_i/\mathrm{U}_{i-1}$, which we can assume to be the highest i appearing in equation (7.1). If i is divisible by p, we can assume that at least one among φ_i and φ_i' is not closed; otherwise we could apply Lemma 5.10 to replace the term $t^i(\varphi_i + (\mathrm{d}t/t) \wedge \varphi_i')$ with something of the form $t^{i/p}\varphi_i''$, and this would imply that $\alpha \in \mathrm{U}_{i-1}$.

If φ_i is not closed, then the element $s^{pi}\varphi_i$ is the highest-degree term in the sum, and by Theorem 5.7 this implies that the equivalence class of α' in $\mathrm{U}_{pi}/\mathrm{U}_{pi-1}$ is non-zero.

If φ_i is closed, we can assume that it is actually zero: indeed, as before, we can apply Lemma 5.10 to rewrite α in such a way that the highest power of t appears in the term $t^i(\mathrm{d}t/t)\wedge\varphi_i'$ with φ_i' not closed. Therefore, the element α' is equivalent to $-s^{p(i-1)+1}(\mathrm{d}s/s) \wedge \varphi_i'$ in $\mathrm{U}_i/\mathrm{U}_{i-1}$. Observe that in $\mathrm{H}_p^{n+1}(k(s))$ we have

$$0 = \mathrm{d}(s^{p(i-1)+1}\varphi_i') = s^{p(i-1)+1}\mathrm{d}(\varphi_i') + (p(i-1)+1)s^{p(i-1)}\mathrm{d}s \wedge \varphi_i'\,,$$

which immediately implies

$$-s^{p(i-1)+1}\frac{\mathrm{d}s}{s} \wedge \varphi_i' = (p(i-1)+1)^{-1}s^{p(i-1)+1}\mathrm{d}(\varphi_i')\,,$$

and the term on the right is non-zero in $\mathrm{U}_{p(i-1)+1}/\mathrm{U}_{p(i-1)}$: this follows from Theorem 5.7 as $\mathrm{d}\varphi_i'$ is by hypothesis non-zero in $\Omega^n_{\mathbf{k}}$ and $p(i-1)+1$ is not divisible by p.

If $i>1$ is not divisible by p (hence $\varphi_i'=0$), we consider two cases. If φ_i is not closed, we can conclude exactly as above. If φ_i is closed, then the term $s^{pi}(\varphi_i)$ is equivalent to some element of degree i, and the next highest-degree term is $-is^{(i-1)p+1}(\varphi_i)$, which is non-zero. As i is not divisible by p, we have that $-is^{(i-1)p+1}(\varphi_i)$ is non-zero in $\mathrm{U}_{(i-1)p+1}/\mathrm{U}_{(i-1)p}$.

We are left with the case $i=1$. In this case $\mathrm{pr}_1^*\alpha-\mathrm{m}^*\alpha=(s^p-s)(\varphi)$. If φ is not closed, then clearly this element cannot be zero. Now assume that φ is closed: by Cartier's theorem we can write $\varphi=\big(\sum_j a_j^p(\psi_j)\big)+\varphi'$, where the forms ψ_j are logarithmic and φ' is exact. Then $(s^p-s)(\varphi)=s\big(\big(\sum_j(a_j-a_j^p)\psi_j\big)+\varphi'\big)$. If the element $\big(\sum_j(a_j-a_j^p)\psi_j\big)+\varphi'$ is equal to zero, then in particular it is equal to zero in $\Omega^n_{\mathbf{k}}/\mathrm{d}\Omega^{n-1}_{\mathbf{k}}$. But this implies that $\sum_j(a_j-a_j^p)\psi_j=\mathcal{P}\big(\sum_j -a_j\psi_j\big)=0$, which implies that $\sum_j -a_j\psi_j$ is logarithmic, and in particular $\sum_j(a_j-a_j^p)\psi_j=\mathcal{P}\big(\sum_j -a_j\psi_j\big)=0$ in $\Omega^n_{\mathbf{k}}$ as well. Then $\varphi'=0$ necessarily.

Finally, note that if $\mathcal{P}\big(\sum_j a_j\psi_j\big)$ is zero, then the same must be true for $\sum_j a_j^p\psi_j$: let $\mathcal{F}$ be the additive morphism defined by $a\varphi\mapsto a^p\varphi$. Then $\mathcal{P}$ and $\mathcal{F}$ commute, so

$$\mathcal{P}\Big(\sum_j a_j^p\psi_j\Big)=\mathcal{P}\Big(\mathcal{F}\Big(\sum_j a_j\psi_j\Big)\Big)=\mathcal{F}\Big(\mathcal{P}\Big(\sum_j a_j\psi_j\Big)\Big)=0\,,$$

showing that if $t(\varphi)$ glues, we must have $\varphi\in\Omega^n_{\mathbf{k},\log}=\mathrm{K}^n_p(\mathbf{k})$, as claimed.

This shows that

$$\mathrm{Inv}^\bullet(\mathrm{B}\mathbb{Z}/p,\mathrm{H}_p)\simeq\mathrm{H}^\bullet_p(\mathbf{k})\oplus\mathrm{K}^{\bullet-1}_p(\mathbf{k})\cdot t\,,$$

and we only have to identify the element t with the pullback of $\alpha\in\mathrm{H}^1(\mathrm{B}\mathbb{Z}/p,\mathbb{Z}/p)$. To do this, consider the $\mathbb{Z}/p$-torsor $E\to\mathbb{A}^1$ induced by taking a root of x^p-x-t. It is immediate to check that the induced map $\mathbb{A}^1\to\mathrm{B}\mathbb{Z}/p$ is the quotient we considered above. Now, the pullback of α to the base of a torsor $E\to\mathrm{Spec}(R)$ defined by adding a root of x^p-x-r is exactly the class $[r]\in\mathrm{H}^1(\mathrm{Spec}(R),\mathbb{Z}/p)$, allowing us to conclude. □

Corollary 7.5. *We have*

$$\mathrm{Inv}^\bullet(\mathrm{B}\mathbb{Z}/p,\mathrm{H}_{p^r})=\mathrm{H}^\bullet_{p^r}(k)\oplus\alpha\cdot\mathrm{K}^{\bullet-1}_p(\mathbf{k})\,,$$

where we identify $\mathrm{K}^\bullet_p=\mathrm{K}^\bullet_{p^r}(k)/p\mathrm{K}^\bullet_{p^r}(k)$.

Proof. Write E_{triv} for the trivial torsor $\mathbb{Z}/p\times\mathrm{Spec}(\mathbf{k})\to\mathrm{Spec}(\mathbf{k})$. An $\alpha\in\mathrm{Inv}^\bullet(\mathrm{B}\mathbb{Z}/p,\mathrm{H}_{p^r})$ is *normalized* if $\alpha(E_{\mathrm{triv}})=0$; given any invariant α, the invariant $\alpha-\alpha(E_{\mathrm{triv}})$ is normalized, where the second element is seen as a constant invariant, so the group of normalized invariants of $\mathrm{B}\mathbb{Z}/p$ is equal to $\mathrm{Inv}^\bullet(\mathrm{B}\mathbb{Z}/p,\mathrm{H}_{p^r})/\mathrm{H}^\bullet_{p^r}(\mathbf{k})$.

The claim is equivalent to showing that all normalized invariants of $\mathrm{B}\mathbb{Z}/p$ with mod p^r coefficients come from normalized invariants with mod p coefficients. The map $\iota_1\colon\mathrm{H}_p\to\mathrm{H}_{p^r}$ induces by composition a map

$$\mathrm{Inv}^\bullet(\mathrm{B}\mathbb{Z}/p,\mathrm{H}^\bullet_p)/\mathrm{H}^\bullet_p(\mathbf{k})\longrightarrow\mathrm{Inv}^\bullet(\mathrm{B}\mathbb{Z}/p,\mathrm{H}^\bullet_{p^r})/\mathrm{H}^\bullet_{p^r}(\mathbf{k})\,.$$

Now, observe that if α is a normalized cohomological invariant of $\mathrm{B}\mathbb{Z}/p$ with any coefficients, then for any $\mathbb{Z}/p$-torsor $E\to\mathrm{Spec}(F)$ the pullback of α to E must be trivial as $E\times_K E\xrightarrow{\pi}E$ is a trivial torsor, and thus by a standard transfer argument, we must have $0=\pi_*\pi^*\alpha(F)=p\alpha(F)$; that is, α is of p-torsion. By Proposition 5.2 we know that the p-torsion of $\mathrm{H}^\bullet_{p^r}(F)$ is exactly

$\mathrm{H}^\bullet_p(F)$: using this identification we get a map

$$\mathrm{Inv}^\bullet(\mathrm{B}\mathbb{Z}/p, \mathrm{H}^\bullet_{p^r})/\mathrm{H}^\bullet_{p^r}(\mathbf{k}) \longrightarrow \mathrm{Inv}^\bullet(\mathrm{B}\mathbb{Z}/p, \mathrm{H}^\bullet_p)/\mathrm{H}^\bullet_p(\mathbf{k})\,.$$

It is immediate that the two maps are inverse to each other, proving our claim. □

Now we consider the relative case, following the same idea as [Tot22, Proposition 8.3].

Corollary 7.6. *Let X be a smooth scheme. Then*

$$\mathrm{Inv}^\bullet(X \times \mathrm{B}\mathbb{Z}/p, \mathrm{H}_{p^r}) = \mathrm{Inv}^\bullet(X, \mathrm{H}_{p^r}) \oplus \alpha \cdot \mathrm{Inv}^\bullet(X, \mathrm{K}_p)\,.$$

Proof. Let β be a cohomological invariant of $[X/(\mathbb{Z}/p)]$. Given a point $x\colon \mathrm{Spec}(F) \to X$, restricting β to the fiber of x, we get an invariant of $\mathrm{B}\mathbb{Z}/p \times \mathrm{Spec}(F)$. In particular, we get unique elements $a_\beta(x) \in \mathrm{H}^\bullet_{p^r}(F), b_\beta(x) \in \mathrm{K}^\bullet_p(F)$. We claim that the associations $x \mapsto a_\beta(x)$, $x \mapsto b_\beta(x)$ are cohomological invariants of X and that $\beta = a_\beta + b_\beta \cdot \alpha$.

First, it is easy to see that a_β is a cohomological invariant of X, as it is equal to the pullback of β through the map $X \to X \times \mathrm{B}\mathbb{Z}/p\mathbb{Z}$ induced by $\mathrm{Spec}(\mathbf{k}) \to \mathrm{B}\mathbb{Z}/p\mathbb{Z}$. Now consider b_β; we want to show that it is functorial and that it satisfies the continuity condition. Functoriality is an immediate consequence of uniqueness.

Given an Henselian DVR (R, v) with a map $\mathrm{Spec}(R) \to X$, we get a map $\mathrm{Spec}(R) \times \mathrm{B}\mathbb{Z}/p \to X \times \mathrm{B}\mathbb{Z}/p$. Taking the usual cover $\mathbb{A}^1 \to \mathrm{B}\mathbb{Z}/p$, we get $\mathrm{Spec}(R) \times \mathbb{A}^1 = \mathrm{Spec}(R[t]) \to X \times \mathrm{B}\mathbb{Z}/p$, and the pullback of β to the generic point is $a_\beta(\mathbf{k}(R)) + b_\beta(\mathbf{k}(R)) \cdot \{t\}$. On the other hand, we have $\beta(\mathbf{k}_v(t)) = a_\beta(\mathbf{k}_v) + b_\beta(\mathbf{k}_v) \cdot \{t\}$ and

$$j(a_\beta(\mathbf{k}_v) + b_\beta(\mathbf{k}_v) \cdot \{t\}) = a_\beta(\mathbf{k}(R)) + b_\beta(\mathbf{k}(R)) \cdot \{t\}\,.$$

Now, as j is a composition of cohomology pullbacks and their inverses, it respects both the group structure and cup product, which implies that

$$j(a_\beta(\mathbf{k}_v) + b_\beta(\mathbf{k}_v) \cdot \{t\}) = j(a_\beta(\mathbf{k})) + j(b_\beta(\mathbf{k}_v)) \cdot t\,,$$

which in turn gives us $j(b_\beta(\mathbf{k}_v)) = b_\beta(\mathbf{k}(R))$, showing that b is a cohomological invariant of X as well.

Finally, consider the inclusion of the subgroup

$$\mathrm{Inv}^\bullet(X, \mathrm{H}_{p^r}) \oplus \mathrm{Inv}^\bullet(X, \mathrm{K}_p) \cdot \alpha \subseteq \mathrm{Inv}^\bullet([X/(\mathbb{Z}/p)], \mathrm{H}_{p^r})\,.$$

Given an invariant β, the element $\beta - a_\beta - b_\beta \cdot \alpha$ is by construction zero at all points, concluding our proof. □

As one would expect, the invariants of $X \times \mathrm{B}\mathbb{Z}/q$ with coefficients in $\mathrm{H}^\bullet_{p^r}$, for q a prime different from p, are only those coming from X.

Lemma 7.7. *Let X be a smooth scheme over a field $\mathbf{k}$ of characteristic p, and let $q \neq p$ be a prime integer. Then the pullback along the projection $X \times \mathrm{B}\mathbb{Z}/q \to X$ induces an isomorphism*

$$\mathrm{Inv}^\bullet(X, \mathrm{H}_{p^r}) \simeq \mathrm{Inv}^\bullet(X \times \mathrm{B}\mathbb{Z}/q, \mathrm{H}_{p^r})\,.$$

Proof. We can regard $X \times \mathrm{B}\mathbb{Z}/q$ as the quotient stack $[X/(\mathbb{Z}/q)]$, where $\mathbb{Z}/q$ acts trivially on X, so that we have a quotient map $\pi\colon X \to X \times \mathrm{B}\mathbb{Z}/q$. Note that the composition $X \to X \times \mathrm{B}\mathbb{Z}/q \xrightarrow{\mathrm{pr}_1} X$ is equal to the identity. This implies that π^* is a section of $\mathrm{pr}_1^*\colon \mathrm{Inv}^\bullet(X, \mathrm{H}_{p^r}) \to \mathrm{Inv}^\bullet(X \times \mathrm{B}\mathbb{Z}/q, \mathrm{H}_{p^r})$; hence it induces a decomposition

$$\mathrm{Inv}^\bullet(X \times \mathrm{B}\mathbb{Z}/q, \mathrm{H}_{p^r}) \simeq \mathrm{Inv}^\bullet(X, \mathrm{H}_{p^r}) \oplus \ker(\pi^*)\,.$$

For every field $F \supset \mathbf{k}$ and every morphism $f\colon \operatorname{Spec}(F) \to X \times \mathrm{B}\mathbb{Z}/q$, we can form the diagram

$$\begin{array}{ccc} E & \xrightarrow{f'} & X \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec}(F) & \xrightarrow{f} & X \times \mathrm{B}\mathbb{Z}/q \\ & \searrow^{g} & \downarrow \\ & & X\,. \end{array}$$

In particular, $E \to \operatorname{Spec}(F)$ is a $\mathbb{Z}/q$-torsor. Let $\beta = \mathrm{pr}_1^* \beta_1 + \beta_0$ be an invariant of $X \times \mathrm{B}\mathbb{Z}/q$, with $\pi^*\beta_0 = 0$. We claim that $f^*\beta = g^*\beta_1$; this would imply that the pullback morphism pr_1^* induces an isomorphism of cohomological invariants. We have

$$\begin{aligned} qf^*\beta &= \pi'_*\pi'^* f^*\beta = \pi'_* f'^* \pi^*(\mathrm{pr}_1^* \beta_1 + \beta_0) \\ &= \pi'_* f'^*(\beta_1) = \pi'_*\pi'^* g^*\beta_1 = qg^*\beta_1\,. \end{aligned}$$

As q is invertible in $\mathrm{H}^\bullet_{p^r}(F)$, we get the claimed equality. □

Lemma 7.8. *Let X be a smooth scheme over a field $\mathbf{k}$ of characteristic $p > 0$, and let G be a smooth affine group over $\mathbf{k}$. Then* $\operatorname{Inv}^\bullet(X \times \mathrm{B}G, \mathrm{K}_{p^r}) \simeq \operatorname{Inv}^\bullet(X, \mathrm{K}_{p^r})$.

Proof. First observe that $X \times \mathrm{B}G \simeq [X/G]$, where the group action is trivial. Let $\pi\colon X \to X \times \mathrm{B}G$ be the quotient map; the same argument as used in the proof of Lemma 7.7 gives us a decomposition

$$\operatorname{Inv}^\bullet(X \times \mathrm{B}G, \mathrm{K}_{p^r}) \simeq \operatorname{Inv}^\bullet(X, \mathrm{K}_{p^r}) \oplus \ker(\pi^*)\,.$$

Let α be an invariant in $\ker(\pi^*)$. For every field F with a map $f\colon \operatorname{Spec}(F) \to X \times \mathrm{B}G$, we claim that $f^*\alpha = 0$ in $\mathrm{K}^\bullet_{p^r}(F)$. If so, this implies that $\ker(\pi^*) = 0$, and we are done.

To prove the claim, observe that if the induced G-torsor $X_F \to F$ is trivial, then we have a factorization

$$\begin{array}{ccc} & \nearrow^{g} & X \\ & & \downarrow{\scriptstyle \pi} \\ \operatorname{Spec}(F) & \xrightarrow{f} & X \times \mathrm{B}G\,, \end{array}$$

hence $f^*\alpha = g^*\pi^*\alpha = 0$. If $X_F \to \operatorname{Spec}(F)$ is not trivial, let F^s be the separable closure of F, and denote by $h\colon \operatorname{Spec}(F^s) \to \operatorname{Spec}(F)$ the induced morphism. Then $X_{F^s} \to \operatorname{Spec}(F^s)$ is a trivial G-torsor because G is smooth and affine; thus $h^*f^*\alpha = 0$. To conclude, observe that h^* is injective because it coincides with the injective homomorphism $W_r\Omega^\bullet_{F,\log} \to W_r\Omega^\bullet_{F^s,\log}$. □

8. Cohomological invariants of $\mathcal{M}_{1,1}$ in positive characteristic

Let $\mathcal{M}_{1,1}$ be the stack of elliptic curves over a base field $\mathbf{k}$ of characteristic $p > 0$. In this section we compute the cohomological invariants of $\mathcal{M}_{1,1}$ with coefficients in H_{p^r} and K_{p^r}. Our main results for coefficients in H_{p^r} can be summarized as follows.

Theorem 8.1. *We have*

$$\mathrm{Inv}^\bullet(\mathcal{M}_{1,1},\mathrm{H}_{p^r}) \simeq \begin{cases} \mathrm{Inv}^\bullet(\mathbb{A}^1,\mathrm{H}_{2^r}) \oplus \mathrm{J}^{\bullet-1}_{2^r}(\mathbf{k}) & \text{if } p=2\,,\\ \mathrm{Inv}^\bullet(\mathbb{A}^1,\mathrm{H}_{3^r}) \oplus \mathrm{H}^{\bullet-1}_{3}(\mathbf{k})\cdot\{\Delta\} & \text{if } p=3\,,\\ \mathrm{Inv}^\bullet(\mathbb{A}^1,\mathrm{H}_{p^r}) & \text{if } p>3\,. \end{cases}$$

The group $\mathrm{J}^\bullet_{2^r}(\mathbf{k})$ is defined in Remark 8.10 (see also the paragraph above the remark), and $\{\Delta\}$ is the invariant that sends an elliptic curve $C \to \mathrm{Spec}(F)$ to its discriminant, regarded as an element in $\mathrm{K}^1_{p^r}(F)$ (for those p for which this quantity is well defined).

The computation is divided into three cases; namely, the case $p=2$ is discussed in Section 8.2, the case $p=3$ in Section 8.3, and finally the case $p>3$ in Section 8.4. The case $p=2$ is the most complicated to deal with, and the most interesting.

Concerning the invariants with coefficients in K_{p^r}, we prove the following.

Theorem 8.2. *We have*

$$\mathrm{Inv}^\bullet(\mathcal{M}_{1,1},\mathrm{K}_{p^r}) \simeq \begin{cases} \mathrm{K}^\bullet_2(\mathbf{k}) \oplus \mathrm{K}^{\bullet-1}_2\cdot\{\Delta\} & \text{if } p=2\,,\ r=1\,,\\ \mathrm{K}^\bullet_{2^r}(\mathbf{k}) \oplus \mathrm{K}^{\bullet-1}_4\cdot\{\Delta\} & \text{if } p=2\,,\ r>1\,,\\ \mathrm{K}^\bullet_{3^r}(\mathbf{k}) \oplus \mathrm{K}^{\bullet-1}_3\cdot\{\Delta\} & \text{if } p=3\,,\\ \mathrm{K}^\bullet_{p^r}(\mathbf{k}) & \text{if } p>3\,. \end{cases}$$

This computation is contained in Section 8.5 and is much easier than the previous ones.

8.1 Setup

Here we recall some basic facts about $\mathcal{M}_{1,1}$. Following [FO10, Sil09], we adopt the following notation: given variables a_1, a_2, a_3, a_4, a_6, we set

$$b_2 = a_1^2 + 4a_4\,, \quad b_4 = 2a_4 + a_1a_3\,, \quad b_6 = a_3^2 + 4a_6$$

and

$$\begin{aligned} b_8 &= a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2\,,\\ c_4 &= b_2^2 - 24b_4\,, \quad c_6 = -b_2^3 - 36b_2b_4 - 216b_6\,,\\ \Delta &= -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6\,,\\ j &= c_4^3/\Delta\,. \end{aligned}$$

We also have $4b_8 = b_2b_6 - b_4^2$ and $1728\Delta = c_4^3 - c_6^2$.

We set $U \overset{\text{def}}{=} \mathrm{Spec}(\mathbb{Z}[a_1,a_2,a_3,a_4,a_6,\Delta^{-1}])$, and we define the group scheme G over $\mathrm{Spec}(\mathbb{Z})$ as the scheme $\mathrm{Spec}\big(\mathbb{Z}\big[u^{\pm1},r,s,t\big]\big)$ together with the group structure

$$(u,r,s,t)\cdot(u',r',s',t') \overset{\text{def}}{=} \big(uu', u^2r'+r, us'+s, u^3t'+u^2r's+t\big)\,.$$

There is an action of G on U defined via the following coaction:

$$\begin{aligned} a_1 &\longmapsto u^{-1}(a_1+2s)\,,\\ a_2 &\longmapsto u^{-2}\big(a_2 - sa_1 + 3r - s^2\big)\,,\\ a_3 &\longmapsto u^{-3}(a_3 + ra_1 + 2t)\,,\\ a_4 &\longmapsto u^{-4}\big(a_4 - sa_3 + 2ra_2 - (t+rs)a_1 + 3r^2 - 2st\big)\,,\\ a_6 &\longmapsto u^{-6}\big(a_6 + ra_4 + r^2a_2 + r^3 - ta_3 - t^2 - rta_1\big)\,. \end{aligned}$$

The quotient stack $[U/G]$ is isomorphic to $\mathcal{M}_{1,1,\mathbb{Z}}$, the stack of elliptic curves over $\operatorname{Spec}(\mathbb{Z})$.

Let $j\colon \mathcal{M}_{1,1,\mathbb{Z}} \to M_{1,1,\mathbb{Z}}$ be the morphism to the coarse moduli space. The scheme $M_{1,1,\mathbb{Z}}$ is isomorphic to $\mathbb{A}^1_{\mathbb{Z}}$, and the induced map $U \to \mathbb{A}^1_{\mathbb{Z}}$ corresponds to the element j defined above.

8.2 Characteristic 2

In this subsection we work over a ground field $\mathbf{k}$ of characteristic 2. Our goal is to compute the group $\operatorname{Inv}^\bullet(\mathcal{M}_{1,1,\mathbf{k}}, \mathrm{H}_{2^r})$. In what follows, as we only deal with stacks and schemes over $\mathbf{k}$, we use the simpler notation $\mathcal{M}_{1,1} \overset{\text{def}}{=\!=} \mathcal{M}_{1,1,\mathbf{k}}$.

The stack $\mathcal{M}_{1,1} \smallsetminus j^{-1}(0)$ is isomorphic to $(\mathbb{A}^1 \smallsetminus \{0\}) \times \mathrm{B}\mathbb{Z}/2$: one way to see this is to observe that $\mathcal{M}_{1,1} \smallsetminus j^{-1}(0) \to \mathbb{A}^1 \smallsetminus \{0\}$ is a gerbe banded by $\mathbb{Z}/2$ and that it has a section, as in [Shi19, Lemma 3.2]. Another one consists in showing that $\mathcal{M}_{1,1} \smallsetminus j^{-1}(0) \simeq \big[\operatorname{Spec}\big(\mathbf{k}\big[a_2', a_6', {a'}_6^{-1}\big]\big)/\mathbb{G}_{\mathrm{a}}\big]$, where the additive group acts by $(a_2', a_6') \mapsto (a_2' + s(s+1), a_6')$. This follows from [Sil09, Appendix A, Proposition 1.1] or, more explicitly, by considering the map

$$\mathbf{k}\big[a_2', {a'}_6^{\pm 1}\big] \longrightarrow \mathbf{k}\big[a_1^{\pm 1}, a_2, a_4, a_6, \Delta^{-1}\big]$$

given by

$$a_2' \longmapsto \frac{a_1 a_2 + a_3}{a_1^3}\,, \quad a_6' \longmapsto \frac{a_1^4 a_2 a_3^2 + a_1^3 a_3^3 + a_3^4 + a_1^5 a_3 a_4 + a_1^4 a_4^2 + a_1^6 a_6}{a_1^{12}} = \frac{\Delta}{a_1^{12}} = j^{-1}\,.$$

Then it is immediate to check that $\big[\operatorname{Spec}\big(\mathbf{k}\big[a_2', {a'}_6^{\pm 1}\big]\big)/\mathbb{G}_{\mathrm{a}}\big] \simeq \operatorname{Spec}\big(\mathbf{k}\big[{a'}_6^{\pm 1}\big]\big) \times \mathrm{B}\mathbb{Z}/2$.

Lemma 8.3. *We have*

$$\operatorname{Inv}^\bullet\big(\mathcal{M}_{1,1} \smallsetminus j^{-1}(0), \mathrm{H}_{2^r}\big) \simeq \operatorname{Inv}^\bullet\big(\mathbb{A}^1 \smallsetminus \{0\}, \mathrm{H}_{2^r}\big) \oplus \operatorname{Inv}^\bullet\big(\mathbb{A}^1 \smallsetminus \{0\}, \mathrm{K}_2\big) \cdot [\alpha]\,,$$

where $[\alpha]$ is the cohomological invariant pulled back from $\mathrm{B}\mathbb{Z}/2$.

Proof. We apply Corollary 7.6 to $\mathcal{M}_{1,1} \smallsetminus j^{-1}(0) \simeq \big(\mathbb{A}^1 \smallsetminus \{0\}\big) \times \mathrm{B}\mathbb{Z}/2$. □

Remark 8.4. More concretely, using the presentation

$$\mathcal{M}_{1,1} \smallsetminus j^{-1}(0) \simeq \big[\operatorname{Spec}\big(\mathbf{k}\big[a_2', {a'}_6^{\pm 1}\big]\big)/\mathbb{G}_{\mathrm{a}}\big]\,,$$

we can look at the pullback of $[\alpha]$ to $\operatorname{Spec}\big(\mathbf{k}\big[a_2', {a'}_6^{\pm 1}\big]\big)$. Given a map $\operatorname{Spec}(F) \to \operatorname{Spec}\big(\mathbf{k}\big[a_2', {a'}_6^{\pm 1}\big]\big)$, there is an induced map $\varphi\colon \mathbf{k}\big[a_2', {a'}_6^{\pm 1}\big] \to F$. Then the pullback of $[\alpha]$ coincides with the invariant

$$\big(\operatorname{Spec}(F) \longrightarrow \operatorname{Spec}\big(\mathbf{k}\big[a_2', {a'}_6^{\pm 1}\big]\big)\big) \longmapsto \big[(0, \ldots, 0, \varphi(a_2'))\big] \in \mathrm{H}^1_{2^r}(\mathrm{F})\,.$$

Observe that this cohomological invariant is $\mathbb{G}_{\mathrm{a}}$-invariant, so it descends to $\mathcal{M}_{1,1} \smallsetminus j^{-1}(0)$, as expected.

Furthermore, we deduce that the pullback of $[\alpha]$ to $\mathrm{H}^1_{p^r}(\mathbf{k}(a_1, \ldots, a_6))$ is equal to $[(0, \ldots, 0, \phi((a_1 a_2 + a_3)/a_1^3)]$.

Knowing the cohomological invariants of $\mathcal{M}_{1,1} \smallsetminus \{j = 0\}$, the next step is to understand which ones extend to cohomological invariants of $\mathcal{M}_{1,1}$. For this, we first investigate what are the invariants of $\mathcal{M}_{1,1} \smallsetminus \{j = 0\}$ that are tamely ramified over $\mathcal{M}_{1,1}$.

Recall that $\mathcal{M}_{1,1} \simeq [U/G]$, where $U \overset{\text{def}}{=\!=} \operatorname{Spec}\big(\mathbb{Z}\big[a_1, a_2, a_3, a_4, a_6, \Delta^{-1}\big]\big)$ and G is a special group. In particular, the morphism $\pi\colon U \to \mathcal{M}_{1,1}$ is smooth-Nisnevich; hence we can check if an element is wildly ramified, tamely ramified or unramified by looking at its pullback to U. In this presentation the substack $\mathcal{M}_{1,1} \smallsetminus \{j = 0\}$ corresponds to the quotient of the complement of the divisor $\{a_1 = 0\}$.

Definition 8.5. We define $\mathrm{Inv}^\bullet(\mathcal{M}_{1,1} \smallsetminus \{j=0\}, \mathrm{H}_{2^r})_{\mathrm{tm}/U}$ as the subgroup of cohomological invariants of $\mathcal{M}_{1,1} \smallsetminus \{j=0\}$ that, once pulled back to $\mathrm{H}^\bullet_{2^r}(\mathbf{k}(U))$, are tamely ramified on U or, equivalently, are tamely ramified at $\{a_1=0\}$. We refer to these elements as *invariants that are tame on* U.

Similarly, we define $\mathrm{Inv}^\bullet\big(\mathbb{A}^1 \smallsetminus \{0\}, \mathrm{H}_{2^r}\big)_{\mathrm{tm}/U}$ as the intersection

$$\mathrm{Inv}^\bullet(\mathcal{M}_{1,1} \smallsetminus \{j=0\}, \mathrm{H}_{2^r})_{\mathrm{tm}/U} \cap \mathrm{Inv}^\bullet\big(\mathbb{A}^1 \smallsetminus \{0\}, \mathrm{H}_{2^r}\big)$$

and $\big(\mathrm{Inv}^\bullet\big(\mathbb{A}^1 \smallsetminus \{0\}, \mathrm{K}_2\big) \cdot [\alpha]\big)_{\mathrm{tm}/U}$ as the intersection

$$\mathrm{Inv}^\bullet(\mathcal{M}_{1,1} \smallsetminus \{j=0\}, \mathrm{H}_{2^r})_{\mathrm{tm}/U} \cap \mathrm{Inv}^\bullet\big(\mathbb{A}^1 \smallsetminus \{0\}, \mathrm{K}_2\big) \cdot [\alpha]\,.$$

Let F be the field $\mathbf{k}(a_1, \ldots, a_6)$, and let v be the discrete valuation determined by a_1. The map $\pi\colon U \to \mathbb{A}^1$ induces a map

$$\tilde{\pi}^*\colon \widetilde{\mathrm{H}}^\bullet_{2^r}(\mathbf{k}(j)) \longrightarrow \widetilde{\mathrm{H}}^\bullet_{2^r}(\mathbf{k}_v(a_1))\,,$$

where $\mathbf{k}_v = \mathbf{k}(a_2, a_3, a_4, a_6)$.

Lemma 8.6. *The map $\tilde{\pi}^*$ is injective, and we have*

$$\mathrm{Inv}^\bullet\big(\mathbb{A}^1 \smallsetminus \{0\}, \mathrm{H}_{2^r}\big)_{\mathrm{tm}/U} = \mathrm{Inv}^\bullet\big(\mathbb{A}^1, \mathrm{H}_{2^r}\big) \oplus \mathrm{H}^{\bullet-1}_{2^r}(\mathbf{k}) \cdot \{j\}\,.$$

Moreover, if $\beta \in \mathrm{H}^{n+1}_p(\mathbf{k}(j))$ belongs to $\mathrm{U}_s \smallsetminus \mathrm{U}_{s-1}$, then $\pi^*\beta$ belongs to $\mathrm{U}_{12s} \smallsetminus \mathrm{U}_{12s-4}$.

Proof. Let $\alpha \in \mathrm{H}^\bullet_{p^r}(\mathbf{k}(j))$ be a wildly ramified element. Let the morphisms π_s and ι_s be defined as in Lemma 5.1. Proposition 5.11 tells us that there must be an i such that $\alpha = \iota_i\alpha' + \alpha_{\mathrm{tm}}$ and $\beta = \pi_{i-1}\alpha' \in \mathrm{H}^\bullet_p(\mathbf{k}(j))$ is wildly ramified. Now, if the pullback of β is wildly ramified, so is the pullback of α. Thus we have reduced our proof to the case $r=1$.

Applying Lemma 5.10 we can rewrite β, up to a tamely ramified element, as

$$\beta = \sum_{i=1}^{s} j^{-i}\varphi_i + j^{-i}\frac{\mathrm{d}j^{-1}}{j^{-1}} \wedge \varphi'_i\,, \quad \varphi_i \in \Omega^n_{\mathbf{k}},\ \varphi'_i \in \Omega^{n-1}_{\mathbf{k}}\,, \tag{8.1}$$

where $\varphi'_i = 0$ if $p \nmid i$. On U, we have

$$j^{-1} = \frac{a_1^4a_2a_3^2 + a_1^3a_3^3 + a_3^4 + a_1^5a_3a_4 + a_1^4a_4^2 + a_1^6a_6}{a_1^{12}} = \frac{a_2a_3^2}{a_1^8} + \frac{a_3^3}{a_1^9} + \frac{a_3^4}{a_1^{12}} + \frac{a_3a_4}{a_1^7} + \frac{a_4^2}{a_1^8} + \frac{a_6}{a_1^6}\,.$$

Note that $\mathrm{d}j^{-1}$ equals

$$\left(\frac{a_3^3}{a_1^9} + \frac{a_3a_4}{a_1^7}\right)\frac{\mathrm{d}a_1}{a_1} + \frac{a_2a_3^2}{a_1^8}\frac{\mathrm{d}a_2}{a_2} + \left(\frac{a_3^3}{a_1^9} + \frac{a_3a_4}{a_1^7}\right)\frac{\mathrm{d}a_3}{a_3} + \frac{a_3a_4}{a_1^7}\frac{\mathrm{d}a_4}{a_4} + \frac{a_6}{a_1^6}\frac{\mathrm{d}a_6}{a_6}\,,$$

so by direct computation

$$\pi^*\left(j^{-i}\frac{\mathrm{d}j^{-1}}{j^{-1}} \wedge \varphi'_i\right) = (\pi^*j)^{-i+1}\mathrm{d}\pi^*j^{-1} \wedge \varphi'_i$$

belongs to U_{12i-3}.

Let $-s$ be the lowest power of j appearing in (8.1). We can assume that j^{-s} appears coupled with an element $\varphi_s + (\mathrm{d}j^{-1}/j^{-1}) \wedge \varphi'_s$ that is non-zero if s is odd, and (φ_s, φ'_s) is not closed in $\Omega^n_{\mathbf{k}} \oplus \Omega^{n-1}_{\mathbf{k}}$ if s is even.

We first consider the case where s is even. If φ_s is not closed, then the lowest-degree term in $\pi^*\beta$ is $a_1^{-12s}(a_3^{4s}\varphi_s)$. Observe that $\mathrm{d}(a_3^{4s}\varphi_s) = a_3^{4s}\mathrm{d}(\varphi_s) \neq 0$. This shows that $\pi^*\beta$ is not 0 in $\mathrm{U}_{12s}/\mathrm{U}_{12s-1}$.

If φ_s is closed, then φ'_s is not closed. By Lemma 5.10, up to elements of degree $-s/2$, we may assume that $\varphi_s = 0$. Then in $\mathrm{U}_{12s-3}/\mathrm{U}_{12s-4}$ the image $\pi^*\beta$ is equal to

$$\pi^* j^{-s+1}\left(\frac{a_3^3}{a_1^9}\frac{\mathrm{d}a_1}{a_1} + \frac{a_3^3}{a_1^9}\frac{\mathrm{d}a_3}{a_3}\right) \wedge \varphi'_s = \frac{a_3^{4s+3}}{a_1^{12s-3}}\left(\frac{\mathrm{d}a_1}{a_1} + \frac{\mathrm{d}a_3}{a_3}\right) \wedge \varphi'_s\,.$$

Observe that as

$$0 = \mathrm{d}\left(\frac{a_3^{4s+3}}{a_1^{12s-3}}\varphi'_s\right) = \frac{a_3^{4s+3}}{a_1^{12s-3}}\mathrm{d}\big(\varphi'_s\big) + \frac{a_3^{4s+3}}{a_1^{12s-3}}\frac{\mathrm{d}a_1}{a_1} \wedge \varphi'_s + \frac{a_3^{4s+3}}{a_1^{12s-3}}\frac{\mathrm{d}a_3}{a_3} \wedge \varphi'_s\,,$$

we get that

$$\pi^*\beta = \frac{a_3^{4s-1}}{a_1^{12s-3}}\mathrm{d}\big(\varphi'_s\big) \neq 0$$

in $\mathrm{U}_{12s-3}/\mathrm{U}_{12s-4}$.

Now assume that s is odd. If φ_s is not closed, the same argument as above allows us to conclude. Assume $\mathrm{d}(\varphi_s) = 0$. In $\mathrm{U}_{12s}/\mathrm{U}_{12s-1}$ we have $\pi^*\beta = (a_3^{4s}/a_1^{12s})\varphi$, which is closed, and by Corollary 5.8 this shows that this term actually belongs to U_{6s}. This shows that in $\mathrm{U}_{12s-3}/\mathrm{U}_{12s-4}$ we have $\pi^*\beta = (a_3^{4s-1}/a_1^{12s-3})\varphi$, which is non-zero as $a_3^{4s-1}\varphi \neq 0$.

By Lemma 8.3 we can regard $\mathrm{Inv}^\bullet\big(\mathbb{A}^1 \smallsetminus \{0\}, \mathrm{H}_{2^r}\big)$ as a subgroup of $\mathrm{Inv}^\bullet(\mathcal{M}_{1,1} \smallsetminus \{j=0\}, \mathrm{H}_{2^r})$. Then the invariants in this subgroup that are tame on U are those that are sent to zero by the map

$$\mathrm{Inv}^\bullet\big(\mathbb{A}^1 \smallsetminus \{0\}, \mathrm{H}_{2^r}\big) \longrightarrow \widetilde{\mathrm{H}}^\bullet_{2^r}(\mathbf{k}(j))\,.$$

By looking at the exact sequence for the invariants of $\mathbb{A}^1$ (see Corollary 7.3), we see that the kernel of this map coincides with the image of the injective homomorphism

$$\mathrm{Inv}^\bullet\big(\mathbb{A}^1, \mathrm{H}_{2^r}\big) \longrightarrow \mathrm{Inv}^\bullet\big(\mathbb{A}^1 \smallsetminus \{0\}, \mathrm{H}_{2^r}\big)\,.$$

This concludes the proof. □

Lemma 8.7. *We have*

$$\big(\mathrm{Inv}^\bullet\big(\mathbb{A}^1 \smallsetminus \{0\}, \mathrm{K}_2\big) \cdot [\alpha]\big)_{\mathrm{tm}/U} = \mathrm{K}_2^{\bullet-1}(\mathbf{k}) \cdot [\alpha, j\}\,.$$

Proof. We have

$$\mathrm{Inv}^\bullet(\mathbb{A}^1 \smallsetminus \{0\}, \mathrm{K}_2) \simeq \mathrm{K}_2^\bullet(\mathbf{k}) \oplus \mathrm{K}_2^{\bullet-1}(\mathbf{k}) \cdot \{j\}$$

due to equation (5.1). We first check that $[\alpha, \Delta\}$ is tame on U: this implies that $[\alpha, j\}$ is tame because $\{\Delta\} = \{j^{-1}\} = -\{j\}$.

Using the formula for the pullback of $[\alpha]$ given in Remark 8.4 and the formula for Δ in characteristic 2, we can write the pullback of $[\alpha, \Delta\}$ to U as a differential form as

$$\left(\frac{a_1a_2+a_3}{a_1^3}\right)\!\left(\frac{\big(a_1^2a_3\big)\big(a_1^2a_4+a_3^2\big)\cdot \mathrm{d}a_1}{\Delta} + \frac{a_1^4\big(a_1^2+a_3^2\big)\cdot \mathrm{d}a_2}{\Delta} + \frac{a_1^3\big(a_1^2a_4+a_3^2\big)\cdot \mathrm{d}a_3}{\Delta} + \frac{a_1^4a_3\cdot \mathrm{d}a_4}{\Delta}\right),$$

which, after some manipulations, can be written as

$$\left(\frac{a_1a_2+a_3}{\Delta}\right)\!\left(a_3\big(a_1^2a_4+a_3^2\big)\frac{\mathrm{d}a_1}{a_1} + a_1a_2\big(a_1^2+a_3^2\big)\cdot\frac{\mathrm{d}a_2}{a_2} + a_3\big(a_1^2a_4+a_3^2\big)\cdot\frac{\mathrm{d}a_3}{a_3} + a_1a_3a_4\cdot\frac{\mathrm{d}a_4}{a_4}\right).$$

All the coefficients in the formula have non-negative valuation at $\{a_1 = 0\}$; hence this element is tamely ramified by Theorem 5.7.

To conclude the proof, we only need to check that $\delta \cdot [\alpha]$ is wildly ramified at U for any $\delta \in \mathrm{K}_2^\bullet(\mathbf{k})$. Again, using the formula of Remark 8.4, we get

$$\pi^*[\alpha] = \left[\left(0, \dots, 0, (a_1a_2 + a_3)/a_1^3\right)\right],$$

which is wildly ramified at $\{a_1 = 0\}$: this follows from Theorem 5.7 as the form

$$\frac{a_1a_2 + a_3}{a_1^3}\varphi, \quad \varphi \in \Omega^n_{\mathbf{k},\log}$$

is sent to $a_3/a_1^3\varphi \neq 0$ in $\mathrm{U}_3/\mathrm{U}_2$, which coincides with the class of $\pi^*[\alpha]$ in this quotient (we are implicitly using the fact that $\pi^*[\alpha]$ belongs by definition to the subgroup $\mathrm{H}^1_2(\mathbf{k}(a_1, \dots, a_6))$ contained in $\mathrm{H}^1_{2^r}(\mathbf{k}(a_1, \dots, a_6))$). □

Before proceeding with the computation of the cohomological invariants of $\mathcal{M}_{1,1}$ in characteristic 2, we need some technical results.

Lemma 8.8. *Let $\pi\colon U \smallsetminus \{a_1 = 0\} \to \mathbb{A}^1 \smallsetminus \{0\}$ be the map given by the j-invariant, and let β be a cohomological invariant of $\mathbb{A}^1 \smallsetminus \{0\}$ that is wildly ramified at $\{j = 0\}$. Then for every $\delta \in \mathrm{K}^n_2(\mathbf{k})$ the element $\pi^*\beta + \delta \cdot [\alpha]$ is wildly ramified at $\{a_1 = 0\}$, where $\alpha = (a_2a_1 + a_3)/a_1^3$.*

Proof. Suppose that the statement holds true for $r = 1$, and let β be an invariant contained in $\mathrm{H}^{n+1}_{p^r}(\mathbf{k}(j))$ with $r > 1$. Let π_1 and ι_1 be defined as in Lemma 5.1. By definition the invariant $\delta \cdot [\alpha]$ belongs to the image of ι_1; hence $\pi_1(\pi^*\beta + \delta \cdot [\alpha]) = \pi_1(\pi^*\beta) = \pi^*(\pi_1(\beta))$, where with a little abuse of notation, we denoted by π_1 both the projection on $\mathrm{H}^{n+1}_p(\mathbf{k}(j))$ and that on $\mathrm{H}^{n+1}_p(\mathbf{k}(a_2, \dots, a_6)(a_1))$.

If $\pi^*\beta + \delta \cdot [\alpha]$ is tamely ramified at $\{a_1 = 0\}$, then so it is its image via π_1; hence by the computation above $\widetilde{\pi}^*(\pi_1(\beta)) = 0$ in $\widetilde{\mathrm{H}}^{n+1}_p(\mathbf{k}(a_2, \dots, a_6)(a_1))$. By Lemma 8.6 this implies that $\pi_1(\beta) = 0$ in $\widetilde{H}^{n+1}_p(\mathbf{k}(j))$; that is, β is equal to $\iota_1(\beta')$ up to tame elements. As the tameness of $\pi^*\beta + \delta \cdot [\alpha]$ is equivalent to the tameness of $\pi^*(\iota_1(\beta')) + \delta \cdot [\alpha]$, we can assume $\beta = \iota_1(\beta')$, that is, β belongs to $\mathrm{H}^{n+1}_p(\mathbf{k}(j))$. Then we have reduced to the case $r = 1$.

Observe that $\delta \cdot [\alpha]$ belongs to $\mathrm{U}_3 \smallsetminus \mathrm{U}_2$ (the U_i are the subgroups appearing in Theorem 5.7), so that if β is tame, we are done.

Suppose that β is in $\mathrm{U}_s \smallsetminus \mathrm{U}_{s-1}$, with $s \geqslant 1$. Then by Lemma 8.6 we have that $\pi^*\beta \in \mathrm{U}_{12s} \smallsetminus \mathrm{U}_{12s-4}$. If $\pi^*\beta + \delta \cdot [\alpha]$ is tame, then it belongs to $\mathrm{U}_0 \subset \mathrm{U}_{12s-4}$ by Theorem 5.7, hence $\pi^*\beta = (\pi^*\beta + \delta \cdot [\alpha]) - \delta \cdot [\alpha]$ is in U_{12s-4} because $3 < 12s - 4$ for $s \geqslant 1$. This contradicts the fact that $\pi^*\beta \in \mathrm{U}_{12s} \smallsetminus \mathrm{U}_{12s-4}$ and concludes the proof. □

Proposition 8.9. *We have*

$$\mathrm{Inv}^\bullet(\mathcal{M}_{1,1} \smallsetminus \{j = 0\}, \mathrm{H}_{2^r})_{\mathrm{tm}/U} \simeq \mathrm{Inv}^\bullet\left(\mathbb{A}^1, \mathrm{H}_{2^r}\right) \oplus \mathrm{H}^{\bullet-1}_{2^r} \cdot \{j\} \oplus \mathrm{K}^{\bullet-1}_2 \cdot [\alpha, j\}\,.$$

Proof. Let β be an invariant of $\mathcal{M}_{1,1} \smallsetminus \{j = 0\}$ with coefficients in H_{2^r}, tame over U. By Lemma 8.3 we can write $\beta = \beta' + \beta'' \cdot [\alpha]$, where β' is (the pullback of) an invariant of $\mathbb{A}^1 \smallsetminus \{0\}$ with coefficients in H_{2^r} and β'' is (the pullback of) an invariant of $\mathbb{A}^1 \smallsetminus \{0\}$ with coefficients in K_2.

We can further rewrite β'' as $\delta + \delta' \cdot \{j\}$, where δ and δ' belong to $\mathrm{K}^\bullet_2(\mathbf{k})$.

If the image of β' is zero in $\widetilde{\mathrm{H}}^{n+1}_{2^r}(\mathbf{k}(a_2, \dots, a_6)(a_1))$, then β' is tamely ramified and hence it belongs to $\mathrm{Inv}^\bullet\left(\mathbb{A}^1, \mathrm{H}_{2^r}\right) \oplus \mathrm{H}^{\bullet-1}_{2^r} \cdot \{j\}$ by Lemma 8.6; moreover, we then have that β is tame if and only if $\delta = 0$ because $[\alpha, j\}$ is tame and $\delta \cdot [\alpha]$ is not (Lemma 8.7).

This implies that β belongs to $\mathrm{Inv}^\bullet\left(\mathbb{A}^1, \mathrm{H}_{2^r}\right) \oplus \mathrm{H}^{\bullet-1}_{2^r} \cdot \{j\} \oplus \mathrm{K}^{\bullet-1}_2 \cdot [\alpha, j\}$, as claimed.

Now suppose that the image of β' is not zero in $\widetilde{\mathrm{H}}^{n+1}_{2^r}(\mathbf{k}(a_2,\dots,a_6)(a_1))$. As β is tamely ramified by hypothesis, this implies that $\beta' + \delta\cdot[\alpha]$ is zero in $\widetilde{\mathrm{H}}^{n+1}_{2^r}(\mathbf{k}(a_2,\dots,a_6)(a_1))$, which contradicts Lemma 8.8. This finishes the proof. □

Let $\mu\colon \mathrm{K}_2^{\bullet-1}(\mathbf{k}) \to \mathrm{H}^{\bullet}_{2^r}(\mathbf{k})$ be the map given by

$$\{x_1,\dots,x_n\} \longmapsto [(0,\dots,0,1),x_1,\dots,x_n\}\,,$$

and let $\mathrm{H}^{\bullet}_{2^r}(\mathbf{k}) \xrightarrow{4} \mathrm{H}^{\bullet}_{2^r}(\mathbf{k})$ be multiplication by 4. In the category of abelian groups, we can form the Cartesian diagram

$$\begin{array}{ccc} \mathrm{J}^{\bullet}_{2^r}(\mathbf{k}) & \longrightarrow & \mathrm{K}_2^{\bullet-1}(\mathbf{k}) \\ \downarrow & & \downarrow{\scriptstyle \mu} \\ \mathrm{H}^{\bullet}_{2^r}(\mathbf{k}) & \xrightarrow{\;4\;} & \mathrm{H}^{\bullet}_{2^r}(\mathbf{k})\,. \end{array}$$

More concretely, one can think of $\mathrm{J}^{\bullet}_{2^r}(\mathbf{k})$ as the subgroup of $\mathrm{H}^{\bullet}_{2^r}(\mathbf{k}) \oplus \mathrm{K}_2^{\bullet-1}(\mathbf{k})$ formed by those pairs $([w,\underline{x}\},\{\underline{y}\})$ such that $4[w,\underline{x}\} = [(0,\dots,0,1),\underline{y}\}$.

Remark 8.10. If $r \leqslant 2$, then multiplication by 4 is equal to the zero map; hence in this range $\mathrm{J}^{\bullet}_{2^r}(\mathbf{k})$ is equal to $\mathrm{H}^{\bullet}_{2^r}(\mathbf{k}) \oplus \ker(\mu)$.

If $r \geqslant 3$, then $\mathrm{J}^{\bullet}_{2^r}(\mathbf{k})$ is equal to $\mathrm{J}^{\bullet}_{8}(\mathbf{k})$: indeed, by definition, for every element in $\mathrm{J}^{\bullet}_{2^r}(\mathbf{k})$, we have $4[w,\underline{x}\} = [(0,\dots,0,1),\underline{y}\}$, which immediately implies that $8[w,\underline{x}\} = 0$. By Proposition 5.2 we deduce that $[w,\underline{x}\}$ belongs to $\mathrm{H}^{\bullet}_{8}(\mathbf{k})$; thus the pair $([w,\underline{x}\},\{\underline{y}\})$ belongs to $\mathrm{J}^{\bullet}_{8}(\mathbf{k})$.

Finally, observe that if x^2+x+1 has a solution in $\mathbf{k}$, we have $[(0,\dots,0,1)] = 0$ in $\mathrm{H}^1_{2^r}(\mathbf{k})$; thus $\mathrm{J}^{\bullet}_{2^r}(\mathbf{k})$ is actually equal to $\mathrm{H}^{\bullet}_{4}(\mathbf{k}) \oplus \mathrm{K}_2^{\bullet-1}(\mathbf{k})$.

THEOREM 8.11. *Suppose that the base field* $\mathbf{k}$ *has characteristic* 2. *Then*

$$\mathrm{Inv}^{\bullet}(\mathcal{M}_{1,1},\mathrm{H}_{2^r}) \simeq \mathrm{Inv}^{\bullet}\big(\mathbb{A}^1,\mathrm{H}_{2^r}\big) \oplus \mathrm{J}^{\bullet-1}_{2^r}(\mathbf{k})\,.$$

Proof. We know from Proposition 8.9 that

$$\mathrm{Inv}^{\bullet}(\mathcal{M}_{1,1} \smallsetminus \{j=0\},\mathrm{H}_{2^r})_{\mathrm{tm}/U} \simeq \mathrm{Inv}^{\bullet}\big(\mathbb{A}^1,\mathrm{H}_{2^r}\big) \oplus \mathrm{H}^{\bullet-1}_{2^r}\cdot\{j\} \oplus \mathrm{K}_2^{\bullet-2}\cdot[\alpha,j\}\,.$$

The cohomological invariants of $\mathcal{M}_{1,1}$ are those elements in the group above whose pullback to U is unramified along $\{a_1=0\}$ (see Corollary 6.9).

The elements coming from $\mathbb{A}^1$ are obviously unramified, so we only have to check when the pullback to U of an element of the form $[w,j,\underline{x}\} + [\alpha,j,\underline{y}\}$ is unramified at $\{a_1=0\}$.

We have seen in the proof of Lemma 8.7 that can rewrite the pullback of $[\alpha,j\}$ to U as

$$\left(\frac{a_1a_2+a_3}{\Delta}\right)\left(a_3\big(a_1^2a_4+a_3^2\big)\frac{\mathrm{d}a_1}{a_1} + a_1a_2\big(a_1^2+a_3^2\big)\cdot\frac{\mathrm{d}a_2}{a_2} + a_3\big(a_1^2a_4+a_3^2\big)\cdot\frac{\mathrm{d}a_3}{a_3} + a_1a_3a_4\cdot\frac{\mathrm{d}a_4}{a_4}\right),$$

from which we deduce by direct computation that the residue of $[\alpha,j,\underline{y}\}$ at $\{a_1=0\}$ is equal to $[(0,\dots,0,1),\underline{y}\}$.

On the other hand, the pullback of $\{j\}$ to U is equal to $\{a_1^{12}/\Delta\}$, whose residue at $\{a_1=0\}$ is 12. This implies that the residue of $[w,j,\underline{x}\}$ is equal to $12[w,\underline{x}\} = 4[w,\underline{x}\}$.

We deduce that $[w,j,\underline{x}\} + [\alpha,\underline{y}\}$ is unramified if and only if $4[w,\underline{x}\} + [(0,\dots,0,1),\underline{y}\}$ is equal to zero. This concludes the proof. □

Remark 8.12. If $\mathbf{k}$ has a primitive cubic root of unity, that is, the polynomial x^2+x+1 is not irreducible in $\mathbf{k}$, then it follows from Remark 8.10 that the invariants of $\mathcal{M}_{1,1}$ are

$$\mathrm{Inv}^{\bullet}\big(\mathbb{A}^1,\mathrm{H}_{2^r}\big) \oplus \mathrm{H}^{\bullet-1}_{4}(\mathbf{k})\cdot\{\Delta\} \oplus \mathrm{K}_2^{\bullet-2}\cdot[\alpha,\Delta\}\,.$$

Indeed, for any $\beta \in \mathrm{H}_4^{\bullet-1}(\mathbf{k})$ the pullback of $\beta\cdot\{j\}$ to U is equal to $\beta\cdot(12\{a_1\}-\{\Delta\}) = -\beta\cdot\{\Delta\}$, from which follows that $\{\Delta\}$ is a generator for this subgroup. Similarly, for $\beta \in \mathrm{K}_2^{\bullet-2}(\mathbf{k})$, we have $\beta'\cdot[\alpha, j\} = \beta'\cdot 12[\alpha, a_1\} - \beta'\cdot[\alpha,\Delta\} = \beta'\cdot[\alpha,\Delta\}$.

Remark 8.13. Theorem 8.11, in conjunction with Proposition 3.14, shows that when $\mathrm{char}(\mathbf{k}) = 2$ we have

$$\mathrm{ed}(\mathcal{M}_{1,1}) \geqslant \mathrm{ed}_2(\mathcal{M}_{1,1}) \geqslant 2$$

as the group $\mathrm{J}^1(\overline{\mathbf{k}})$ is isomorphic to $\mathrm{K}_2^0(\overline{\mathbf{k}}) = \mathbb{Z}/2$.

The essential dimension of $\mathcal{M}_{1,1}$ is known to be 2 over any field [BRV11, Section 9.5], so Theorem 8.1 shows that $\mathrm{char}(\mathbf{k}) = 2$ is the only characteristic where cohomological invariants provide a sharp lower bound for the essential dimension of $\mathcal{M}_{1,1}$.

8.3 Characteristic 3

In this subsection we assume that the base field $\mathbf{k}$ has characteristic 3, and our goal is to compute $\mathrm{Inv}^\bullet(\mathcal{M}_{1,1}, \mathrm{H}_{3^r})$. Recall [Sil09, Section III.1, p. 42] that if $\mathrm{char}(\mathbf{k}) = 3$, then

$$\mathcal{M}_{1,1} \simeq \left[\mathrm{Spec}\big(\mathbf{k}\big[b_2, b_4, b_6, \Delta^{-1}\big]\big)/\mathbb{G}_\mathrm{m} \ltimes \mathbb{G}_\mathrm{a}\right],$$

where the group $\mathbb{G}_\mathrm{m} \ltimes \mathbb{G}_\mathrm{a}$ acts on $\mathrm{Spec}\big(\mathbf{k}\big[b_2, b_4, b_6, \Delta^{-1}\big]\big)$ as

$$b_2 \longmapsto u^{-2}b_2\,, \quad b_4 \longmapsto u^{-4}(b_4 + rb_2)\,, \quad b_6 \longmapsto u^{-6}\big(b_6 - rb_4 + r^2 b_2 + r^3\big)\,.$$

The isomorphism $[U/G] \simeq \left[\mathrm{Spec}\big(\mathbf{k}\big[b_2, b_4, b_6, \Delta^{-1}\big]\big)/\mathbb{G}_\mathrm{m} \ltimes \mathbb{G}_\mathrm{a}\right]$ is given by expressing the coordinates b_i in terms of the coordinates a_i via the formulas of Section 8.1. In particular, we have

$$\Delta = -b_2^3 b_6 + b_2^2 b_4^2 + b_4^3\,, \quad j = b_2^6/\Delta\,.$$

Similarly to what happens in characteristic 2, in characteristic 3 we also have

$$\mathcal{M}_{1,1} \smallsetminus \{j=0\} \simeq \big(\mathbb{A}^1 \smallsetminus \{0\}\big) \times \mathrm{B}\mathbb{Z}/2\,.$$

This can be proven exactly as in the characteristic 2 case, using the fact that for every elliptic curve C with $j(C) \neq 0$, the only non-trivial automorphism is the involution. We can also write

$$\mathcal{M}_{1,1} \smallsetminus \{j=0\} \simeq \left[\mathrm{Spec}\big(\mathbf{k}\big[b_2^{\pm 1}, b_4, b_6, \Delta^{-1}\big]\big)/\mathbb{G}_\mathrm{m} \ltimes \mathbb{G}_\mathrm{a}\right].$$

In other terms, the invariant subscheme in $\mathrm{Spec}\big(\mathbf{k}\big[b_2, b_4, b_6, \Delta^{-1}\big]\big)$ that parametrizes curves with j-invariant equal to zero is the divisor $\{b_2 = 0\}$.

We can apply Lemma 7.7 to deduce

$$\mathrm{Inv}^\bullet(\mathcal{M}_{1,1} \smallsetminus \{j=0\}, \mathrm{H}_{3^r}) \simeq \mathrm{Inv}^\bullet\big(\mathbb{A}^1 \smallsetminus \{0\}, \mathrm{H}_{3^r}\big)\,.$$

Set $U' \overset{\mathrm{def}}{=\!=} \mathrm{Spec}\big(\mathbf{k}\big[b_2, b_4, b_6, \Delta^{-1}\big]\big)$, so that $U' \to \mathcal{M}_{1,1}$ is a $\mathbb{G}_\mathrm{m} \ltimes \mathbb{G}_\mathrm{a}$-torsor.

Definition 8.14. We define $\mathrm{Inv}^\bullet(\mathcal{M}_{1,1} \smallsetminus \{j=0\}, \mathrm{H}_{3^r})_{\mathrm{tm}/U'}$ as the subgroup of cohomological invariants of $\mathcal{M}_{1,1} \smallsetminus \{j=0\}$ that, once pulled back to $\mathrm{H}_{3^r}^\bullet(\mathbf{k}(U'))$, are tamely ramified on U' or, equivalently, are tamely ramified at $\{b_2 = 0\}$. We refer to these elements as *invariants that are tame on* U'.

Analogously to what we did in characteristic 2, our strategy consists in first finding the invariants of $\mathcal{M}_{1,1} \smallsetminus \{j=0\}$ that are tamely ramified on U' and then checking what are the ones that are unramified using the residue homomorphism.

Let $\pi\colon (U' \smallsetminus \{b_2 = 0\}) \to \mathbb{A}^1$ be the composition of the quotient map and the j-invariant, so that there is an induced morphism

$$\widetilde{\pi}^*\colon \widetilde{\mathrm{H}}_{3^r}(\mathbf{k}(j)) \longrightarrow \widetilde{\mathrm{H}}_{3^r}(\mathbf{k}(b_4, b_6)(b_2))\,.$$

Lemma 8.15. *The homomorphism $\widetilde{\pi}^*$ is injective, and*

$$\mathrm{Inv}^\bullet(\mathcal{M}_{1,1} \smallsetminus \{j = 0\}, \mathrm{H}_{3^r})_{\mathrm{tm}/U'} \simeq \mathrm{Inv}^\bullet\big(\mathbb{A}^1, \mathrm{H}_{3^r}\big) \oplus \mathrm{H}^{\bullet-1}_{3^r}(\mathbf{k}) \cdot \{j\}\,.$$

Proof. The proof adopts the same strategy as that of Lemma 8.6, and the computations are quite similar. First observe that

$$j^{-1} = -\frac{b_6}{b_2^3} + \frac{b_4^2}{b_2^4} + \frac{b_4^3}{b_2^6}\,, \quad \mathrm{d}j^{-1} = \left(\frac{2b_6}{b_2^3}\right)\frac{\mathrm{d}b_6}{b_6} + \left(\frac{2b_4^2}{b_2^4}\right)\frac{\mathrm{d}b_4}{b_4} + \left(\frac{2b_4^2}{b_2^4}\right)\frac{\mathrm{d}b_2}{b_2}\,.$$

To prove that $\widetilde{\pi}^*$ is injective, we can reduce to the case $r = 1$ using exactly the same strategy as adopted in the proof of Lemma 8.6. Let β be a wildly ramified element of $\mathrm{H}_3^{n+1}(\mathbf{k}(j))$. We can apply Lemma 5.10 to write it down, up to a tamely ramified element, as

$$\beta = \sum_{i \geqslant 1} j^{-i} \wedge \varphi_i + j^{-i}\frac{\mathrm{d}j^{-1}}{j^{-1}} \wedge \varphi_i'\,, \tag{8.2}$$

where φ_i and φ_i' come from, respectively, $\Omega^n_{\mathbf{k}}$ and $\Omega^{n-1}_{\mathbf{k}}$. Moreover, we can assume that if $3 \nmid i$, then $\varphi_i \neq 0$ and $\varphi_i' = 0$, whereas if $3 \mid i$, then at least one of φ_i and φ_i' is not closed.

Let s be the highest power of j^{-1} appearing in (8.2). Suppose $3 \mid s$ and that φ_s is not closed. Then in $\mathrm{U}_{6s}/\mathrm{U}_{6s-1}$ we have $\pi^*\beta = b_2^{-6s}(b_4^{3s}\varphi_s)$; hence if $b_4^{3s}\varphi_s$ is not closed, we can conclude by Theorem 5.7 that $\pi^*\beta$ is non-zero in $\mathrm{U}_{6s}/\mathrm{U}_{6s-1}$. We have $\mathrm{d}(b_4^{3s}\phi_s) = b_4^{3s}\mathrm{d}\phi_s \neq 0$ because $\mathrm{d}\phi_s \neq 0$ by hypothesis.

Now suppose $3 \mid s$, that we have $\varphi_s = 0$ but that φ_s' is not closed. Then in $\mathrm{U}_{6s-2}/\mathrm{U}_{6s-3}$ we have

$$b_2^{-(6s-2)}\left(2b_4^2\frac{\mathrm{d}b_4}{b_4} \wedge \phi_s'\right) + b_2^{-(6s-2)}\left(2b_4^2\frac{\mathrm{d}b_2}{b_2} \wedge \phi_s'\right). \tag{8.3}$$

Observe that in $\mathrm{H}_3^{n+1}(\mathbf{k}(b_2, b_4, b_6))$ we have

$$0 = \mathrm{d}\big(b_2^{-(6s-2)}b_4^2\varphi_s'\big) = b_2^{-(6s-2)}\left(2b_4^2\frac{\mathrm{d}b_4}{b_4} \wedge \phi_s'\right) + b_2^{-(6s-2)}\left(2b_4^2\frac{\mathrm{d}b_2}{b_2} \wedge \phi_s'\right) + b_2^{-(6s-2)}\big(b_4^2\mathrm{d}\phi_s'\big)\,;$$

hence (8.3) is equal to $-b_2^{-(6s-2)}\big(b_4^2\mathrm{d}\phi_s'\big)$, and the latter is non-zero in $\mathrm{U}_{6s-2}/\mathrm{U}_{6s-3}$ by Theorem 5.7 because $b_4^2\mathrm{d}\phi_s' \neq 0$.

Finally, suppose $3 \nmid s$, so that we only know that $\varphi_s \neq 0$. The highest term in the pullback of β is $b_2^{-6s}\big(b_4^{3s}\varphi_s\big)$. If φ_s is not closed, then we can conclude as before, so suppose the contrary. In this case, applying Corollary 5.8 we see that this term can be rewritten as $b_2^{-3s}\varphi_s$, so that now $\pi^*\beta$ is equal to $b_2^{-6s+2}\big(sb_4^{3s-1}\varphi_s\big)$ in $\mathrm{U}_{6s-2}/\mathrm{U}_{6s-3}$. As $-6s+2$ is not divisible by 3, the form $b_4^{3s-1}\varphi_s$ is non-zero and s is invertible, we can again conclude that the pullback of β is non-zero in $\mathrm{U}_{6s-2}/\mathrm{U}_{6s-3}$.

In summary we have proved that $\widetilde{\pi}^*$ is injective.

From Corollary 7.3 we have that the invariants of $\mathbb{A}^1 \smallsetminus \{j = 0\}$ fit into the exact sequence

$$0 \longrightarrow \mathrm{Inv}^\bullet\big(\mathbb{A}^1, \mathrm{H}_{3^r}\big) \oplus \mathrm{H}^{\bullet-1}_{3^r} \cdot \{j\} \longrightarrow \mathrm{Inv}^\bullet\big(\mathbb{A}^1 \smallsetminus \{0\}, \mathrm{H}_{3^r}\big) \longrightarrow \widetilde{\mathrm{H}}^\bullet_{3^r}(\mathbf{k}(j)) \longrightarrow 0\,.$$

Let $\pi\colon (U' \smallsetminus \{b_2 = 0\}) \to \mathbb{A}^1$ be the composition of the quotient map and the j-invariant. Then

$$\pi^*[w, j, \underline{x}\} = \big[w, b_2^6/\Delta, \underline{x}\big\} = \big[w, b_2^6, \underline{x}\big\} - [w, \Delta, \underline{x}\}\,,$$

from which we deduce that $\pi^*(\delta\cdot\{j\})$ is tamely ramified at $\{b_2=0\}$ for every $\delta\in \mathrm{H}^{\bullet-1}_{3^r}(\mathbf{k})$. The invariants of $\mathbb{A}^1$ are obviously tamely ramified on U'. This, together with the injectivity of $\widetilde{\pi}^*$, gives the claimed description of the subgroup of invariants that are tame on U'. □

Theorem 8.16. *Suppose that the base field* $\mathbf{k}$ *has characteristic* 3. *Then we have*

$$\mathrm{Inv}^\bullet(\mathcal{M}_{1,1},\mathrm{H}_{3^r})\simeq \mathrm{Inv}^\bullet\big(\mathbb{A}^1,\mathrm{H}_{3^r}\big)\oplus \mathrm{H}^{\bullet-1}_{3}(\mathbf{k})\cdot\{\Delta\}\,.$$

Proof. We know from Lemma 8.15 that the cohomological invariants of $\mathcal{M}_{1,1}$ correspond to the unramified elements in

$$\mathrm{Inv}^\bullet\big(\mathbb{A}^1,\mathrm{H}_{3^r}\big)\oplus \mathrm{H}^{\bullet-1}_{3^r}(\mathbf{k})\cdot\{j\}\,.$$

Of course, the invariants coming from $\mathbb{A}^1$ are unramified, so that we only have to check the invariants of the form $\delta\cdot\{j\}$, where $\delta\in \mathrm{H}^{\bullet-1}_{3^r}$.

This can be done on the cover $U'=\mathrm{Spec}\big(\mathbf{k}\big[b_2,b_4,b_6,\Delta^{-1}\big]\big)\to\mathcal{M}_{1,1}$, where we only have to check the unramifiedness at $\{b_2=0\}$. When we pull $\{j\}$ back to U', we get $\big\{b_2^6/\Delta\big\}$, so that the residue of $\delta\cdot\{j\}$ at $\{b_2=0\}$ is equal to 6δ, which is zero if and only if δ is of 3-torsion.

By Proposition 5.2 the elements of 3-torsion in $\mathrm{H}^{\bullet-1}_{3^r}(\mathbf{k})$ are those that belong to $\mathrm{H}^{\bullet-1}_{3}(\mathbf{k})$. This, together with the fact that $\{j\}=\big\{b_2^6/\Delta\big\}=-\{\Delta\}$ in $\mathrm{K}^1_3(\mathbf{k})$, concludes the proof. □

8.4 Characteristic $p>3$

In this section we assume that the base field $\mathbf{k}$ has characteristic $p>3$. Our goal is to compute the cohomological invariants of $\mathcal{M}_{1,1}$ with coefficients in H_{p^r}.

Recall [Sil09, Section III.1] that with these hypotheses on $\mathbf{k}$, the stack $\mathcal{M}_{1,1}$ has the following presentation as a quotient stack:

$$\mathcal{M}_{1,1}\simeq\big[\mathrm{Spec}\big(\mathbf{k}\big[c_4,c_6,\Delta^{-1}\big]\big)/\mathbb{G}_\mathrm{m}\big]\,,$$

where the multiplicative group acts by $(c_4,c_6)\mapsto\big(u^{-4}c_4,u^{-6}c_6\big)$ and $\Delta=(c_4^3-c_6^2)/1728$. The j-invariant in this case is $j=1728c_4^3/(c_4^3-c_6^2)$, and we have

$$\mathcal{M}_{1,1}\smallsetminus\{j=0,1728\}\simeq\big(\mathbb{A}^1\smallsetminus\{0,1728\}\big)\times \mathrm{B}\mathbb{Z}/2\,.$$

As we are in characteristic $p>3$, we can apply Lemma 7.7 to deduce

$$\mathrm{Inv}^\bullet(\mathcal{M}_{1,1}\smallsetminus\{j=0,1728\},\mathrm{H}_{p^r})\simeq\mathrm{Inv}^\bullet\big(\mathbb{A}^1\smallsetminus\{0,1728\},\mathrm{H}_{p^r}\big)\,.$$

Using Theorem 7.1 we immediately deduce that there is a short exact sequence

$$0\longrightarrow \mathrm{N}^\bullet\longrightarrow \mathrm{Inv}^\bullet\big(\mathbb{A}^1\smallsetminus\{0,1728\},\mathrm{H}_{p^r}\big)\longrightarrow \mathrm{P}^\bullet\longrightarrow 0\,,$$

where

$$\mathrm{N}^\bullet\overset{\mathrm{def}}{=\!=}\mathrm{Inv}^\bullet\big(\mathbb{A}^1,\mathrm{H}_{p^r}\big)\oplus\mathrm{H}^{\bullet-1}_{p^r}(\mathbf{k})\cdot\{j\}\oplus\mathrm{H}^{\bullet-1}_{p^r}(\mathbf{k})\cdot\{j-1728\}\,,$$

$$\mathrm{P}^\bullet\overset{\mathrm{def}}{=\!=}\widetilde{\mathrm{H}}^\bullet_{p^r}(\mathbf{k}(j))\oplus\widetilde{\mathrm{H}}^\bullet_{p^r}(\mathbf{k}(j-1728))\,.$$

Set $U''\overset{\mathrm{def}}{=\!=}\mathrm{Spec}\big(\mathbf{k}\big[c_4,c_6,\Delta^{-1}\big]\big)$. As before, we introduce the definition of elements *tame on* U''.

Definition 8.17. We define $\mathrm{Inv}^\bullet(\mathcal{M}_{1,1}\smallsetminus\{j=0,1728\},\mathrm{H}_{p^r})_{\mathrm{tm}/U''}$ as the subgroup of cohomological invariants of $\mathcal{M}_{1,1}\smallsetminus\{j=0,1728\}$ that, once pulled back to $\mathrm{H}^\bullet_{p^r}(\mathbf{k}(U''))$, are tamely ramified on U''. We refer to these elements as *invariants that are tame on* U''.

To compute the invariants of $\mathcal{M}_{1,1}$, we first determine which invariants are tame on U''. For this, let us denote by $\pi\colon U''\to\mathbb{A}^1$ the composition of the quotient map together with the morphism given by the j-invariant. Observe that the preimage of 0 in U'' corresponds to the divisor $\{c_4 = 0\}$, and the preimage of 1728 corresponds to $\{c_6 = 0\}$, so that we have pullback homomorphisms

$$\widetilde{\pi}^*_{1728} \colon \widetilde{\mathrm{H}}^{\bullet}_{p^r}(\mathbf{k}(j-1728)) \longrightarrow \widetilde{\mathrm{H}}^{\bullet}_{p^r}(\mathbf{k}(c_4)(c_6))\,, \quad \widetilde{\pi}^*_0 \colon \widetilde{\mathrm{H}}^{\bullet}_{p^r}(\mathbf{k}(j)) \longrightarrow \widetilde{\mathrm{H}}^{\bullet}_{p^r}(\mathbf{k}(c_6)(c_4))\,.$$

We use these maps to determine the invariants that are tame on U''.

Lemma 8.18. *The pullback homomorphisms $\widetilde{\pi}_0$ and $\widetilde{\pi}_{1728}$ are both injective, and*

$$\begin{aligned}&\mathrm{Inv}^{\bullet}(\mathcal{M}_{1,1} \smallsetminus \{j = 0, 1728\}, \mathrm{H}_{p^r})_{\mathrm{tm}/U''}\\ &\quad \simeq \mathrm{Inv}^{\bullet}(\mathbb{A}^1, \mathrm{H}_{p^r}) \oplus \mathrm{H}^{\bullet-1}_{p^r}(\mathbf{k}) \cdot \{j\} \oplus \mathrm{H}^{\bullet-1}_{p^r}(\mathbf{k}) \cdot \{j-1728\}\,.\end{aligned}$$

Proof. Using exactly the same argument as adopted in the proof of Lemma 8.6, we reduce to the case $r=1$. First we deal with the morphism $\widetilde{\pi}^*_0$. Using Lemma 5.10, given a wildly ramified element β in $\mathrm{H}^{\bullet}_{p^r}(\mathbf{k}(j))$, we can rewrite it as

$$\beta = \sum_{i \geqslant 1} j^{-i} \wedge \varphi_i + j^{-i}\frac{\mathrm{d}j^{-1}}{j^{-1}} \wedge \varphi'_i\,, \tag{8.4}$$

where φ_i and φ'_i come from, respectively, $\Omega^n_{\mathbf{k}}$ and $\Omega^{n-1}_{\mathbf{k}}$. Moreover, we can assume that if $p \nmid i$ then $\varphi_i \neq 0$ and $\varphi'_i = 0$, whereas if $p \mid i$, then at least one of φ_i and φ'_i is not closed. Observe that

$$j^{-1} = (1728)^{-1}\left(1 - \frac{c_6^2}{c_4^3}\right)\,, \quad \mathrm{d}j^{-1} = -(1728)^{-1}\left(\left(\frac{2c_6^2}{c_4^3}\right)\frac{\mathrm{d}c_6}{c_6} - \left(\frac{3c_6^2}{c_4^3}\right)\frac{\mathrm{d}c_4}{c_4}\right)\,.$$

Let s be the highest power of j^{-1} appearing in (8.4), and suppose $p \mid s$; then in $\mathrm{U}_{3s}/\mathrm{U}_{3s-1}$ we have

$$\pi_0^*\beta = (1728)^{-s}c_4^{-3s}\big(c_6^{2s}\left(\varphi_s - 2\mathrm{d}c_6 \wedge \varphi'_s\right)\big) + (-1728)^{-s}c_4^{-3s}\left(3c_6^{2s}\frac{\mathrm{d}c_4}{c_4} \wedge \varphi'_s\right)\,,$$

which by Theorem 5.7 is non-zero if and only if the pair

$$\big((1728)^{-s}c_6^{2s}(\varphi_s - 2\mathrm{d}c_6 \wedge \varphi'_s), (1728)^{-s}3c_6^{2s}\varphi'_s\big) \in \Omega^n/Z^n \oplus \Omega^{n-1}/Z^{n-1}$$

is non-zero. We know by hypothesis that φ_s and φ'_s cannot both be closed. If φ'_s is not closed, then we deduce that the element in (8.4) is not zero because the second entry is not zero. Suppose that φ'_s is closed: then the first entry of (8.4) is equal to $(1728)^s c_6^{2s}\varphi_s$, and we have $\mathrm{d}\big((1728)^s c_6^{2s}\varphi_s\big) = (1728)^s c_6^{2s}\mathrm{d}\varphi_s \neq 0$ by hypothesis. We have shown that the pullback of β is not zero in $\mathrm{U}_{3s}/\mathrm{U}_{3s-1}$ when $p \mid s$.

If $p \nmid s$, then $p \nmid 3s$, the form φ_s is not zero, and again by Theorem 5.7 we can conclude that the element $\pi_0^*\beta = (1728)^{-s}c_4^{-3s}\big(c_6^{2s}\varphi_s\big)$ is not zero in $\mathrm{U}_{3s}/\mathrm{U}_{3s-1}$ because $(1728)^s c_6^{2s}\varphi_s \neq 0$. This implies that $\widetilde{\pi}_0^*\beta \neq 0$, as claimed.

We deal with $\widetilde{\pi}^*_{1728}$ basically in the same way. Observe that

$$(j-1728)^{-1} = (1728)^{-1}\left(\frac{c_4^3}{c_6^2} - 1\right)\,, \quad \mathrm{d}j^{-1} = (1728)^{-1}\frac{c_4^3}{c_6^2}\left(3\frac{\mathrm{d}c_4}{c_4} - 2\frac{\mathrm{d}c_6}{c_6}\right)\,.$$

Write

$$\beta = \sum_{i \geqslant 1}(j-1728)^{-i}\varphi_i + (j-1728)^{-i}\frac{\mathrm{d}(j-1728)^{-1}}{(j-1728)^{-1}} \wedge \varphi'_i\,, \tag{8.5}$$

and pull it back to U''.

Let s be the highest power of $(j-1728)^{-1}$ appearing in (8.5); then in $\mathrm{U}_{2s}/\mathrm{U}_{2s-1}$ we have

$$\pi_{1728}^{*}\beta = 1728^{-s}c_6^{-2s}\big(c_4^{3s}(\varphi_s + 3\mathrm{d}c_4\wedge\varphi_s')\big) - (1728)^{-s}c_6^{-2s}\left(2c_4^{3s}\frac{\mathrm{d}c_6}{c_6}\wedge\varphi_s'\right).$$

If $p \mid s$, by Theorem 5.7 the element above is non-zero if and only if the pair

$$\big((1728)^{s}c_4^{3s}(\varphi_s + 3\mathrm{d}c_4\wedge\varphi_s'), -(1728)^{s}2c_6^{2s}\varphi_s'\big) \in \Omega^n/Z^n \oplus \Omega^{n-1}/Z^{n-1}$$

is non-zero. Arguing exactly as in the case of $\widetilde{\pi}_0^{*}$, we can conclude that $\pi_{1728}^{*}\beta \neq 0$ in $\mathrm{U}_{2s}/\mathrm{U}_{2s-1}$. If $p \nmid s$, then $p \nmid 2s$, the form φ_s is not zero, and again by Theorem 5.7 we can conclude that $\pi_{1728}^{*}\beta \neq 0$ in $\mathrm{U}_{2s}/\mathrm{U}_{2s-1}$. This implies that $\widetilde{\pi}_{1728}^{*}\beta \neq 0$, as claimed. □

We are ready to compute the cohomological invariants of $\mathcal{M}_{1,1}$ with coefficients in H_{p^r}.

Theorem 8.19. *Suppose that the base field* $\mathbf{k}$ *has characteristic* $p > 3$. *Then we have*

$$\mathrm{Inv}^{\bullet}(\mathcal{M}_{1,1}, \mathrm{H}_{p^r}) \simeq \mathrm{Inv}^{\bullet}\big(\mathbb{A}^1, \mathrm{H}_{p^r}\big).$$

Proof. The cohomological invariants of $\mathcal{M}_{1,1}$ coincide with the invariants of $\mathcal{M}_{1,1} \smallsetminus \{j=0,1728\}$ that are unramified on U''. By Lemma 8.18 the tamely ramified elements are of the form $\beta + \delta\cdot\{j\} + \delta'\cdot\{j-1728\}$, where β is an invariant of $\mathbb{A}^1$ and δ, δ' belong to $\mathrm{H}_{p^r}^{n+1}(\mathbf{k})$. The term β is then unramified, so our claim would follow by showing that $\delta\cdot\{j\} + \delta'\cdot\{j-1728\}$ is unramified on U'' if and only if $\delta = \delta' = 0$.

The only divisors where the ramification can be non-zero are $\{c_4 = 0\}$ and $\{c_6 = 0\}$. The pullback of $\{j\}$ is equal to $\{c_4^3/\Delta\}$, whereas the pullback of $\{j-1728\}$ is equal to $\{c_6^2/\Delta\}$. This implies that the ramification at $\{c_4 = 0\}$ of the pullback of $\delta\cdot\{j\} + \delta'\cdot\{j-1728\}$ is equal to 3δ, whereas the ramification at $\{c_6 = 0\}$ is $2\delta'$. As both 2 and 3 are invertible in $\mathrm{H}_{p^r}^{\bullet}(F)$ for every field F, we deduce that $\delta\cdot\{j\} + \delta'\cdot\{j-1728\}$ is unramified if and only if $\delta = \delta' = 0$. □

8.5 Invariants with coefficients in $\mathbf{K}_{p^r}$

We conclude this section with the computation of the invariants of $\mathcal{M}_{1,1}$ with coefficients in K_{p^r}, contained in Theorem 8.2, stated at the beginning of the section.

Proof of Theorem 8.2. We divide the proof into three cases, depending on the characteristic of the ground field $\mathbf{k}$. Basically all the arguments used here already appeared in the computation of the cohomological invariants with coefficients in $\mathrm{H}_{p^r}^{\bullet}$.

For $\mathbf{k}$ of characteristic 2 or 3, we have that $\mathcal{M}_{1,1} \smallsetminus \{j=0\} \simeq \big(\mathbb{A}^1 \smallsetminus \{0\}\big) \times \mathrm{B}\mathbb{Z}/2$. It follows from Lemma 7.8 that

$$\mathrm{Inv}^{\bullet}\big(\big(\mathbb{A}^1 \smallsetminus \{0\}\big) \times \mathrm{B}\mathbb{Z}/2, \mathrm{K}_{p^r}\big) \simeq \mathrm{Inv}^{\bullet}(\mathbb{A}^1 \smallsetminus \{0\}, \mathrm{K}_{p^r}).$$

By (5.1) the invariants of $\mathbb{A}^1 \smallsetminus \{0\}$ are equal to

$$\mathrm{K}_{p^r}^{\bullet}(\mathbf{k}) \oplus \mathrm{K}_{p^r}^{\bullet-1}(\mathbf{k})\cdot\{j\}.$$

Suppose that $\mathbf{k}$ has characteristic 2. Let $U \to \mathcal{M}_{1,1}$ be the cover introduced in Section 8.2, and let $\pi\colon U \to \mathbb{A}^1$ be the map obtained by composing with the j-invariant $\mathcal{M}_{1,1} \to \mathbb{A}^1$. Then $\pi^{*}(\beta\cdot\{j\}) = \beta\cdot\{a_1^12/\Delta\}$, whose ramification along $\{a_1 = 0\}$ is equal to 12β. We deduce that $\pi^{*}\beta\cdot\{j\}$ is unramified if and only if $12\beta = 0$, which implies that

$$\mathrm{Inv}^{\bullet}(\mathcal{M}_{1,1}, \mathrm{K}_{2^r}) \simeq \begin{cases} \mathrm{K}_{2^r}^{\bullet}(\mathbf{k}) \oplus \mathrm{K}_2^{\bullet-1}\cdot\{\Delta\} & \text{if } r = 1, \\ \mathrm{K}_{2^r}^{\bullet}(\mathbf{k}) \oplus \mathrm{K}_4^{\bullet-1}\cdot\{\Delta\} & \text{if } r > 1. \end{cases}$$

Now suppose that $\mathbf{k}$ has characteristic 3. Then we have a cover $U' \to \mathcal{M}_{1,1}$ (see Section 8.3) and an induced map $\pi\colon U' \to \mathbb{A}^1$ such that $\pi^*(\beta \cdot \{j\}) = \beta \cdot \{b_2^6/\Delta\}$. The residue of this element at $\{b_2 = 0\}$ is 6β; hence we deduce

$$\mathrm{Inv}^\bullet(\mathcal{M}_{1,1}, \mathrm{K}_{3^r}) \simeq \mathrm{K}^\bullet_{3^r}(\mathbf{k}) \oplus \mathrm{K}_3^{\bullet-1} \cdot \{\Delta\}\,.$$

Finally, we deal with the case $p > 3$. We have

$$\mathcal{M}_{1,1} \smallsetminus \{j = 0, 1728\} \simeq \mathbb{A}^1 \smallsetminus \{0, 1728\} \times \mathrm{B}\mathbb{Z}/2\,.$$

Again by (5.1), we have

$$\mathrm{Inv}^\bullet\big(\mathbb{A}^1 \smallsetminus \{0, 1728\}, \mathrm{K}_{p^r}\big) \simeq \mathrm{K}^\bullet_{p^r}(\mathbf{k}) \oplus \mathrm{K}^{\bullet-1}_{p^r}(\mathbf{k}) \cdot \{j\} \oplus \mathrm{K}^{\bullet-1}_{p^r}(\mathbf{k}) \cdot \{j - 1728\}\,.$$

Denote by $U'' \to \mathcal{M}_{1,1}$ the cover introduced in Section 8.4, and let $\pi\colon U'' \to \mathbb{A}^1$ be the composition of the cover with the map given by the j-invariant. We have

$$\pi^*(\beta_0 \cdot \{j\} + \beta_1 \cdot \{j - 1728\}) = \beta_0 \cdot \{c_4^3/\Delta\} + \beta_1 \cdot \{c_6^2/\Delta\}\,.$$

For this element to be unramified at both $\{c_4 = 0\}$ and $\{c_6 = 0\}$, we must have $\beta_0 = \beta_1 = 0$, from which we conclude $\mathrm{Inv}^\bullet(\mathcal{M}_{1,1}, \mathrm{K}_{p^r}) = \mathrm{K}^\bullet_{p^r}(\mathbf{k})$. This finishes the proof. □

9. Mod ℓ computations

In this section we complete the computation of mod ℓ cohomological invariants of $\mathcal{M}_{1,1}$ from [DLP21a, Section 3]. This is needed for our description of $\mathrm{Br}(\mathcal{M}_{1,1})$. As in [DLP21a] we will be working with cohomological invariants with coefficients in a general cycle module, which were developed in the classical case by Gille and Hirsch [GH22]. The reader can refer to [DLP21a, Section 2,4,5] for an introduction to the mod ℓ theory.

We begin with a simple lemma, which in a way mirrors Lemma 7.7.

Lemma 9.1. *Let ℓ be a positive integer that is coprime to p, and let M be an ℓ-torsion cycle module. Then for any smooth scheme X, we have*

$$\mathrm{Inv}^\bullet(X \times \mathrm{B}\mathbb{Z}/p, \mathrm{M}) = \mathrm{Inv}^\bullet(X, \mathrm{M})\,.$$

Proof. This is an immediate consequence of homotopy invariance: $X \times \mathbb{A}^1 \to X \times \mathrm{B}\mathbb{Z}/p$ is a $\mathbb{G}_\mathrm{a}$-torsor, and consequently a smooth-Nisnevich cover, but on the other hand, $\mathrm{Inv}^\bullet\big(X \times \mathbb{A}^1, \mathrm{M}\big) \simeq \mathrm{Inv}^\bullet(X, \mathrm{M})$. □

The presentations of $\mathcal{M}_{1,1}$ as a quotient stack in characteristics 2 and 3 that we used for the mod p invariants will be enough to go through our computations in the mod ℓ case as well.

Proposition 9.2. *Let $\mathbf{k}$ be a field of characteristic* 2, *and let $\ell > 2$ be coprime to* 2. *Let M be an ℓ-torsion cycle module. Then*

$$\mathrm{Inv}^\bullet(\mathcal{M}_{1,1}, \mathrm{M}) = \mathrm{M}^\bullet(\mathbf{k}) \oplus \mathrm{M}^\bullet(\mathbf{k})_3 \cdot \{\Delta\}\,.$$

Proof. First observe that by Lemma 9.1 we have $\mathrm{Inv}^\bullet\big(\mathcal{M}_{1,1} \smallsetminus j^{-1}(0), \mathrm{M}\big) \simeq \mathrm{Inv}^\bullet\big(\mathbb{A}^1 \smallsetminus \{0\}, \mathrm{M}\big)$, which has been computed in [DLP21a, Lemma 2.22]. Specifically, if we write $\mathbb{A}^1 \smallsetminus \{0\} = \mathrm{Spec}\big(\mathbf{k}[j, j^{-1}]\big)$, we get that

$$\mathrm{Inv}^\bullet\big(\mathbb{A}^1 \smallsetminus \{0\}, \mathrm{M}\big) \simeq \mathrm{M}(\mathbf{k}) \oplus \mathrm{M}^\bullet(\mathbf{k}) \cdot \{j\}\,.$$

After identifying $(a_6')^{-1}$ with j, we deduce that

$$\mathrm{Inv}^\bullet\big(\mathcal{M}_{1,1} \smallsetminus j^{-1}(0), \mathrm{M}\big) \simeq \mathrm{M}^\bullet(\mathbf{k}) \oplus \mathrm{M}^\bullet(\mathbf{k}) \cdot \{j\}\,.$$

Consider the smooth-Nisnevich cover $U \to \mathcal{M}_{1,1}$ (notation as in Section 8.1): an invariant γ of $\mathcal{M}_{1,1} \smallsetminus j^{-1}(0)$ pulls back to an invariant of $U \setminus \{j = 0\}$. If it is unramified on $\{j = 0\}$, then it must come from $\mathcal{M}_{1,1}$ as it trivially glues. On the other hand, if γ comes from $\mathcal{M}_{1,1}$, it has to be unramified on $\{j = 0\}$ by definition.

The element $\{j\} \cdot \beta$ with $\beta \in \mathrm{M}^{\bullet-1}(\mathbf{k})$ pulls back to $\{a_1^{12}/\Delta \cdot \beta\}$, whose residue at $\{a_1 = 0\}$ is equal to 12β. As ℓ is coprime to 2, we get that $\{j\} \cdot \beta$ is unramified if and only if β is of 3-torsion. Finally, by the same reasoning, we have $\{j\} \cdot \beta = \{\Delta\} \cdot \beta$. □

Proposition 9.3. *Let $\mathbf{k}$ be a field of characteristic 3, and let $\ell > 1$ be coprime to 3. Let M be an ℓ-torsion cycle module. Then*

$$\mathrm{Inv}^\bullet(\mathcal{M}_{1,1}, \mathrm{M}) = \mathrm{M}^\bullet(\mathbf{k}) \oplus \{\Delta\} \cdot \mathrm{M}^\bullet(\mathbf{k})_4 \,.$$

Proof. Consider the smooth-Nisnevich cover $U \to \mathcal{M}_{1,1}$. First we observe that $\{\Delta\} \cdot \beta$ is clearly unramified on U and glues whenever β is of 4-torsion as $\mathrm{m}^*\{\Delta\} = \{u^{12}\Delta\} = 12\{u\} + \{\Delta\}$.

Now recall that we have an isomorphism

$$\mathcal{M}_{1,1} \setminus \{j = 0\} \simeq \left(\mathbb{A}^1 \smallsetminus \{0\}\right) \times \mathrm{B}\mathbb{Z}/2 \simeq \left(\mathbb{A}^1 \smallsetminus \{0\}\right) \times \mathrm{B}\mu_2$$

as $\mathbb{Z}/2 \simeq \mu_2$ when the characteristic of $\mathbf{k}$ is not 2. Applying [DLP21a, Proposition 4.3] we get

$$\mathrm{Inv}^\bullet(\mathcal{M}_{1,1}, \mathrm{M}) = (\mathrm{M}^\bullet(\mathbf{k}) \oplus \{j\} \cdot \mathrm{M}^\bullet) \oplus \alpha \cdot \mathrm{M}^\bullet(\mathbf{k})_2 \oplus \alpha \cdot \{j\} \cdot \mathrm{M}^\bullet(\mathbf{k})_2 \,.$$

Using the presentation

$$\mathcal{M}_{1,1}\smallsetminus \simeq \left[\mathrm{Spec}\left(\mathbf{k}\left[b_2, b_4, b_6, \Delta^{-1}\right]\right)/\mathbb{G}_\mathrm{m} \rtimes \mathbb{G}_\mathrm{a}\right],\ j = b_2^6/\Delta\,,$$

we see that $\{j\} = 6\{b_2\} - \{\Delta\}$ and $\alpha = \{b_2\}$. Note that $\alpha(F)$ should be seen as an element in $\mathrm{K}_2^1(F) = F^*/(F^*)^2$ and can only multiply elements of 2-torsion in $\mathrm{M}^\bullet(F)$.

Now, the cohomological invariants of $\mathcal{M}_{1,1} \smallsetminus \{j = 0\}$ coming from $\mathcal{M}_{1,1}$ are exactly those that, when pulled back to U, are unramified at $b_2 = 0$.

Consider a general element

$$\gamma = \beta_0 + \{j\} \cdot \beta_1 + \alpha \cdot \beta_2 + \alpha \cdot \{j\} \cdot \beta_3$$

with $\beta_0, \beta_1 \in \mathrm{M}^\bullet(\mathbf{k})$ and $\beta_2, \beta_3 \in \mathrm{M}^\bullet(\mathbf{k})_2$. We have

$$\alpha \cdot \{j\} \cdot \beta_3 = 6\{b_2\} \cdot \{b_2\} \cdot \beta - \{\Delta\} \cdot \{b_2\} \cdot \beta = \{\Delta\} \cdot \{b_2\} \cdot \beta\,,$$

which shows that the ramification of γ at $b_2 = 0$ is $6\beta_1 + \beta_2 + \{\Delta\} \cdot \beta_3$ as there are elliptic curves over $\mathbf{k}\left[t, t^{-1}\right]$ with $\Delta = t^{-1}$. The element $\{\Delta\} \cdot \beta_3$ can never cancel out with the other two, which come from the base field. Then we must have $\beta_3 = 0$. The only possibility for the remaining elements is that β_1 is of 4-torsion and $\beta_2 = -6\beta_1$. In other words, all unramified elements are of the form $\beta_0 + \{j\} \cdot \beta_1 + \alpha \cdot (6\beta_1)$ for some $\beta_0 \in \mathrm{M}^\bullet(\mathbf{k})$, $\beta_1 \in \mathrm{M}^\bullet(\mathbf{k})_4$. Finally, we have

$$\{j\} \cdot \beta_1 - \alpha \cdot (6\beta_1) = (6\{b_2\} - \{\Delta\}) \cdot \beta_1 - \{b_2\} \cdot (6\beta_1) = -\{\Delta\} \cdot \beta_1\,,$$

proving our claim. □

Corollary 9.4. *Write $c = \mathrm{char}(\mathbf{k})$ and ${}^c\mathrm{Br}'(\mathcal{M}_{1,1})$ for the subgroup of $\mathrm{Br}'(\mathcal{M}_{1,1})$ whose elements have order not divisible by c. Then*

$$\begin{cases} {}^2\mathrm{Br}'(\mathcal{M}_{1,1}) = {}^2\mathrm{Br}'(\mathbf{k}) \oplus \mathrm{H}^1(\mathbf{k}, \mathbb{Z}/3) & \text{if } c = 2\,, \\ {}^3\mathrm{Br}'(\mathcal{M}_{1,1}) = {}^3\mathrm{Br}'(\mathbf{k}) \oplus \mathrm{H}^1(\mathbf{k}, \mathbb{Z}/4) & \text{if } c = 3\,. \end{cases}$$

Proof. This is an immediate consequence of the computations above applied for $\mathrm{M} = \mathrm{H}_{\mathbb{Z}/\ell(-1)}$ and degree 2. □

10. The Brauer group of $\mathcal{M}_{1,1}$

We now have all the tools to compute the Brauer group of $\mathcal{M}_{1,1}$ over any field.

THEOREM 10.1. *Let $\mathcal{M}_{1,1}$ be the stack over $\mathrm{Spec}(\mathbf{k})$ parametrizing elliptic curves. If $\mathrm{char}(\mathbf{k}) = c$, the group $\mathrm{Br}(\mathcal{M}_{1,1})$ is*

$$\begin{cases} \mathrm{Br}(\mathbb{A}^1_{\mathbf{k}}) \oplus \mathrm{H}^1(\mathbf{k}, \mathbb{Z}/12\mathbb{Z}) & \text{if } c \neq 2\,, \\ \mathrm{Br}(\mathbb{A}^1_{\mathbf{k}}) \oplus \mathrm{H}^1(\mathbf{k}, \mathbb{Z}/3\mathbb{Z}) \oplus \mathrm{J} & \text{if } c = 2 \text{ and } x^2+x+1 \text{ irreducible over } \mathbf{k}\,, \\ \mathrm{Br}(\mathbb{A}^1_{\mathbf{k}}) \oplus \mathrm{H}^1(\mathbf{k}, \mathbb{Z}/12\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } c = 2 \text{ and } x^2+x+1 \text{ has a root in } \mathbf{k}\,, \end{cases}$$

where $\mathrm{H}^1(\mathbf{k}, \mathbb{Z}/4) \subset \mathrm{J} \subseteq \mathrm{H}^1(\mathbf{k}, \mathbb{Z}/8)$ sits in an exact sequence

$$0 \longrightarrow \mathrm{H}^1(\mathbf{k}, \mathbb{Z}/4) \longrightarrow \mathrm{J} \longrightarrow \mathbb{Z}/2 \longrightarrow 0\,.$$

Proof. First we note that by [Shi19, Lemma 3.1] the Brauer map $\mathrm{Br}(\mathcal{M}_{1,1}) \to \mathrm{Br}'(\mathcal{M}_{1,1})$ is surjective, so we only need to worry about computing the latter.

The ℓ-torsion is computed in [DLP21a, Corollary 3.2] for $c \neq 2$, 3 and in Corollary 9.4 for $c = 2$, 3. The p-torsion for c equal to, respectively, 2, 3 and $p > 3$ is obtained by restricting Theorem 8.1 to degree 2. □

Now let us look at the generators. For $c \neq 2$ the group is generated by the elements coming from the base field and elements of the form $\{\Delta\} \cdot \beta$ with $\beta \in \mathrm{H}^1(\mathbf{k}, \mathbb{Z}/12)$. Write $\mathrm{H}^1(\mathbf{k}, \mathbb{Z}/12) = \mathrm{H}^1(\mathbf{k}, \mathbb{Z}/4) \oplus \mathrm{H}^1(\mathbf{k}, \mathbb{Z}/3)$ and $\{\Delta\} = \{\Delta\}_4 + \{\Delta\}_3$. Then by either [DLP21a, Lemma 2.18] or Proposition 4.4, we know that $\{\Delta\}_4$ and $\{\Delta\}_3$ come from $\mathrm{H}^1_{\mathrm{fl}}(\mathcal{M}_{1,1}, \mu_n)$, $n = 3, 4$ (note that for n not divisible by c, that is just the regular étale cohomology group), and consequently $\{\Delta\}$ comes from $\mathrm{H}^1_{\mathrm{fl}}(\mathcal{M}_{1,1}, \mu_{12})$. Thus we can conclude that the elements of the form $\{\Delta\} \cdot \beta$ are all cyclic algebras by [AM20, Lemma 2.10].

When $c = 2$, the generators are of the form $\{\Delta\} \cdot \beta$, $[\alpha, \Delta\}$ (here we are identifying the generator with its pullback to U) or a sum of the two. Any element of the first type is a cyclic algebra by the same reasoning as above.

We claim that $[\alpha, \Delta\}$ is not a cyclic algebra. First, note that $[\alpha, \Delta\}$ does not go to zero if we pass to $\overline{\mathbf{k}}$, so it suffices to show that it is not a cyclic algebra when the base field is algebraically closed. It is a 2-torsion element, so saying that it is a cyclic algebra is equivalent to saying that there must exist $h \in \mathrm{H}^1_{\mathrm{fl}}(\mathcal{M}_{1,1}, \mu_2)$ and $h' \in \mathrm{H}^1(\mathcal{M}_{1,1}, \mathbb{Z}/2)$ with $h \cdot h' = [\alpha, \Delta\}$. When $\mathbf{k} = \overline{\mathbf{k}}$, we have

$$\mathrm{H}^1_{\mathrm{fl}}(\mathcal{M}_{1,1}, \mu_2) = \{\Delta\} \cdot \mathbb{Z}/2\,, \quad \mathrm{H}^1(\mathcal{M}_{1,1}, \mathbb{Z}/2) = \mathrm{H}^1(\mathbb{A}^1_{\mathbf{k}}, \mathbb{Z}/2)\,.$$

If we pull everything back to $\mathcal{M}_{1,1} \smallsetminus \{j = 0\} \simeq (\mathbb{A}^1 \smallsetminus \{0\}) \times \mathrm{B}\mathbb{Z}/2$, we immediately see that $\{\Delta\} = \{j\} \in \mathrm{H}^1_{\mathrm{fl}}(\mathcal{M}_{1,1}, \mu_2)$, which shows that

$$\mathrm{H}^1_{\mathrm{fl}}(\mathcal{M}_{1,1}, \mu_2) \subset \mathrm{H}^1_{\mathrm{fl}}(\mathbb{A}^1 \smallsetminus \{0\}, \mu_2) \subset \mathrm{H}^1_{\mathrm{fl}}(\mathcal{M}_{1,1} \smallsetminus \{j = 0\}, \mu_2)$$

and

$$\mathrm{H}^1(\mathcal{M}_{1,1}, \mathbb{Z}/2) \subset \mathrm{H}^1(\mathbb{A}^1 \smallsetminus \{0\}, \mathbb{Z}/2) \subset \mathrm{H}^1(\mathcal{M}_{1,1} \smallsetminus \{j = 0\}, \mathbb{Z}/2)\,.$$

But then we must have $h \cdot h' \in \mathrm{Br}'(\mathbb{A}^1 \smallsetminus \{0\})_2$, which is zero when $\mathbf{k} = \overline{\mathbf{k}}$ by Corollary 7.3 (note that $\Omega_{\mathbf{k}} = 0$ as $\Omega_{\mathbf{k},\log}$ is 2-divisible), giving a contradiction.

Remark 10.2. A natural question left open in Shin's results [Shi19] is, given a finite field $\mathbf{k}$ of characteristic 2 not containing a third root of unit ζ, to compute the restriction map

$$\mathrm{Br}(\mathcal{M}_{1,1,\mathbf{k}}) \longrightarrow \mathrm{Br}(\mathcal{M}_{1,1,\mathbf{k}(\zeta)})\,.$$

Our description makes the task quite simple: we just need to understand the restriction map

$$\mathrm{H}^1(\mathbf{k}, \mathbb{Z}/8\mathbb{Z}) \longrightarrow \mathrm{H}^1(\mathbf{k}(\zeta), \mathbb{Z}/8\mathbb{Z})\,.$$

Recall that $\mathrm{H}^1(F, \mathbb{Z}/8\mathbb{Z})$ is equal to the quotient of $W_3(F)$ by the subgroup of elements of the form $\left[a_1^2, a_2^2, a_3^2\right] - [a_1, a_2, a_3]$. Using the definition of Witt vectors, we get the formula

$$[x_1, x_2, x_3] + [y_1, y_2, y_3] = [x_1 + y_1, x_2 + y_2 + x_1y_1, S_3(\underline{x}, \underline{y})]\,,$$

where

$$S_3(\underline{x}, \underline{y}) = x_3 + y_3 + x_2y_2 + x_2x_1y_1 + y_2x_1y_1 + x_1^2y_1^2 + x_1^3y_1 + y_1^3x_1\,.$$

We know by standard Galois theory that both groups are isomorphic to $\mathbb{Z}/8\mathbb{Z}$, and the first group is generated by $[1, 0, 0]$ as $4[1, 0, 0] = [0, 0, 1] \neq 0$. So to understand the map, we just have to check the class of $[1, 0, 0]$ in $\mathrm{H}^1(\mathbf{k}(\zeta), \mathbb{Z}/8\mathbb{Z})$. Now, observe that

$$\left[\zeta^2, \zeta^2, \zeta^2\right] - [\zeta, \zeta, \zeta] = [\zeta + 1, \zeta + 1, \zeta + 1] + [\zeta, 1, \zeta + 1] = [1, 0, 0]\,,$$

which implies that the class of $[1, 0, 0]$ maps to zero, and as a degree 2 extension will induce an isomorphism on mod 3 Galois cohomology, we conclude that the map of Brauer groups maps $\mathbb{Z}/24\mathbb{Z} = \mathrm{Br}(\mathcal{M}_{1,1,\mathbf{k}})$ to $\mathbb{Z}/3\mathbb{Z} \subset \mathrm{Br}(\mathcal{M}_{1,1,\mathbf{k}(\zeta)}) = \mathbb{Z}/12\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

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