

Research Article

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Characterization of sets of finite local and non-local perimeter via non-local heat equation

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Abstract: In this paper we provide a characterization of sets with finite local and non-local perimeter via a Γ -convergence result. As an application, we give a short proof of the isoperimetric inequality, both in the local and in the non-local cases.

Keywords: Gamma convergence, fractional heat equation, sets of finite perimeter

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Introduction

In this article we analyze the asymptotic behavior of the energy

$$\mathcal{E}_t^s(E) := \int_E \int_{E^c} P^s(x-y, t) dy dx \quad (0.1)$$

as $t \rightarrow 0^+$ for all $s \in (0, 1)$ and where $P^s(z, t)$ is the fundamental solution of the fractional heat equation. Since the kernel satisfies $\|P^s(\cdot, t)\|_{L^1(\mathbb{R}^n)} = 1$ for all $t > 0$, it is straightforward to verify that the functional in (0.1) is well-defined for all measurable sets E such that E or E^c has finite measure. A simple computation shows that $\mathcal{E}_t^s(E) \rightarrow 0$ as $t \rightarrow 0$ (see the proof of Theorem 3.1) whenever E is a set of finite measure. Our scope in this article is to determine the first non-trivial terms in the power series expansion of such a functional under suitable assumptions. To this end, we introduce the function

$$g_s(t) := \begin{cases} t & \text{if } s \in (0, \frac{1}{2}), \\ t \log t & \text{if } s = \frac{1}{2}, \\ t^{\frac{1}{2s}} & \text{if } s \in (\frac{1}{2}, 1). \end{cases} \quad (0.2)$$

The function g_s describes the leading order of $\mathcal{E}_t^s$ as $t \rightarrow 0$. Let us just synthetically state our (meta-)theorem as follows.

Theorem 0.1. *Let $n \geq 2$ and $s \in (0, 1)$ and define $\mathbb{X}^s := \text{BV}(\mathbb{R}^n)$ if $s \geq \frac{1}{2}$ and $\mathbb{X}^s := \text{H}^s(\mathbb{R}^n)$ if $s < \frac{1}{2}$. Then*

$$\Gamma\text{-}\lim_{t \rightarrow 0} \frac{\mathcal{E}_t^s(E)}{g_s(t)} = \begin{cases} \Gamma^{n,s} P_{\min\{2s, 1\}}(E) & \text{if } \chi_E \in \mathbb{X}^s, \\ +\infty & \text{otherwise.} \end{cases}$$

In the above theorem, $P_{2s}(E)$ denotes the fractional perimeter of E , defined in (1.3), $P_1(E)$ is the De Giorgi perimeter of the set E defined in (1.1) and the constant $\Gamma^{n,s}$ is given in (4.1).

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History of the problem and motivation

Without aiming at completeness, we now briefly recall related results from the literature. Our result is inspired on an idea of De Giorgi (see [11]), where the author used the heat kernel to define the perimeter of a set. Later, Ledoux in [24] build upon this idea to provide a new proof of the isoperimetric inequality using the heat kernel semigroup, in both the Euclidean and Gaussian setting. The result in [11] was generalized in the context of Riemannian manifolds in [27] (see also [8] and [15] for Carnot groups), where the authors provided a generalization of the asymptotic behavior of the heat semigroup shown in [24] and also a characterization for sets of finite perimeter via the heat semigroup. This line of research has been developed in several directions. We mention the contribution in [20], where the authors consider the case of Riemannian manifolds with unbounded Ricci curvature, and [5], where the authors prove a characterization of Sobolev space using, as energy, the s -Gagliardo seminorms, also providing pointwise convergence result as the parameter $s \rightarrow 1$. In [10], the authors analyze the Γ -convergence for the s -Gagliardo seminorms as $s \rightarrow 1$, while in [21] the second-order Γ -convergence for the second order of these seminorms is investigated. In [13], it is shown that fractional perimeters can be rigorously obtained as limits of renormalized Riesz energies by subtracting the infinite core energy and letting the core radius go to zero (see also [22]). We also mention the very recent manuscript [19], where the authors study a problem closely related to the subject of the present paper.

When $s = 1$, the functions $\mathcal{E}_r^1$ has been used to construct weak solution to the mean curvature equation. More precisely, in Bence, Merriman, and Osher in [26] constructed a time discrete approximation of the mean curvature flow, which is known in literature as thresholding scheme or MBO scheme. Laux and Otto in [23] proved the convergence of the thresholding scheme towards a variational solution of the mean curvature flow under suitable assumptions. It seems natural to try to extend such a result when considering the non-local heat equation. Our Γ -convergence result is a first step towards the understanding of the non-local thresholding scheme from a variational point of view. We also mention that in [7], Caffarelli and Souganidis studied the convergence of non-local threshold dynamics, showing that they approximate viscosity solutions to the mean curvature and non-local mean curvature flow for $s \in [\frac{1}{2}, 1)$ and $s \in (0, \frac{1}{2})$ respectively.

In this paper we extend the result of De Giorgi in [11]. In fact, we also provide a characterization of sets of finite local and non-local perimeter via the fractional heat semigroup.

Outline of the proof

We now describe the strategy used to prove Theorem 0.1. Since we aim to prove a Γ -convergence result, we need to prove: compactness, the Γ -lim inf inequality, the Γ -lim sup inequality.

In Section 2, we prove compactness properties. Here we encounter the first difficulty, as the first non-trivial order in the power expansion of $\mathcal{E}_t^s$ depends on the parameter s (see the definition of the function $g_s(t)$ in (0.2)). Therefore the first step to do is to find the correct space-time scaling that ensures some basic properties, such as compactness. Once the function g_s is detected, we prove the compactness Theorem 2.1. When $s \neq \frac{1}{2}$, the proof relies on properties of the solution to the fractional heat equation. When $s = \frac{1}{2}$, due to the logarithmic behavior (see the definition of the function g_s), we adopt a completely different strategy, inspired by [12].

After establishing good compactness properties, we prove Γ -lim inf inequality and the Γ -lim sup inequality. To this end, we distinguish between two cases, as the proofs differ significantly.

In Section 3 we study the case $s < \frac{1}{2}$. In this range, we do not need any tool geometric measure theory, as the proof relies Fourier analysis. The main reason is that the fractional Laplacian of a characteristic function, for $s < \frac{1}{2}$, is not merely a measure but an actual function. Therefore, a strategy based on Fourier analysis provides a more direct approach – indeed, this is the case.

We continue in Section 4, where we address the problem with $s \geq \frac{1}{2}$. In this range, the nature of the functional changes, losing part of its non-local character. Consequently, we must employ tools from geometric measure theory. To prove the Γ -lim inf inequality, we follow a well established strategy based on a blow up argument, originally used in this context by Fonseca and Muller in [16]. For the Γ -lim sup inequality, we use the approach from [2], which studies the Γ -convergence of the fractional perimeter as the differentiation

parameter s tends to one (see also [9], [14] for the Gaussian case). Note that the fractional heat kernel is not explicit when $s > \frac{1}{2}$. Therefore, the computation of the constant $\Gamma^{n,s}$ is not straightforward and to deal with it we once again need to carefully use Fourier analysis. We stress that, although some decay properties are known (see (1.7)), such a knowledge is not enough to prove Γ -convergence, because the sole use (1.7) would provide different constant in the Γ -lim inf and Γ -lim sup.

The case $s > \frac{1}{2}$ could, in principle, be recovered from [1], but in our case we are able to provide the explicit form of the limit functional. For this reason, we believe it is worth highlighting this result here. We then proceed to prove our convergence result. We do not enter into details of the proof of the equality $\Gamma^{n,s} = \Gamma_{n,s}$ (see (4.1) and (4.14) for the definition of these constants) as the proof follows without modification from a standard argument (gluing lemma and calibration) as in [4].

Finally, in Section 5 we use our result to prove the isoperimetric inequality, both in the local and non-local framework. The proof relies on Hardy rearrangement inequality (the approximating kernels are L^1 normalized-functions in the space variable) and our Γ -convergence result.

We conclude this introduction with a brief heuristic interpretation of our result. By Bochner subordination formula, one way to obtain the fractional laplacian of a function is

$$(-\Delta)^s u = \int_0^\infty \frac{e^{t\Delta} u - u}{t^{1+s}} dt,$$

where $e^{t\Delta} u$ is the heat semigroup. Thus, if $s < \frac{1}{2}$ and $u = \chi_E$,

$$\begin{aligned} P_{2s}(E) &= \int_{\mathbb{R}^n} u(-\Delta)^s u dx = \frac{1}{\Gamma(-s)} \int_0^\infty \frac{1}{t^{1+s}} \int_{\mathbb{R}^n} u(e^{t\Delta} u - u) dx dt \\ &= \frac{1}{\Gamma(-s)} \int_0^\infty \frac{\|e^{\frac{t}{2}\Delta} u\|_{L^2}^2 - \|u\|_{L^2}^2}{t^{1+s}} dt. \end{aligned}$$

Hence, the fractional perimeter of a set E can be computed by evolving the characteristic function of E via the heat equation and then computing the s -derivative (with respect to time) at time 0 of the energy of the evolving solution at time $\frac{t}{2}$. One of the results of this paper can be formally written as:

$$\Gamma_{n,s} P(E) = \lim_{t \rightarrow 0^+} \frac{1}{t} \int_{\mathbb{R}^n} (u - e^{-t^{2s}(-\Delta)^s} u) dx = \lim_{t \rightarrow 0^+} \frac{\|u\|_{L^2}^2 - \|e^{-\frac{t^{2s}}{2}(-\Delta)^s} u\|_{L^2}^2}{t}$$

for $s > \frac{1}{2}$. In this sense, our result resembles the famous Bochner subordination formula for the perimeter of a set: we show that the perimeter of a set E can be recovered by evolving the characteristic function of E via the fractional heat equation, evaluated at time $t = t^{2s}$, and then computing the derivative at time 0 of the energy of the evolving solution at time $\frac{1}{2}t^{2s}$. Thus, we can say that for a sufficiently smooth measurable set E , if we first compute the evolution via *local* heat equation and then take a *non-local* time derivative of an appropriate energy, we recover the *non-local* perimeter. Conversely, if we first compute the evolution of χ_E via *non-local* heat equation and then take a *local* time derivative of an appropriate energy, we recover the *local* perimeter.

1 Notation and preliminary results

1.1 Sets of finite perimeter

We recall some notation and basic results of geometric measure theory from [3]. We denote with $M(\mathbb{R}^n)$ the set of all Lebesgue measurable subset of $\mathbb{R}^n$. For all $k \in \mathbb{N}$ we denote with ω_k the Lebesgue measure of the unit ball of $\mathbb{R}^k$. For every $E \in M(\mathbb{R}^n)$ we denote by $P(E)$ the De Giorgi perimeter of E defined by

$$P(E) = \sup \left\{ \int_E \operatorname{div} \phi(x) dx : \phi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n) \text{ and } \|\phi\|_\infty \leq 1 \right\}. \quad (1.1)$$

This means that a set E is a finite perimeter if $\chi_E \in \text{BV}(\mathbb{R}^n)$. We also recall the classic compactness theorem in BV.

Theorem 1.1 (Compactness in BV). *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\{u_i\}_{i \in \mathbb{N}} \subset \text{BV}_{loc}(\Omega)$ with*

$$\sup_{i \in \mathbb{N}} \left\{ \int_A |u_i(x)| dx + |Du_i|(A) \right\} < +\infty \quad \text{for all } A \subset\subset \Omega.$$

Then, there exist a subsequence $\{i_k\}_{k \in \mathbb{N}}$ and a function $u \in \text{BV}_{loc}(\Omega)$ such that $u_{i_k} \rightarrow u$ in $L^1_{loc}(\Omega)$ as $k \rightarrow +\infty$.

For every $E \subset \mathbb{R}^n$ with finite perimeter, the set $\partial^* E \subset \mathbb{R}^n$ identifies the reduced boundary of E and the Borel measurable map $\nu_E : \partial^* E \rightarrow \mathbb{R}^n$ the measure theoretic outer normal vector field.

1.2 Non-local perimeter

The non-local perimeter of a set E is defined via the fractional laplacian of the function χ_E . Given a function $u \in L^2(\mathbb{R}^n)$, the Fourier transform of u , denoted as $\mathcal{F}[u(\cdot)](\cdot)$, is defined as

$$\mathcal{F}[u(\cdot)](\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} u(x) e^{-i\langle x, \xi \rangle} dx \quad \text{for all } \xi \in \mathbb{R}^n.$$

The Fourier transform is invertible $L^2(\mathbb{R}^n)$ and we denote its inverse by

$$\mathcal{F}^{-1}[\nu(\cdot)](x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \nu(\xi) e^{i\langle x, \xi \rangle} d\xi.$$

Let $s \in (0, 1)$ and $u \in C^\infty(\mathbb{R}^n)$ we define the s -fractional Laplacian of u as

$$(-\Delta)^s u(x) := C_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy,$$

where

$$C_{n,s} := \left(\int_{\mathbb{R}^n} \frac{1 - \cos(h_1)}{|h|^{n+2s}} dh \right)^{-1} \quad (1.2)$$

with $h = (h_1, \dots, h_n) \in \mathbb{R}^n$. Observe that for all $u : \mathbb{R}^n \rightarrow \mathbb{R}$ smooth, the fractional Laplace admits a representation in terms of Fourier transform via the following formula:

$$\mathcal{F}[(-\Delta)^s u(\cdot)](\xi) = |\xi|^{2s} \mathcal{F}[u(\cdot)](\xi).$$

For all $s \in (0, 1)$ we define the s -fractional Gagliardo seminorm of a function $u : \mathbb{R}^n \rightarrow \mathbb{R}$ as

$$[u]_{H^s(\mathbb{R}^n)}^2 := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy = \frac{1}{2C_{n,s}} \int_{\mathbb{R}^n} u(-\Delta)^s u dx$$

and we denote with $H^s(\mathbb{R}^n)$ the space of the functions $u \in L^2(\mathbb{R}^n)$ such that $[u]_{H^s(\mathbb{R}^n)} < +\infty$. The fractional Gagliardo seminorm can be written in Fourier as

$$[u]_{H^s(\mathbb{R}^n)}^2 = 2C_{n,s}^{-1} \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}[u](\xi)|^2 d\xi.$$

If $s \in (0, \frac{1}{2})$ and $E \in \mathcal{M}(\mathbb{R}^n)$, we define the $2s$ -fractional perimeter as

$$P_{2s}(E) := \int_E \int_{E^c} \frac{1}{|x - y|^{n+2s}} dx dy \quad (1.3)$$

and we observe that $P_{2s}(E) = \frac{1}{2} [\chi_E]_{H^s(\mathbb{R}^n)}^2$. To conclude this subsection, we recall a compactness theorem for fractional Sobolev spaces (see [25, Theorem 6.13] for a proof).

Theorem 1.2. *Let $\{u_n\}_{n \in \mathbb{N}}$ be a bounded sequence in $H^s(\mathbb{R}^n)$, i.e.,*

$$\sup_{n \in \mathbb{N}} \{ \|u_n\|_{L^2(\mathbb{R}^n)} + [u_n]_{H^s(\mathbb{R}^n)} \} < +\infty.$$

There exist a subsequence $n_k \subset \mathbb{N}$ and a function $u \in H^s(\mathbb{R}^n)$ such that $u_{n_k} \rightarrow u$ in L^2_{loc} .

1.3 Fractional heat equation

Here we collect some known facts about fractional heat equation and refer to [17] for a detailed discussion. Given $u_0 \in L^\infty(\mathbb{R}^n)$ let us consider the solution to fractional heat equation

$$\begin{cases} \partial_t v + (-\Delta)^s v = 0, \\ v(x, 0) = u_0(x). \end{cases} \quad (1.4)$$

We denote by $P^s(z, t)$ the fundamental solution of the fractional heat equation. More precisely, $P^s(z, t)$ is the solution of the Cauchy problem

$$\begin{cases} \partial_t P^s(z, t) + (-\Delta)^s P^s(z, t) = 0 & t > 0, z \in \mathbb{R}^n, \\ P^s(\cdot, 0) = \delta_0, \end{cases} \quad \int_{\mathbb{R}^n} P^s(z, t) dz = 1,$$

where δ_0 is the Dirac delta measure in $\mathbb{R}^n$ with center in $0 \in \mathbb{R}^n$. It is well known that for $u_0 \in H^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ the solution of the problem (1.4) can be written as a convolution between the fundamental solution and the initial data, i.e.,

$$v(x, t) = \int_{\mathbb{R}^n} P^s(x - y, t) u_0(y) dy \quad (1.5)$$

and $v \in C([0, +\infty), H^s(\mathbb{R}^n)) \cap C^1([0, +\infty), L^2(\mathbb{R}^n)) \cap C^\infty((0, +\infty), C^\infty(\mathbb{R}^n))$. Moreover, the function $P^s(z, t)$ satisfies

$$\mathcal{F}[P^s(\cdot, t)](\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-t|\xi|^{2s}}$$

for $s \in (0, 1)$. The function P^s cannot be explicitly computed unless in the particular case $s = \frac{1}{2}$, for which we know

$$P^{\frac{1}{2}}(z, t) = \frac{t}{(|z|^2 + t^2)^{\frac{n+1}{2}}}. \quad (1.6)$$

When $s \neq \frac{1}{2}$, it is well known that $P^s(z, t)$ satisfies the decay estimates

$$c(n, s) \left(\frac{t}{|z|^{n+2s}} \wedge t^{-\frac{n}{2s}} \right) \leq P^s(z, t) \leq C(n, s) \left(\frac{t}{|z|^{n+2s}} \wedge t^{-\frac{n}{2s}} \right) \quad (1.7)$$

and the scaling property

$$P^s(z, t) = t^{-\frac{n}{2s}} P^s(z t^{-\frac{1}{2s}}, 1). \quad (1.8)$$

Finally, we also need the estimate for the gradient of the fractional heat kernel

$$|\nabla P^s(x - y, t)| \leq \frac{C}{t^{\frac{1}{2s}}} P^s(x - y, t). \quad (1.9)$$

1.4 Technical results

In order to prove compactness when $s = \frac{1}{2}$, we recall the following technical lemma.

Lemma 1.3 ([18, Lemma 15]). *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz continuous boundary and $|\Omega| = 1$. Let $\{\rho_\varepsilon\}_{\varepsilon \in (0,1)}$ be the standard family of Friedrichs mollifiers with support in $B(0, 1)$. For every $\eta \in (0, 1)$ there exists $C = C(n, \Omega, \rho, \eta) > 0$ such that*

$$\inf \left\{ \frac{1}{\varepsilon} \int_A \int_{\Omega \setminus A} \rho_\varepsilon(|x - y|) dx dy : A \subset \Omega, |A| \in (\eta, 1 - \eta), t \in (0, 1) \right\} \geq C.$$

Lemma 1.3 it is proved in [18, Lemma 15] for $n = 2$ and $\Omega = (-\frac{1}{2}, \frac{1}{2})^2$, but the argument works in every dimension and for all Ω with Lipschitz continuous boundary.

Finally, in order to compute the constant in the Γ -limit for $s = \frac{1}{2}$, we need the following simple lemma.

Lemma 1.4. Let $a, b \in \mathbb{R}$ and $k \in \mathbb{N}$ such that $b > -k$ and $a > k + b$. Then

$$\int_{\mathbb{R}^k} \frac{|x|^b}{(1 + |x|^2)^{\frac{a}{2}}} dx = \frac{\omega_{k-1}}{2} \frac{\Gamma(\frac{a+k}{2})\Gamma(\frac{a-b-k}{2})}{\Gamma(\frac{a+b}{2})}. \quad (1.10)$$

In particular,

$$c_n := \int_{\mathbb{R}^{n-1}} \frac{1}{(1 + |x|^2)^{\frac{n+1}{2}}} dx = \frac{\omega_{n-2}}{2} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} := \Gamma^{n, \frac{1}{2}}, \quad (1.11)$$

where $\Gamma(t) := \int_0^{+\infty} r^{t-1} e^{-r} dr$ is the Gamma function.

Before proving Lemma 1.4, we recall the definition of Euler's beta function

$$B : (0, +\infty) \times (0, +\infty) \rightarrow \mathbb{R}, \quad B(x, y) := 2 \int_0^{\frac{\pi}{2}} (\cos(\theta))^{2x-1} (\sin(\theta))^{2y-1} d\theta.$$

The beta and the gamma function are connected by the identity

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \quad (1.12)$$

Proof of Lemma 1.4. We observe that the function $h(x) := \frac{|x|^b}{(1+|x|^2)^{\frac{a}{2}}}$ belong $L^1(\mathbb{R}^k)$. Integrating in polar coordinate and by (1.12) we have that

$$\begin{aligned} \int_{\mathbb{R}^k} \frac{|x|^b}{(1 + |x|^2)^{\frac{a}{2}}} dx &= \omega_{k-1} \int_0^{+\infty} \frac{\rho^{b+k-1}}{(1 + \rho^2)^{\frac{a}{2}}} d\rho \\ &\stackrel{\rho=\tan(\eta)}{=} \omega_{k-1} \int_0^{\frac{\pi}{2}} \frac{(\tan(\eta))^{b+k-1}}{(1 + \tan^2(\eta))^{\frac{a-2}{2}}} d\eta = \omega_{k-1} \int_0^{\frac{\pi}{2}} (\sin(\eta))^{b+k-1} (\cos(\eta))^{a-b-k-1} d\eta \\ &= \frac{\omega_{k-1}}{2} B\left(\frac{b+k}{2}, \frac{a-b-k}{2}\right) = \frac{\omega_{k-1}}{2} \frac{\Gamma(\frac{a+k}{2})\Gamma(\frac{a-b-k}{2})}{\Gamma(\frac{a+b}{2})}, \end{aligned}$$

hence (1.10). Formula (1.11) follows by (1.10) with $b = 0$ and $k = n - 1$ and $a = n + 1$. $\square$

2 Compactness

In this section we study compactness properties of sequences $\{E_i\}_{i \in \mathbb{N}}$ of equibounded satisfying

$$\sup_i \frac{1}{g_s(t_i)} \mathcal{E}_{t_i}^s(E_i) < \infty, \quad t_i \rightarrow 0^+ \quad \text{as } i \rightarrow +\infty,$$

where $\mathcal{E}_t^s(\cdot)$ is defined in (0.1) and $g_s(t)$ is defined in (0.2).

This compactness properties suggests the candidate Γ -limit: for $s \in (0, \frac{1}{2})$ such a candidate is the $2s$ -fractional perimeter defined in (1.3) while for $s \geq \frac{1}{2}$ is the classical perimeter, defined in (1.1).

The main theorem of this section is the following.

Theorem 2.1. Let $U \subset \mathbb{R}^n$ be an open bounded set and $E_{t_i} \subset U$ for all $i \in \mathbb{N}$, with $t_i \rightarrow 0^+$. Assume that

$$\frac{\mathcal{E}_{t_i}^s(\chi_{E_{t_i}})}{g_s(t_i)} \leq M. \quad (2.1)$$

There exists $E \subset U$ such that, up to a subsequence, $\chi_{E_{t_i}} \rightarrow \chi_E$ in $L^1(\mathbb{R}^n)$. Moreover, if $s \in (0, \frac{1}{2})$, we have $P_{2s}(E) < \infty$, while for $s \in [\frac{1}{2}, 1)$ it holds $P(E) < \infty$.

As already observed in the introduction, for $s \neq \frac{1}{2}$ our proof strongly relies on properties of solutions to the fractional heat equation, while for $s = \frac{1}{2}$ a more delicate geometric argument is needed to catch the logarithmic behavior of $\mathcal{E}_{t_i}(E_i)$ as the sequence $t_i \rightarrow 0$.

2.1 Proof of Theorem 2.1 when $s \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$

Compactness for $2s < 1$. Let $E_i \subset \mathbb{R}^n$ and t_i as in (2.1) and set $u_i = \chi_{E_i}$, $u_i(\cdot, t) = P^s(\cdot, t) * \chi_{E_i}$, and $v_i(\cdot) = u_i(\cdot, t_i)$. We start observing two basic properties of solutions to (1.4): the functions

$$t \in (0, \infty) \rightarrow \|u(x, t)\|_{L^2(\mathbb{R}^n)} \quad \text{and} \quad t \in (0, \infty) \rightarrow \|u(x, t)\|_{H^s(\mathbb{R}^n)} \quad (2.2)$$

are not increasing. The proof of this fact is quite immediate, and we skip it. Observe that

$$\begin{aligned} \mathcal{E}_{t_i}^s(\chi_{E_i}) &= \int_{\mathbb{R}^n} (1 - \chi_{E_i}(x)) u_i(x, t_i) dx = \int_{\mathbb{R}^n} u_i(x, 0)^2 - u_i(x, \frac{t_i}{2})^2 dx \\ &= - \int_0^{\frac{t_i}{2}} \frac{d}{d\tau} \int_{\mathbb{R}^n} u_i^2(x, \tau) dx d\tau = -2 \int_0^{\frac{t_i}{2}} \int_{\mathbb{R}^n} u_i(x, \tau) \partial_\tau u_i(x, \tau) dx d\tau \\ &= 2 \int_0^{\frac{t_i}{2}} [u_i(\cdot, \tau)]_{H^s(\mathbb{R}^n)}^2 d\tau, \end{aligned}$$

where in the last equality we have used $[f]_{H^s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} f(x)(-\Delta)^s f(x) dx$. Therefore, using (2.2) we obtain

$$\mathcal{E}_{t_i}^s(\chi_{E_i}) \geq \frac{t_i}{2} [u_i(\cdot, t_i)]_{H^s(\mathbb{R}^n)}^2.$$

Recalling the definition of v_i , the above inequality joint with the assumption (2.1) yields

$$\|v_i\|_{H^s(\mathbb{R}^n)} \leq \frac{\mathcal{E}_{t_i}^s(\chi_{E_i})}{t_i} \leq M.$$

Moreover, since $E_i \subset U$ and U is bounded, we also have

$$\|v_i\|_{L^2(\mathbb{R}^n)} \leq \|u_i\|_{L^2(\mathbb{R}^n)} \leq |U|^{\frac{1}{2}}.$$

Therefore, v_i is a bounded sequence in $H^s(\mathbb{R}^n)$. Thus, by Theorem 1.2 it contains a subsequence converging in $L_{loc}^2(\mathbb{R}^n)$. With an abuse of notation, we still denote such a subsequence as v_i . It is immediate to show that also $u_i \rightarrow v$ in L_{loc}^2 and therefore, still up to a subsequence, $\chi_{E_i} \rightarrow v$ almost everywhere, hence v takes only the values 0 and 1 and is the characteristic function of a set E . To conclude, we observe that $P_{2s}(E) < \infty$ is a simple consequence of the semicontinuity of $u \rightarrow \|u\|_{H^s(\mathbb{R}^n)}$ with respect to the weak topology. $\square$

Compactness for $2s > 1$. In this case the proof is similar. In fact, setting again $v_i(\cdot) = P^s(\cdot, t_i) * \chi_{E_i}$, we have that v_i is smooth and

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla v_i| &= \int_{\mathbb{R}^n} |\nabla P^s(\cdot, t_i) * \chi_{E_i}| dy dx = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \nabla P^s(x - y, t_i) \chi_{E_i}(y) dy \right| dx \\ &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \nabla P^s(x - y, t_i) (\chi_{E_i}(y) - \chi_{E_i}(x)) dy \right| dx \\ &\leq \frac{1}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\nabla P^s(x - y, t_i)| (\chi_{E_i}(y) - \chi_{E_i}(x))^2 dx dy \\ &\leq \frac{C_n}{t_i^{\frac{1}{2s}}} \int_{E_i^c} \int_{E_i} P^s(x - y, t_i) dx \leq C, \end{aligned}$$

where in the second inequality we have used the gradient kernel estimate (1.9), while in the last inequality we have used the assumption (2.1). Thus, there exists $v \in L^1(\mathbb{R}^n)$ such that, up to a subsequence, $v_i \rightarrow v$ in L_{loc}^1 . Observe that it is easy to show that $\chi_{E_i} \rightarrow v$ in L^1 as well (this is because $E_i \subset U$ where U is a open bounded set) and, arguing as in the case $s < \frac{1}{2}$ we can extract a further subsequence to show that $v = \chi_E$. Finally, by semicontinuity of the total variation, it is immediate that $v = \chi_E$ is a BV($\mathbb{R}^n$) function, which means that E has finite perimeter. $\square$

2.2 Proof of Theorem 2.1 when $s = \frac{1}{2}$

This subsection is devoted to the proof of Theorem 2.1 when $s = \frac{1}{2}$. Our proof is inspired by the compactness result in [12]. The first step is to prove a non-local Poincaré–Wirtinger-type inequality. In what follows we set $Q := [0, 1]^n$.

Lemma 2.2. *Let $\xi \in \mathbb{R}^n$ and $u \in L^1(lQ + \xi)$. Then, for all $0 < t < l$, we have*

$$\int_{lQ+\xi} \left| u(y) - \frac{1}{l^n} \int_{lQ+\xi} u(x) dx \right| dy \leq lC(n) \int_{lQ+\xi} \int_{lQ+\xi} \frac{|u(x) - u(y)|}{(|x - y|^2 + t^2)^{\frac{n+1}{2}}} dx dy. \quad (2.3)$$

Proof. By translational invariance, it is enough to prove the claim only for $\xi = 0$. By assumption, we have

$$\begin{aligned} \int_Q \left| u(y) - \frac{1}{l^n} \int_Q u(x) dx \right| dy &\leq \frac{1}{l^n} \int_Q \int_Q |u(x) - u(y)| dx dy \\ &= \frac{1}{l^n} \int_Q \int_Q \frac{|u(x) - u(y)|}{(|x - y|^2 + t^2)^{\frac{n+1}{2}}} (|x - y|^2 + t^2)^{\frac{n+1}{2}} dx dy \\ &\leq lC(n) \int_Q \int_Q \frac{|u(x) - u(y)|}{(|x - y|^2 + t^2)^{\frac{n+1}{2}}} dx dy, \end{aligned}$$

i.e., (2.3). □

Lemma 2.3. *For every $\xi \in \mathbb{R}^n$ and for every $E \in \mathcal{M}(\mathbb{R}^n)$, it holds*

$$\frac{1}{l^n} |(lQ + \xi) \setminus E| |(lQ + \xi) \cap E| \leq \int_{lQ+\xi} \left| \chi_E(x) - \frac{1}{l^n} \int_{lQ+\xi} \chi_E(y) dy \right| dx.$$

Proof. We can assume without loss of generality that $\xi = 0$. Then we have that

$$\begin{aligned} \int_Q \left| \chi_E(x) - \frac{1}{l^n} \int_Q \chi_E(y) dy \right| dx &= \int_{lQ \cap E} \left| 1 - \frac{|lQ \cap E|}{l^n} \right| dx + \int_{lQ \setminus E} \frac{|lQ \cap E|}{l^n} dx \\ &= \frac{1}{l^n} \left(\int_{lQ \cap E} |lQ \setminus E| dx + \int_{lQ \setminus E} |lQ \cap E| dx \right) \\ &= \frac{2}{l^n} \int_{lQ \setminus E} |lQ \cap E| dx \geq \frac{|lQ \cap E| |lQ \setminus E|}{l^n}. \end{aligned} \quad \square$$

The following result is a localized isoperimetric inequality for the non-local energy (0.1) when $s = \frac{1}{2}$.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^n$ be a open bounded set with Lipschitz continuous boundary and $|\Omega| = 1$. For every $\eta \in (0, 1)$ there exist $t_0 > 0$ and $C = C(n, \Omega, \eta) > 0$ such that*

$$\inf \left\{ \frac{1}{g_{\frac{1}{2}}(t)} \int_A \int_{\Omega \setminus A} P^{\frac{1}{2}}(x - y, t) dx dy : A \subset \Omega, |A| \in (\eta, 1 - \eta), t \in (0, t_0) \right\} \geq C. \quad (2.4)$$

Proof. Fix $\eta \in (0, 1)$, $t \in (0, 1)$ and let $I \in \mathbb{N}$ be such that $2^{-I-1} \leq t \leq 2^{-I}$. We have that

$$\text{if } 0 \leq |z| \leq 2^{-i}, i \in \mathbb{N}, \text{ then } \frac{1}{(|z|^2 + t^2)^{\frac{n+1}{2}}} \geq \frac{2^{(n+1)\min\{i, I\}}}{2^{\frac{n+1}{2}}}. \quad (2.5)$$

Let ρ and ρ_ε (for $\varepsilon \in (0, 1)$) be as in Lemma (1.3).

We claim that there exists $C(n, \rho) > 0$ such that

$$\sum_{i=1}^I 2^i \rho_{2^{-i}}(z) \leq \frac{C(n, \rho)}{(|z|^2 + t^2)^{\frac{n+1}{2}}} \quad \text{for every } z \in \mathbb{R}^n. \quad (2.6)$$

Indeed, if $0 \leq |z| \leq 2^{-I}$, then we get

$$\begin{aligned} \sum_{i=0}^I 2^i \rho_{2^{-i}}(z) &\leq \|\rho\|_\infty \sum_{i=0}^I (2^{n+1})^i = \|\rho\|_\infty \sum_{i=0}^I \frac{(2^{n+1})^I}{(2^{n+1})^{I-i}} \\ &\leq \|\rho\|_\infty (2^{n+1})^I \sum_{i=0}^{+\infty} \frac{1}{(2^{n+1})^i} = (2^{n+1})^I C(\rho, n) \leq \frac{C(\rho, n)}{(|z|^2 + t^2)^{\frac{n+1}{2}}}, \end{aligned}$$

where in the last step we have used (2.5) for $i = I$. If $2^{-\tilde{i}-1} \leq |z| \leq 2^{-\tilde{i}}$ for some $\tilde{i} = 0, 1, \dots, I-1$, using that $\rho_{2^{-i}}(z) = 0$ for every $i = \tilde{i} + 1, \dots, I$ we have

$$\begin{aligned} \sum_{i=0}^I 2^i \rho_{2^{-i}}(z) &= \sum_{i=0}^{\tilde{i}} 2^i \rho_{2^{-i}}(z) \leq \|\rho\|_\infty \sum_{i=0}^{\tilde{i}} (2^{n+1})^i \\ &\leq (2^{n+1})^{\tilde{i}} \|\rho\|_\infty \sum_{i=0}^{+\infty} (2^{n+1})^{-i} = (2^{n+1})^{\tilde{i}} C(\rho, n) \\ &\leq \frac{C(\rho, n)}{(|z|^2 + t^2)^{\frac{n+1}{2}}}, \end{aligned}$$

where in the last inequality we used (2.5). If $|z| \geq 1$, we have $\rho_{2^{-i}}(z) = 0$ for all $i \in \mathbb{N}$ and

$$\sum_{i=0}^I 2^i \rho_{2^{-i}}(z) = 0 \leq \frac{C(\rho, n)}{(|z|^2 + t^2)^{\frac{n+1}{2}}}.$$

Hence (2.6) holds also in case $|z| \geq 1$, which concludes the proof of the claim.

We prove that (2.6) implies (2.4). We observe that

$$\frac{|\log(t)|}{\log(2)} - 1 \leq I \leq \frac{|\log(t)|}{\log(2)}$$

and hence

$$\sum_{i=0}^I 1 = I + 1 \geq \frac{|\log(t)|}{\log(2)}.$$

Therefore by (2.6) and Lemma 1.3, with ε replaced by 2^{-i} , we obtain

$$\begin{aligned} \int_A \int_{\Omega \setminus A} \frac{1}{(|x-y|^2 + t^2)^{\frac{n+1}{2}}} dx dy &\geq C(\Omega, \rho, n) \sum_{i=0}^I 2^i \int_A \int_{\Omega \setminus A} \rho_{2^{-i}}(x-y) dx dy \\ &\geq C(\Omega, \rho, n, \eta) \sum_{i=0}^I 1 = C(\Omega, \rho, n, \eta) |\log(t)| \end{aligned}$$

and hence recalling the very definition of $g_{\frac{1}{2}}(t)$ and of $z \rightarrow P^{\frac{1}{2}}(z, t)$ we obtain (2.4). $\square$

We are now in a position to prove Theorem 2.1 when $s = \frac{1}{2}$.

Proof. We divide the proof into three steps. Throughout the proof, we write E_i instead of E_{t_i} .

Step 1. Let $\alpha \in (0, 1)$ and set $l_i := t_i^\alpha$ for every $i \in \mathbb{N}$. Let $\{Q_h^i\}_{h \in \mathbb{N}}$ be a disjoint family of cubes of sidelength l_i such that $\mathbb{R}^n = \bigcup_{h \in \mathbb{N}} Q_h^i$. Since $|E_i| \leq |U|$, there exists $H(i) \in \mathbb{N}$ such that, up to permutation of indices,

$$\begin{aligned} |Q_h^i \cap E_i| &\geq \frac{l_i^n}{2} \quad \text{for every } h = 1, \dots, H(i), \\ |Q_h^i \setminus E_i| &> \frac{l_i^n}{2} \quad \text{for every } h > H(i). \end{aligned} \tag{2.7}$$

For every $i \in \mathbb{N}$, we set

$$\tilde{E}_i := \bigcup_{h=1}^{H(i)} Q_h^i.$$

We claim that there exists a constant $C(n) > 0$ such that

$$|\tilde{E}_i \Delta E_i| \leq C(n) M l_i |\log(t_i)| \quad \text{for every } i \in \mathbb{N}, \quad (2.8)$$

where M is the constant in (2.1). Indeed, we have

$$\begin{aligned} |E_i \Delta \tilde{E}_i| &= |\tilde{E}_i \setminus E_i| + |E_i \setminus \tilde{E}_i| = \sum_{h=1}^{H(i)} |Q_h^i \setminus E_i| + \sum_{h=H(i)+1}^{+\infty} |E_i \cap Q_h^i| \\ &= 2 \sum_{h=1}^{H(i)} \frac{1}{l_i^n} |Q_h^i \setminus E_i| \frac{l_i^n}{2} + 2 \sum_{h=H(i)+1}^{+\infty} \frac{1}{l_i^n} |E_i \cap Q_h^i| \frac{l_i^n}{2} \\ &\leq 2 \sum_{h=1}^{+\infty} \frac{1}{l_i^n} |Q_h^i \setminus E_i| |Q_h^n \cap E_i| \\ &\leq 2 \sum_{h=1}^{+\infty} \int_{Q_h^i} \left| \chi_{E_i}(x) - \frac{1}{l_i^n} \int_{Q_h^i} \chi_{E_i}(y) dy \right| dx \\ &\leq C(n) \sum_{h=1}^{+\infty} l_i \int_{Q_h^i \cap E_i} \int_{Q_h^i \setminus E_i} \frac{1}{(|x-y|^2 + t_i^2)^{\frac{n+1}{2}}} dx dy \\ &\leq C(n) l_i |\log(t_i)| \frac{\mathcal{E}_{\frac{1}{2}}^{\frac{1}{2}}(E_i)}{g_{\frac{1}{2}}(t_i)} \leq C(n) M l_i |\log(t_i)|, \end{aligned}$$

where the first inequality follows by (2.7), the second from Lemma 2.3, the third from (2.3) and the last one follows directly by (2.1).

Step 2. For every $i \in \mathbb{N}$, let l_i and $\tilde{E}_i := \bigcup_{h=1}^{H(i)} Q_h^i$ be as in Step 1. We claim that there exists a constant $C(\alpha, n)$ such that

$$P(\tilde{E}_i) \leq C(\alpha, n) \frac{\mathcal{E}_{\frac{1}{2}}^{\frac{1}{2}}(E_i)}{g_{\frac{1}{2}}(t_i)} \quad (2.9)$$

holds for i large enough. We omit the dependence on i by setting $t := t_i$, $l := l_i$, $E := E_i$, $Q_h := Q_h^i$, $H := H(i)$, $\tilde{E} := \tilde{E}_i$. Let $\mathcal{R}$ be the family of all the rectangles $R = \tilde{Q} \cup \hat{Q}$ such that $\tilde{Q}$ and $\hat{Q}$ are adjacent cubes (of the type Q_h) and $\tilde{Q} \subset \tilde{E}$, $\hat{Q} \subset \tilde{E}^c$. We have

$$P(\tilde{E}) \leq C(n) l^{n-1} \#\mathcal{R} \quad \text{and} \quad \sum_{R \in \mathcal{R}} \int_{R \cap E} \int_{R \setminus E} P^{\frac{1}{2}}(|x-y|, t) dy dx \leq \mathcal{E}_t^{\frac{1}{2}}(E). \quad (2.10)$$

By Lemma 2.4, there exist $t_0 > 0$ and $C(n) > 0$ such that for every rectangle $\bar{R}$ given by the union of two adjacent unitary cubes in $\mathbb{R}^n$, it holds

$$\inf \left\{ \frac{1}{g_{\frac{1}{2}}(t)} \int_F \int_{\bar{R} \setminus F} P^{\frac{1}{2}}(|x-y|, t) dy dx : 0 < t < t_0, F \subset \bar{R}, \frac{1}{2} \leq |F| \leq \frac{3}{2} \right\} := C(n) > 0. \quad (2.11)$$

For every $A \subset \mathbb{R}^n$ we set $A^l := \frac{A}{l}$. By formula (2.10) and (2.11), with $\bar{R} = R^l$ for every $R \in \mathcal{R}$, for t small enough we have

$$\begin{aligned} \frac{\mathcal{E}_t^{\frac{1}{2}}(E)}{g_{\frac{1}{2}}(t)} &\geq \frac{l^{2n}}{g_{\frac{1}{2}}(t)} \sum_{R \in \mathcal{R}} \int_{R^l \cap E^l} \int_{R^l \setminus E^l} P^{\frac{1}{2}}(l|x-y|, t) dy dx \\ &= \frac{C(n, \alpha) l^{n-1}}{g_{\frac{1}{2}}(\frac{t}{l})} \sum_{R \in \mathcal{R}} \int_{R^l \cap E^l} \int_{R^l \setminus E^l} P^{\frac{1}{2}}\left(|x-y|, \frac{t}{l}\right) dx dy \\ &\geq C(n, \alpha) l^{n-1} \#\mathcal{R} \geq C(n, \alpha) P(\tilde{E}), \end{aligned}$$

i.e., (2.9).

Step 3. Here we conclude the proof of the compactness result. Fix $\alpha \in (0, 1)$, by (2.8) we get

$$|E_i \Delta \tilde{E}_i| \rightarrow 0 \text{ as } i \rightarrow +\infty. \quad (2.12)$$

By step 2, we have that $\{\tilde{E}_i\}_{i \in \mathbb{N}}$ satisfies the assumption of Theorem 1.1. Therefore, $\chi_{\tilde{E}_i} \rightarrow \chi_E$, up to subsequence, in $L^1(\mathbb{R}^n)$ with $P(E) < +\infty$. Hence, by (2.12) we have $\chi_{E_i} \rightarrow \chi_E$ in $L^1(\mathbb{R}^n)$. $\square$

3 Γ -convergence for $s \in (0, \frac{1}{2})$

The main result of this section is the following.

Theorem 3.1. *Let $s \in (0, \frac{1}{2})$ and let $\{t_i\}_{i \in \mathbb{N}} \subset (0, 1)$ such that $t_i \rightarrow 0^+$ as $i \rightarrow +\infty$. The following Γ -convergence result holds true.*

(i) (Lower bound) *Let $E \in M(\mathbb{R}^n)$ be a Lebesgue measurable set. For every $\{E_i\}_{i \in \mathbb{N}} \subset M(\mathbb{R}^n)$ with $\chi_{E_i} \rightarrow \chi_E$ strongly in $L^1(\mathbb{R}^n)$, it holds*

$$C_{n,s} P_{2s}(E) \leq \liminf_{i \rightarrow +\infty} \frac{\mathcal{E}_{t_i}^s(E_i)}{g_s(t_i)}. \quad (3.1)$$

(ii) (Upper bound) *For every $E \in M(\mathbb{R}^n)$ there exists $\{E_i\}_{i \in \mathbb{N}} \subset M(\mathbb{R}^n)$ such that $\chi_{E_i} \rightarrow \chi_E$ strongly in $L^1(\mathbb{R}^n)$ and*

$$C_{n,s} P_{2s}(E) \geq \limsup_{i \rightarrow +\infty} \frac{\mathcal{E}_{t_i}^s(E_i)}{g_s(t_i)}. \quad (3.2)$$

Here $C_{n,s}$ is the constant defined in (1.2).

3.1 Proof of Theorem 3.1 (i)

Recall that $g_s(t) = t$ for $s \in (0, \frac{1}{2})$. Let $\{E_i\}_{i \in \mathbb{N}} \subset M(\mathbb{R}^n)$ such that $\chi_{E_i} \rightarrow \chi_E$ in $L^1(\mathbb{R}^n)$ and assume

$$\liminf_{i \rightarrow +\infty} \frac{\mathcal{E}_{t_i}(E_i)}{t_i} < +\infty$$

(otherwise (3.1) is trivial). To prove (3.1), we observe

$$\begin{aligned} \frac{\mathcal{E}_{t_i}^s(E_i)}{t_i} &= \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{P^s(x-y, t_i)}{t_i} |\chi_{E_i}(x) - \chi_{E_i}(y)|^2 dx dy \\ &= \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{P^s(h, t_i)}{t_i} |\chi_{E_i}(x+h) - \chi_{E_i}(x)|^2 dx dh \\ &= \int_{\mathbb{R}^n} \frac{P^s(h, t_i)}{t_i} \int_{\mathbb{R}^n} (1 - \cos(\xi \cdot h)) |\hat{\chi}_{E_i}(\xi)|^2 d\xi dh, \end{aligned} \quad (3.3)$$

where in the last step we used the Plancherel theorem and

$$\int_{\mathbb{R}^n} |u(x+h) - u(x)|^2 dx = \int_{\mathbb{R}^n} |1 - e^{i\xi \cdot h}|^2 |\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^n} 2(1 - \cos(\xi \cdot h)) |\hat{u}(\xi)|^2 d\xi.$$

We recall that $h \mapsto P^s(h, t)$ is a Schwartz function and $\mathcal{F}[P^s(\cdot, t)](\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-t|\xi|^{2s}}$. We define

$$\mathcal{J}(\xi) := \int_{\mathbb{R}^n} (1 - \cos(\xi \cdot h)) P^s(h, t) dh.$$

We observe that the function $\mathcal{J}$ is rotationally invariant, that is,

$$\mathcal{J}(\xi) = \mathcal{J}(|\xi|e_1),$$

where e_1 denotes the first direction vector in $\mathbb{R}^n$. Therefore, we have

$$\int_{\mathbb{R}^n} (1 - \cos(\xi \cdot h)) P^s(h, t) dh = \int_{\mathbb{R}^n} (1 - \cos(|\xi|h_1)) P^s(h, t) dh. \quad (3.4)$$

The Fourier transform of the function $a \mapsto \cos(aa)$ for $a \in \mathbb{R}$ is given by $\mathcal{F}[\cos(a \cdot)](\beta) = \frac{\sqrt{\pi}}{\sqrt{2}} [\delta(\beta - a) + \delta(\beta + a)]$. This Fourier transform should be read in the sense of the tempered distributions (that is the dual space of the Schwartz function). Therefore, using the Plancherel theorem, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{P^s(h, t)}{t} (1 - \cos(|\xi|h_1)) dh &= \frac{1}{t} \left[1 - \int_{\mathbb{R}^n} \cos(|\xi|h_1) P^s(h, t) dh \right] \\ &= \frac{1}{t} \left[1 - \int_{\mathbb{R}^n} \frac{1}{2} (\delta_0(\eta_1 - |\xi|) + \delta_0(\eta_1 + |\xi|)) \delta_0(\eta_2) \cdots \delta_0(\eta_n) e^{-t|\eta|^{2s}} d\eta \right] \\ &= \frac{1}{t} [1 - e^{-t|\xi|^{2s}}], \end{aligned} \quad (3.5)$$

where δ_0 is the one-dimensional Dirac delta with center in $0 \in \mathbb{R}$. By formulas (3.3), (3.4), (3.5) and for all $R > 0$ we obtain

$$\begin{aligned} \liminf_{i \rightarrow +\infty} \frac{\mathcal{E}_{t_i}^s(E_i)}{t_i} &= \liminf_{i \rightarrow +\infty} \int_{\mathbb{R}^n} |\mathcal{F}[\chi_{E_i}](\xi)|^2 \frac{1}{t_i} [1 - e^{-t_i|\xi|^{2s}}] d\xi \\ &\geq \liminf_{i \rightarrow +\infty} \int_{B(0, R)} |\mathcal{F}[\chi_{E_i}](\xi)|^2 \frac{1}{t_i} [1 - e^{-t_i|\xi|^{2s}}] d\xi \\ &= \int_{B(0, R)} |\mathcal{F}[\chi_E](\xi)|^2 |\xi|^{2s} d\xi, \end{aligned} \quad (3.6)$$

where in the last step we used that

$$\lim_{t \rightarrow 0^+} \sup_{\xi \in B(0, R)} \left| \frac{1}{t} [1 - e^{-t|\xi|^{2s}}] - |\xi|^{2s} \right| = 0.$$

Sending $R \rightarrow +\infty$ in (3.6), we obtain (3.1).

3.2 Proof of Theorem 3.1 (ii)

Let $E \in \mathcal{M}(\mathbb{R}^n)$ such that $P_{2s}(E) < +\infty$ (otherwise the formula (3.2) is trivial) we prove that the limsup in (3.2) is realized by the sequence $E_i = E$ for all $i \in \mathbb{N}$ and it is a limit. We recall that E has finite $2s$ -fractional perimeter if $\chi_E \in H^s(\mathbb{R}^n)$. Let $v : \mathbb{R}^n \times [0, +\infty) \rightarrow \mathbb{R}$ be the solution to the Cauchy problem in (1.4) with $u_0 = \chi_E$. By the very definition of $\mathcal{E}_t(E)$, see (0.1), and by formula (1.5) we have

$$\begin{aligned} \mathcal{E}_t^s(u_0) &= \int_{\mathbb{R}^n} (1 - u_0(x)) v(x, t) dx = \int_{\mathbb{R}^n} v(x, t) dx - \int_{\mathbb{R}^n} v(x, t) u_0(x) dx \\ &= \int_{\mathbb{R}^n} u_0(x) dx - \int_{\mathbb{R}^n} v(x, t) u_0(x) dx \\ &= \int_{\mathbb{R}^n} u_0(x)^2 dx - \int_{\mathbb{R}^n} v(x, t) u_0(x) dx, \end{aligned}$$

where we used that

$$\int_{\mathbb{R}^n} v(x, t) dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} P^s(x - y, t) u_0(y) dy dx = \int_{\mathbb{R}^n} u_0(y) dy$$

and the fact that $u_0(x) = \chi_E(x) \in \{0, 1\}$. Since $v \in C^1([0, +\infty), L^2(\mathbb{R}^n))$, we get

$$\begin{aligned} \int_{\mathbb{R}^n} u_0(x)^2 dx - \int_{\mathbb{R}^n} v(x, t) u_0(x) dx &= - \int_{\mathbb{R}^n} u_0(x) \int_0^t \partial_h v(x, h) dh dx \\ &= \int_{\mathbb{R}^n} \int_0^t u_0(x) (-\Delta)^s v(x, h) dh dx. \end{aligned}$$

At this point, we just need to use that the Fourier transform is an isometry and

$$\mathcal{F}[v(\cdot, t)](\xi) = \mathcal{F}[u_0(\cdot)](\xi)e^{-t|\xi|^{2s}}$$

and therefore

$$\int_{\mathbb{R}^n} u_0(x)(-\Delta)^s v(x, t) dx = \int_{\mathbb{R}^n} |\xi|^{2s} \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-t|\xi|^{2s}} |\mathcal{F}[u_0(\cdot)](\xi)|^2 d\xi.$$

Since $u_0 \in H^s(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} |\xi|^{2s} \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-t|\xi|^{2s}} |\mathcal{F}[u_0(\cdot)](\xi)|^2 d\xi \leq \int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}[u_0(\cdot)](\xi)|^2 d\xi = C_{n,s} P_{2s}(E).$$

Finally, to conclude we just need to observe that

$$\lim_{t \rightarrow 0^+} \frac{\mathcal{E}_t(u_0)}{t} = \int_{\mathbb{R}^n} u_0(-\Delta)^s u_0 = C_{n,s} P_{2s}(E).$$

3.3 Characterization of sets of finite 2s-fractional perimeter

As a byproduct of our Γ -convergence analysis, we prove that a set $E \in \mathcal{M}(\mathbb{R}^n)$ has finite 2s-fractional perimeter if and only if

$$\limsup_{t \rightarrow 0^+} \frac{1}{t} \mathcal{E}_t^s(E) < +\infty.$$

Theorem 3.2. *Let $E \subset \mathcal{M}(\mathbb{R}^n)$ and $s \in (0, \frac{1}{2})$. Then*

$$\limsup_{t \rightarrow 0^+} \frac{1}{t} \mathcal{E}_t^s(E) < +\infty \iff P_{2s}(E) < +\infty.$$

Proof. We notice that the implication $(\Rightarrow)$ is a consequence of Theorem 2.1. To prove the reverse implication, we apply Theorem 3.1 (ii) with $E_i := E$ for all $i \in \mathbb{N}$. $\square$

4 Γ -convergence for $s \in [\frac{1}{2}, 1)$

The main result of this section is the following Γ -convergence theorem.

Theorem 4.1. *Let $s \in [\frac{1}{2}, 1)$ and let $\{t_i\}_{i \in \mathbb{N}} \subset (0, 1)$ such that $t_i \rightarrow 0^+$ as $i \rightarrow +\infty$. The following Γ -convergence result holds true.*

(i) (Lower bound) *Let $E \in \mathcal{M}(\mathbb{R}^n)$ be a Lebesgue measurable set. For every $\{E_i\}_{i \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^n)$ with $\chi_{E_i} \rightarrow \chi_E$ strongly in $L^1(\mathbb{R}^n)$ it holds*

$$\Gamma^{n,s}P(E) \leq \liminf_{i \rightarrow +\infty} \frac{\mathcal{E}_{t_i}^s(E_i)}{g_s(t_i)}.$$

(ii) (Upper bound) *For every $E \in \mathcal{M}(\mathbb{R}^n)$ there exists $\{E_i\}_{i \in \mathbb{N}} \subset \mathcal{M}(\mathbb{R}^n)$ such that $\chi_{E_i} \rightarrow \chi_E$ strongly in $L^1(\mathbb{R}^n)$ and*

$$\Gamma^{n,s}P(E) \geq \limsup_{i \rightarrow +\infty} \frac{\mathcal{E}_{t_i}^s(E_i)}{g_s(t_i)}.$$

Here

$$\Gamma^{n,s} := \begin{cases} \frac{1}{2\pi} \Gamma(1 - \frac{1}{2s}), & s \in (\frac{1}{2}, 1), \\ \frac{\omega_{n-2}}{2} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}, & s = \frac{1}{2}. \end{cases} \quad (4.1)$$

The argument to prove Theorem 4.1 is completely different from the used to show Theorem 3.1, as it relies on geometric measure theory. We start computing the limit as $t \rightarrow 0$ of our functional on hyperplanes.

4.1 Estimates on cubes

Throughout this subsection, we set

$$Q'_\delta := [-\delta, \delta]^{n-1}, \quad Q_\delta^+ := [0, \delta] \times Q'_\delta \quad \text{and} \quad Q_\delta^- := [-\delta, 0] \times Q'_\delta. \quad (4.2)$$

The main goal of this subsection is to prove the following result.

Proposition 4.2. *Let $s \in [\frac{1}{2}, 1)$, let g_s be the function defined in (0.2) and let $\Gamma^{n,s}$ be the constant defined in (4.1). Then*

$$\lim_{t \rightarrow 0} \frac{1}{g_s(t)} \int_{Q_\delta^-} \int_{Q_\delta^+} P^s(x-y, t) \, dx \, dy = \Gamma^{n,s} \mathcal{H}^{n-1}(Q'_\delta). \quad (4.3)$$

To begin, we consider the case when $s = \frac{1}{2}$. This is the only instance where we have a precise formula for the fundamental solution of the fractional heat equation.

Lemma 4.3. *For any $\delta > 0$ it holds*

$$\int_{Q_\delta^-} \int_{Q_\delta^+} \frac{t}{(|x-y|^2 + t^2)^{\frac{n+1}{2}}} \, dx \, dy \leq c_n \mathcal{H}^{n-1}(Q'_\delta) t |\log t| + o(t |\log t|), \quad (4.4)$$

where c_n is given in (4.6)

Proof. Using Fubini Theorem, we have

$$\int_{Q_\delta^-} \int_{Q_\delta^+} \frac{t}{(|x-y|^2 + t^2)^{\frac{n+1}{2}}} \, dx \, dy = \int_{Q_\delta^+} \int_0^\delta \int_{-\delta}^0 \int_{-\delta}^\delta \frac{t}{(|x' - y'|^2 + (x_n - y_n)^2 + t^2)^{\frac{n+1}{2}}} \, dy' \, dx_n \, dy_n \, dx'.$$

We observe that

$$\int_0^\delta \int_{-\delta}^0 \int_{-\delta}^\delta P^{\frac{1}{2}}(x-y, t) \, dy' \, dx_n \, dy_n = c_n t \int_0^\delta \int_{-\delta}^0 \frac{1}{(x_n - y_n)^2 + t^2} \, dx_n \, dy_n, \quad (4.5)$$

where we used the change of variable

$$z = \frac{y' - x'}{\sqrt{(x_n - y_n)^2 + t^2}}$$

and we have set

$$c_n := \int_{\mathbb{R}^{n-1}} \frac{1}{(1 + |z|^2)^{\frac{n+1}{2}}} \, dz. \quad (4.6)$$

A straightforward computation gives

$$\begin{aligned} t \int_0^\delta \int_{-\delta}^0 \frac{1}{(x_n - y_n)^2 + t^2} \, dx_n \, dy_n &= 2\delta \left(\arctan \frac{t}{\delta} - \arctan \frac{t}{2\delta} \right) - t \log t + t \log \left(\frac{\delta^2 + t^2}{t^2 + 4\delta^2} \right) \\ &=: g(t, \delta) + t |\log t| \end{aligned} \quad (4.7)$$

with

$$c_1(\delta)t \leq g(t, \delta) \leq tc_2(\delta)$$

and $c_1(\delta), c_2(\delta) \rightarrow 0$ as $\delta \rightarrow 0^+$. Therefore, using the monotonicity of the integral and (4.7) we have

$$\int_{Q_\delta^-} \int_{Q_\delta^+} \frac{t}{(|x-y|^2 + t^2)^{\frac{n+1}{2}}} \, dx \, dy \leq \int_{Q_\delta^+} \int_{\mathbb{R}^{n-1}} \frac{t}{(|x-y|^2 + t^2)^{\frac{n+1}{2}}} \, dx \, dy = c_n t |\log t| + o(t \log t),$$

which is exactly (4.4). $\square$

We now estimate the integral on cubes from below.

Lemma 4.4. For any $\delta > 0$ and for t small enough it holds true

$$\int_{Q_\delta^-} \int_{Q_\delta^+} \frac{t}{(|x-y|^2 + t^2)^{\frac{n+1}{2}}} dx dy \geq c_n \mathcal{H}^{n-1}(Q'_\delta)(t|\log t| + o(t \log t)),$$

where c_n is define in (4.6).

Proof. To prove this lemma, we just need to show that the error passing from a small cube to the hyperplane $\mathbb{R}^{n-1}$ is negligible when t is small. To this aim, let $x \in Q_\delta^-$ and observe that

$$\int_0^\delta \int_{-Q'_\delta} \int P^{\frac{1}{2}}(x-y, t) dy' dx_n dy_n = \int_0^\delta \int_{-Q'_\delta} \left(\int_{\mathbb{R}^{n-1}} P^{\frac{1}{2}}(x-y, t) dy' - \int_{(Q'_\delta)^c} P^{\frac{1}{2}}(x-y, t) dy' \right) dx_n dy_n$$

to observe that, by using the change of variable $z = \frac{y'-x'}{\sqrt{(x_n-y_n)^2 + t^2}}$, we get

$$\int_0^\delta \int_{-Q'_\delta} \int P^s(x-y, t) dy' dx_n dy_n = \frac{1}{t} \int_0^\delta \int_{\mathbb{R}^{n-1} \setminus Q'_\delta(x')} \int \frac{1}{(|z|^2 + \frac{(x_n-y_n)^2}{t^2} + 1)^{\frac{n+1}{2}}} dz dx_n dy_n.$$

Using the integration in polar coordinate, we obtain

$$\begin{aligned} \int_0^\delta \int_{\mathbb{R}^{n-1} \setminus Q'_\delta(x')} \int \frac{dz dx_n dy_n}{(|z|^2 + \frac{(x_n-y_n)^2}{t^2} + 1)^{\frac{n+1}{2}}} &\leq \int_0^\delta \int_{\mathbb{R}^{n-1} \setminus Q'_\delta(x')} \int \frac{dz dx_n dy_n}{(|z|^2 + 1)^{\frac{n+1}{2}}} \\ &\leq (n-1)\omega_{n-1} \int_0^\delta \int_{\frac{\delta}{t}}^\infty \int \frac{\rho^{n-2}}{(\rho^2 + 1)^{\frac{n+1}{2}}} d\rho dx_n dy_n \\ &\leq (n-1)\omega_{n-1} \delta^2 \int_{\frac{\delta}{t^2}}^\infty \frac{\rho}{(\rho^2 + 1)^2} d\rho \\ &= \frac{(n-1)\omega_{n-1}\delta^2}{\delta^2 + t^2} t^2. \end{aligned} \quad (4.8)$$

The conclusion follows by (4.5), (4.7) and (4.8). $\square$

Remark 4.5. Thanks to Lemma 1.4 we get $\Gamma^{n, \frac{1}{2}} = c_n$.

We now proceed performing the same computation for $s > \frac{1}{2}$. We stress that in this instance we do not have the precise expression of the kernel, as it is not known. However, the Fourier transform of the function $P^s(\cdot, t)$ is explicit and can be used to perform the computations that we need. In what follows, we set $\mathbb{R}_+^n := \mathbb{R}^{n-1} \times [0, +\infty)$ and $R_\delta := \{(x', x_n) : x' \in \mathbb{R}^{n-1}, x_n \in (0, \delta)\}$ for all $\delta > 0$.

Lemma 4.6. Let $\delta > 0$, $s \in (\frac{1}{2}, 1)$. For $t > 0$ it holds true

$$\int_{Q_\delta^-} \int_{\mathbb{R}_+^n} P^s(x-y, t) dx dy = \frac{1}{2\pi} \Gamma\left(1 - \frac{1}{2s}\right) t^{\frac{1}{2s}} (\mathcal{H}^{n-1}(Q'_\delta) + o(1)). \quad (4.9)$$

Proof. We start by observing that

$$\int_{Q_\delta^-} \int_{\mathbb{R}_+^n \setminus R_\delta} P^s(x-y, t) dx dy = o(1).$$

We claim that

$$\int_{\mathbb{R}^{n-1}} P^s((z', z_n), t) dz' = 2 \int_0^\infty e^{-t|r|^{2s}} \cos(z_n r) dr. \quad (4.10)$$

For all $z_n \in \mathbb{R}$, we set $v_{z_n}(z', t) := P^s((z', z_n), t)$. We have

$$\begin{aligned} P^s(z, t) &= \mathcal{F}^{-1} \left[\frac{1}{(2\pi)^{\frac{n}{2}}} e^{-t|\cdot|^{2s}} \right] (z) = \int_{\mathbb{R}^n} \frac{1}{(2\pi)^n} e^{-t|\xi|^{2s}} e^{i\langle z, \xi \rangle} d\xi \\ &= \int_{\mathbb{R}^{n-1}} \frac{1}{(2\pi)^{\frac{n-1}{2}}} e^{i\langle z', \xi' \rangle} \int_{\mathbb{R}} \frac{1}{(2\pi)^{1+\frac{n-1}{2}}} e^{-t|\xi|^{2s}} e^{iz_n \xi_n} d\xi_n d\xi'. \end{aligned}$$

Since the Fourier transform is an isometry, we have

$$v_{z_n}(z', t) = \mathcal{F}^{-1}[\mathcal{F}[v_{z_n}(\cdot, t)]](z') = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{i\langle z', \xi' \rangle} \mathcal{F}[v_{z_n}(\cdot, t)](\xi') d\xi',$$

where with a little abuse of notation we denoted the Fourier transform in $\mathbb{R}^{n-1}$ still by $\mathcal{F}[\cdot]$. Therefore, by uniqueness of the Fourier transform we get

$$\begin{aligned} \mathcal{F}[v_{z_n}(\cdot, t)](\xi') &= \frac{1}{(2\pi)^{1+\frac{n-1}{2}}} \int_{\mathbb{R}} e^{-t|\xi|^{2s}} e^{iz_n \xi_n} d\xi_n \\ &= \frac{1}{(2\pi)^{1+\frac{n-1}{2}}} \int_{\mathbb{R}} e^{-t|\xi|^{2s}} \cos(z_n \xi_n) d\xi_n \\ &= \frac{1}{(2\pi)^{1+\frac{n-1}{2}}} \int_{\mathbb{R}} e^{-t(|\xi'|^2 + \xi_n^2)^s} \cos(z_n \xi_n) d\xi_n. \end{aligned}$$

From the above formula and by

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} P^s((z', z_n), t) dz' &= \int_{\mathbb{R}^{n-1}} v_{z_n}(z', t) dz' \\ &= (2\pi)^{\frac{n-1}{2}} \mathcal{F}[v_{z_n}(\cdot, t)](0) \\ &= \frac{1}{\pi} \int_0^\infty e^{-t|\xi_n|^{2s}} \cos(z_n \xi_n) d\xi_n, \end{aligned}$$

we obtain (4.10).

Now, using Fubini Theorem and (4.10) with $z_n = x_n - y_n$, we obtain

$$\begin{aligned} \int_{Q'_\delta} \int_{R_\delta} P^s(x - y, t) dx dy &= \int_{Q'_\delta} \int_{-\delta}^0 \int_{\mathbb{R}^{n-1}} \int_0^\delta P^s((x' - y', x_n - y_n), t) dx' dy' dx_n dy_n \\ &= \frac{\mathcal{H}^{n-1}(Q'_\delta)}{\pi} \int_{-\delta}^0 \int_0^\delta \int_0^\infty e^{-tr^{2s}} \cos(r(x_n - y_n)) dr dx_n dy_n. \end{aligned} \quad (4.11)$$

We use basic trigonometry to get

$$\begin{aligned} \int_{-\delta}^0 \int_0^\delta \cos(r(x_n - y_n)) dr dx_n dy_n &= \frac{\sin^2(\delta r) - (1 - \cos(\delta r))^2}{r^2} \\ &= 2 \cos(\delta r) \frac{(1 - \cos(\delta r))}{r^2}. \end{aligned}$$

Thus, using the above formula, we have that (4.11) becomes

$$\int_{Q'_\delta} \int_{R_\delta} P^s(x - y, t) dx dy = \frac{\mathcal{H}^{n-1}(Q'_\delta)}{\pi} \int_0^\infty e^{-tr^{2s}} \cos(\delta r) \frac{(1 - \cos(\delta r))}{r^2} dr.$$

We observe that

$$\begin{aligned}
 \int_0^\infty e^{-tr^{2s}} \cos(\delta r) \frac{(1 - \cos(\delta r))}{r^2} dr &= \int_0^\infty e^{-tr^{2s}} \frac{(\cos(\delta r)) - 1}{r^2} dr + \int_0^\infty e^{-tr^{2s}} \frac{(1 - \cos^2(\delta r))}{r^2} dr \\
 &= \int_0^\infty e^{-tr^{2s}} \frac{(\cos(\delta r)) - 1}{r^2} dr + \frac{1}{2} \int_0^\infty e^{-tr^{2s}} \frac{(1 - \cos(2\delta r))}{r^2} dr \\
 &= \int_0^\infty e^{-tr^{2s}} \frac{(\cos(\delta r)) - 1}{r^2} dr + \int_0^\infty e^{-\frac{tr^{2s}}{2}} \frac{(1 - \cos(\delta r))}{r^2} dr \\
 &= \int_0^\infty \frac{(1 - \cos(\delta r))}{r^2} (e^{-\frac{tr^{2s}}{2}} - e^{-tr^{2s}}) dr = t^{\frac{1}{2s}} h(t),
 \end{aligned}$$

where in the last equality we have used a change of variable $r' = rt^{\frac{1}{2s}}$ and we have set

$$h(t) := \int_0^\infty \frac{1 - \cos\left(\frac{\delta r}{t^{\frac{1}{2s}}}\right)}{r^2} (e^{-\frac{r^{2s}}{2}} - e^{-r^{2s}}) dr.$$

It is quite easy to show that h is bounded and $h \geq 0$. Applying the Riemann–Lebesgue lemma, we finally have

$$\lim_{t \rightarrow 0} h(t) = \int_0^\infty \frac{e^{-\frac{r^{2s}}{2}} - e^{-r^{2s}}}{r^2} dr = \frac{1}{2} \int_0^\infty \frac{1 - e^{-r^{2s}}}{r^2} dr = \frac{1}{4s} \int_0^\infty \frac{1 - e^{-r}}{r^{1+\frac{1}{2}s}} dr = \frac{\Gamma(1 - \frac{1}{2s})}{4},$$

which gives (4.9). □

The next lemma is an obvious consequence of the previous one.

Lemma 4.7. *For $s \in (\frac{1}{2}, 1)$ and for t small enough it holds true*

$$\int_{Q_\delta^-} \int_{Q_\delta^+} P^s(x - y, t) dx dy \leq \frac{\Gamma(1 - \frac{1}{2s})}{2\pi} t^{\frac{1}{2s}} (\mathcal{H}^{n-1}(Q'_\delta) + o(1)),$$

where Q_δ^+ , Q_δ^- and Q'_δ are defined in (4.2).

Finally, we prove the crucial estimate to prove the Γ -lim inf inequality when $s \in (\frac{1}{2}, 1)$.

Lemma 4.8. *For $s \in (\frac{1}{2}, 1)$ it holds true*

$$\int_{Q_\delta^-} \int_{Q_\delta^+} P^s(x - y, t) dx dy \geq \frac{\Gamma(1 - \frac{1}{2s})}{2\pi} t^{\frac{1}{2s}} (\mathcal{H}^{n-1}(Q'_\delta) + o(1)) \quad (4.12)$$

where Q_δ^+ , Q_δ^- and Q'_δ are defined in (4.2).

Proof. To prove this lemma, we just need to show that the error passing from a small cube to the hyperplane $\mathbb{R}^{n-1}$ is negligible when t is small. To do that, we start decomposing the integral

$$\int_0^\delta \int_{-\delta}^0 \int_{Q'_\delta} P^s(x - y, t) dx_n dy = \int_0^\delta \int_{-\delta}^0 \int_{\mathbb{R}^{n-1}} P^s(x - y, t) dx_n dy - \int_0^\delta \int_{-\delta}^0 \int_{\mathbb{R}^{n-1} \setminus Q'_\delta} P^s(x - y, t) dx_n dy$$

and observe that, by (1.8) and using a change of variable, we have

$$\begin{aligned}
 \int_0^\delta \int_{-\delta}^0 \int_{\mathbb{R}^{n-1} \setminus Q'_\delta} P(x - y, t) dx_n dy &= t^{-\frac{n}{2s}} \int_0^\delta \int_{-\delta}^0 \int_{\mathbb{R}^{n-1} \setminus Q'_\delta} P^s\left(\frac{x - y}{t^{\frac{1}{2s}}}, 1\right) dx_n dy \\
 &= \frac{1}{t^{\frac{1}{2s}}} \int_0^\delta \int_{-\delta}^0 \int_{\mathbb{R}^{n-1} \setminus Q'_\delta} P^s\left(\left(z, \frac{y_n - x_n}{t^{\frac{1}{2s}}}\right), 1\right) dz dx_n dy_n.
 \end{aligned}$$

Using (1.7), we have

$$\begin{aligned} t^{-\frac{1}{2s}} \int_0^\delta \int_{-\delta}^0 \int_{\mathbb{R}^{n-1} \setminus Q'_{\frac{\delta}{t^{\frac{1}{2s}}}}(x')} P^s\left(z, \frac{y_n - x_n}{t^{\frac{1}{2s}}}, 1\right) dz dx_n dy_n &\leq t^{-\frac{1}{2s}} \int_0^\delta \int_{-\delta}^0 \int_{\mathbb{R}^{n-1} \setminus B_{\frac{\delta}{t^{\frac{1}{2s}}}}(x')} \frac{1}{|z|^{n+2s}} dz dx_n dy_n \\ &\leq (n-1)\omega_{n-1} t^{-\frac{1}{2s}} \int_0^\delta \int_{-\delta}^0 \int_{\frac{\delta}{t^{\frac{1}{2s}}}}^\infty \rho^{-2-2s} d\rho dx_n dy_n \\ &= (n-1)\omega_{n-1} \delta^{1-2s} t = o(t^{\frac{1}{2s}}). \end{aligned} \quad (4.13)$$

The proof of (4.12) is now direct consequence of Lemma 4.6 and inequality (4.13). $\square$

Proof of Proposition 4.2. Formula (4.3) is now a direct consequence of Lemma 4.3 and Lemma 4.4 when $s = \frac{1}{2}$, while it comes from Lemma 4.7 and Lemma 4.8 when $s \in (\frac{1}{2}, 1)$. $\square$

We are now ready to prove our Γ -convergence theorem.

4.2 Proof of Theorem 4.1 (i)

Proposition 4.9. Let $s \in [\frac{1}{2}, 1)$, $Q = [-1, 1]^n$, $g_s(t)$ defined in (0.2) and let E be a set of finite perimeter. Set

$$\Gamma_{n,s} = \inf \left\{ \liminf_{t \rightarrow 0} \frac{1}{g_s(t)} \int_{E_t^c \cap Q} \int_{E_t \cap Q} P^s(x-y, t) dx dy : E_t \rightarrow \mathbb{R}_+^n \text{ in } L^1(\mathbb{R}^n) \right\}. \quad (4.14)$$

Then, for any sequence $E_i \rightarrow E$ and $t_i \rightarrow 0$ as $i \rightarrow +\infty$ we have

$$\liminf_{i \rightarrow \infty} \frac{1}{g_s(t_i)} \int_{E_i^c} \int_{E_i} P^s(x-y, t) dx dy \geq \Gamma_{n,s} P(E), \quad (4.15)$$

where the constant $\Gamma_{n,s}$ is defined in (4.14).

Proof. We set $H := \mathbb{R}_+^n = \mathbb{R}^{n-1} \times [0, +\infty)$. We prove (4.15) using a blow up argument. Denote by $\mathcal{C}$ the family of all n -cubes in $\mathbb{R}^n$

$$\mathcal{C} := \{R(x+rQ) : x \in \mathbb{R}^n, r > 0, R \in \text{SO}(n)\}.$$

Since the set E is of finite perimeter, for $|\nabla \chi_E|$ -a.e. $x_0 \in \mathbb{R}^n$ there exists $R_{x_0} \in \text{SO}(n)$ such that $\frac{1}{r}(E - x_0)$ locally converge in measure to $R_{x_0}H$ as $r \rightarrow 0$ and

$$\lim_{r \rightarrow 0} \frac{|\nabla \chi_E|(x_0 + rR_{x_0}Q)}{r^{n-1}} = 1 \quad \text{for } |\nabla \chi_E| \text{-a.e. } x_0. \quad (4.16)$$

Now, given a cube $C \in \mathcal{C}$, we set

$$\alpha_i(C) := \frac{1}{g(t_i)} \int_{E_i^c \cap C} \int_{E_i \cap C} P^s(x-y, t_i) dx dy$$

and

$$\alpha(C) := \liminf_{i \rightarrow \infty} \alpha_i(C).$$

We claim that, setting $C_r(x_0) := x_0 + rR_{x_0}Q$, where R_{x_0} is as in (4.16), for $|\nabla \chi_E|$ -a.e. x_0 we have

$$\Gamma_{n,s} \leq \liminf_{r \rightarrow 0} \frac{\alpha(C_r(x_0))}{r^{n-1}} \quad \text{for } |\nabla \chi_E| \text{-a.e. } x \in \mathbb{R}^n. \quad (4.17)$$

Without loss of generality, we assume $R_{x_0} = I$, so that the limit hyperplane is H and the cubes $C_r(x_0)$ are the standard ones $x_0 + rQ$. Let us choose a sequence $r_k \rightarrow 0$ such that

$$\liminf_{r \rightarrow 0} \frac{\alpha(C_r(x_0))}{r^{n-1}} = \lim_{k \rightarrow \infty} \frac{\alpha(C_{r_k}(x_0))}{r_k^{n-1}}.$$

For all $k \in \mathbb{N}$, we can pick a subsequence i_k so large that the following conditions hold:

$$\left\{ \begin{array}{l} \alpha_{i_k}(C_{r_k}(x_0)) \leq \alpha(C_{r_k}(x_0)) + r_k^n, \\ \int_{C_{r_k}(x)} |\chi_{E_{i_k}} - \chi_E| dx < \frac{|C_{r_k}|}{k}. \end{array} \right.$$

Then we infer

$$\begin{aligned} \frac{\alpha(C_{r_k}(x_0))}{r_k^{n-1}} &\geq \frac{\alpha_{i(k)}(C_{r_k}(x_0))}{r_k^{n-1}} - r_k \\ &= \frac{1}{g(t_{i_k})r_k^{n-1}} \int_{E_{i_k}^c \cap C_{r_k}(x_0)} \int_{E_{i_k} \cap C_{r_k}(x_0)} P^s(x-y, t_{i_k}) dx dy - r_k. \end{aligned} \quad (4.18)$$

Now, we distinguish between the case $s = \frac{1}{2}$ and $s \in (\frac{1}{2}, 1)$.

Case $s = \frac{1}{2}$. The inequality (4.18) becomes

$$\frac{\alpha(C_{r_k}(x_0))}{r_k^{n-1}} \geq \frac{1}{|\log t_{i_k}|} \int_{(E_{i_k}-x_0)/r_k \cap C} \int_{(E_{i_k}-x_0)/r_k \cap C} \frac{1}{(|x-y|^2 + \frac{t_{i_k}^2}{r_k^2})^{\frac{n+1}{2}}} dx dy. \quad (4.19)$$

Up to extract a further subsequence, we assume that $t_{i_k} \leq r_k^{k+1}$ and therefore

$$\log \frac{1}{t_{i_k}} = \log \frac{r_k}{t_{i_k}} + \frac{1}{k+1} \log \frac{1}{r_k^k} \leq \log \frac{r_k}{t_{i_k}} + \frac{1}{k} \log \frac{1}{r_k^k} \leq \left(1 + \frac{1}{k}\right) \log \frac{r_k}{t_{i_k}}.$$

Using that $\frac{1}{r_k}(E_{i_k} - x) \rightarrow H$, by the above inequality and (4.19) we obtain

$$\liminf_{r \rightarrow 0} \frac{\alpha(C_r(x_0))}{r^{n-1}} \geq \Gamma_{n,s},$$

where the constant $\Gamma_{n,s}$ is defined in (4.14). This concludes the proof of (4.17) when $s = \frac{1}{2}$.

Case $s \in (\frac{1}{2}, 1)$. In this case, we recall that $g(t) = t^{\frac{1}{2s}}$ (see (0.2)). Using the scaling property (1.8), we have that inequality (4.18) becomes

$$\begin{aligned} \frac{\alpha(C_{r_k}(x_0))}{r_k^{n-1}} &= \frac{t_{i_k}^{-\frac{n+1}{2s}}}{r_k^{n-1}} \int_{E_{i_k}^c \cap C_{r_k}(x_0)} \int_{E_{i_k} \cap C_{r_k}(x_0)} P^s((x-y)t^{-\frac{1}{2s}}, 1) dx dy - r_k \\ &= (t_{i_k}^{-\frac{1}{2s}} r_k)^{n+1} \int_{(E_{i_k}-x_0)/r_k \cap C} \int_{(E_{i_k}-x_0)/r_k \cap C} P^s((x-y)r_k t_{i_k}^{-\frac{1}{2s}}, 1) dx dy - r_k \\ &= (t_{i_k} r_k^{-2s})^{-\frac{1}{2s}} \int_{(E_{i_k}-x_0)/r_k \cap C} \int_{(E_{i_k}-x_0)/r_k \cap C} P^s((x-y), t_{i_k} r_k^{-2s}) dx dy - r_k. \end{aligned}$$

Up to extract a further subsequence, we can assume $t_{i_k} r_k^{-2s} \rightarrow 0$. Thus, we get

$$\frac{\alpha(C_{r_k}(x_0))}{r_k^{n-1}} \geq \Gamma_{n,s}$$

with $\Gamma_{n,s}$ defined in (4.14). Hence, we have (4.17) also when $s \in (\frac{1}{2}, 1)$. The proof of Proposition 4.9 follows from (4.17) and using a suitable version of the Vitali Covering Theorem. $\square$

We are now in a position to prove Theorem 4.1 (i).

Proof of Theorem 4.1 (i). To conclude the proof, we need to show that $\Gamma_{n,s} = \Gamma^{n,s}$. The proof of this fact is the same as in [2, Lemma 13]. We decide to not reproduce it here in details but to just sketch it. First, via a gluing argument, which is possible since a coarea formula holds for our functional, one can show that the infimum in the definition of $\Gamma_{n,s}$ can be taken among sets which coincide with H outside a smaller cube Q_δ and then, via a non-local calibration argument, one also gets the minimality of the halfspace (see [6, 28]), which readily provides $\Gamma^{n,s} = \Gamma_{n,s}$. $\square$

4.3 Proof of Theorem 4.1 (ii)

Proof of Theorem 4.1 (ii). We start by proving that for a polyhedron Π it holds

$$\limsup_{t \rightarrow 0} \frac{1}{g_s(t)} \int_{\Pi} \int_{\Pi^c} P^s(x-y, t) dx dy \leq \Gamma^{n,s} P(\Pi). \quad (4.20)$$

For any $\varepsilon > 0$ there exists δ_0 such that for any $\delta \in (0, \delta_0)$ there exists a collection of N_δ cubes of volume $(2\delta)^n$ centered at $x_i \in \partial\Pi$ such that each cubes $Q_\delta(x_i)$ intersect one and only one face of $\partial\Pi$ and

- if $\tilde{Q}_\delta(x_i)$ denotes the dilation of $Q_\delta(x_i)$ by a factor $(1 + \delta)$, then each cube $\tilde{Q}_\delta(x_i)$ intersects exactly one face Σ of $\partial\Pi$ and each of its sides is either parallel or orthogonal to Σ ,
- $|\mathcal{H}^{n-1}(\partial\Pi) - N_\delta(2\delta)^{n-1}| < \varepsilon$.

Moreover, since we are interested in the limit as $t \rightarrow 0$, we can assume that $t < \delta$. For all $\delta > 0$, we set

$$(\partial\Pi)_\delta := \{x \in \mathbb{R}^n : d(x, \partial\Pi) < \delta\}, \quad (\partial\Pi)_\delta^- := (\partial\Pi)_\delta \cap \Pi.$$

For $x \in \Pi$, we define

$$I(x, t) = \int_{\Pi^c} P^s(x-y, t) dy.$$

We now have to distinguish among many cases.

Case one: $x \in \Pi \setminus (\Pi)_\delta^-$. In this case, we note that (1.7) implies the existence of a constant $c_{n,s}$ such that

$$P^s(x-y, t) \leq \frac{c_{n,s} t}{(|x-y|^2 + t^{\frac{1}{s}})^{\frac{n+2s}{2}}}$$

and therefore

$$\begin{aligned} I_s(x, t) &\leq c_{n,s} \int_{\Pi^c} \frac{t}{(|x-y|^2 + t^{\frac{1}{s}})^{\frac{n+2s}{2}}} \leq n\omega_n t \int_{\delta}^{\infty} \frac{\rho^{n-1}}{(\rho^2 + t^{\frac{1}{s}})^{\frac{n+2s}{2}}} d\rho \\ &\leq n\omega_n t \int_{\delta}^{\infty} \frac{\rho}{(\rho^2 + t^{\frac{1}{s}})^{1+s}} d\rho \\ &= \frac{\delta t}{2s(\delta^2 + t^{\frac{1}{s}})^s}. \end{aligned} \quad (4.21)$$

Case two: $x \in \Pi_\delta^- \setminus \bigcup_j Q_\delta(x_j)$. In this case, we write

$$\partial\Pi = \bigcup_{j=1}^J \Sigma_j,$$

and define

$$(\partial\Pi)_{\delta,j}^- := \{x \in (\partial\Pi)_\delta^- : d(x, \Pi^c) = d(x, \Sigma_j)\}.$$

Clearly, $(\partial\Pi)_\delta^- = \bigcup_{j=1}^J (\partial\Pi)_{\delta,j}^-$ and

$$(\partial\Pi)_{\delta,j}^- \subset \{x + t\nu : x \in \Sigma_{\delta,j}, t \in (0, \delta), \nu \text{ is the interior unit normal to } \Sigma_{\delta,j}\},$$

and $\Sigma_{\delta,j}$ is the set of points x belonging to the same hyperplane as Σ_j and with $d(x, \Sigma_j) \leq \delta$. Set $d(x) := \text{dist}(x, \Pi^c)$ and observe that $B_{d(x)}(x) \subset \Pi$. Therefore,

$$\begin{aligned} I(x, t) &\leq \int_{B_{d(x)}^c(x)} P^s(x-y, t) dy \leq n\omega_n t \int_{d(x)}^{\infty} \frac{\rho^{n-1}}{(\rho^2 + t^{\frac{1}{s}})^{\frac{n+2s}{2}}} d\rho \\ &\leq \frac{n\omega_n}{2s} \frac{t}{(d(x)^2 + t^{\frac{1}{s}})^s}. \end{aligned}$$

Integrating the above inequality on the set $(\partial\Pi)_\delta^- \setminus \bigcup_{i=1}^{N_\delta} Q_i^\delta$ gives

$$\begin{aligned}
 \int_{(\partial\Pi)_\delta^- \setminus \bigcup_{i=1}^{N_\delta} Q_\delta(x_i)} I(x, t) dx &\leq \frac{n\omega_n}{s} \sum_{j=1}^J \int_{(\partial\Pi)_{\delta,j}^- \setminus \bigcup_{i=1}^{N_\delta} Q_\delta(x_i)} \frac{t}{(d(x)^2 + t^{\frac{1}{s}})^s} dx \\
 &\leq \frac{n\omega_n}{s} \sum_{j=1}^J \int_{(\partial\Pi)_{\delta,j}^- \setminus \bigcup_{i=1}^{N_\delta} Q_\delta(x_i)} \frac{t}{[\text{dist}(x, \Sigma_{\delta,j})]^s} dx \\
 &\leq \frac{n\omega_n t}{s} \sum_{j=1}^J \int_{(\Sigma_{\delta,j}) \setminus \bigcup_{i=1}^{N_\delta} Q_\delta(x_i)} \left(\int_0^\delta \frac{dr}{(r^2 + t^{\frac{1}{s}})^s} \right) d\mathcal{H}^{n-1} \\
 &= \frac{n\omega_n t^{\frac{1}{2s}}}{s} \sum_{j=1}^J \int_{(\Sigma_{\delta,j}) \setminus \bigcup_{i=1}^{N_\delta} Q_\delta(x_i)} \left(\int_0^{\delta t^{-\frac{1}{2s}}} \frac{dr}{(r^2 + 1)^s} \right) d\mathcal{H}^{n-1}.
 \end{aligned} \tag{4.22}$$

Now we need to distinguish between the case $s = \frac{1}{2}$ and $s > \frac{1}{2}$. In the former case we have

$$\int_0^{\delta t^{-1}} \frac{1}{r^2 + 1} dr = \log(\delta + \sqrt{\delta^2 + t^2}) - \log(t) \leq |\log t|.$$

Therefore, (4.22) becomes

$$\int_{(\partial\Pi)_\delta^- \setminus \bigcup_{i=1}^{N_\delta} Q_\delta(x_i)} I(x, t) dx \leq 2n\omega_n t |\log t| \mathcal{H}^{n-1} \left(\left(\bigcup_{j=1}^J \Sigma_{\delta,j} \right) \setminus \bigcup_{i=1}^{N_\delta} Q_\delta(x_i) \right).$$

While, for $s > \frac{1}{2}$, from (4.22) we get

$$\int_{(\partial\Pi)_\delta^- \setminus \bigcup_{i=1}^{N_\delta} Q_i^\delta} I(x, t) dx \leq \frac{n\omega_n C t^{\frac{1}{2s}}}{s} \mathcal{H}^{n-1} \left(\left(\bigcup_{j=1}^J \Sigma_{\delta,j} \right) \setminus \bigcup_{i=1}^{N_\delta} Q_i^\delta \right),$$

where

$$C = \int_0^\infty \frac{1}{(r^2 + 1)^s} dr.$$

Therefore, in any case we found that for $s \geq \frac{1}{2}$ it holds

$$\frac{1}{g_s(t)} \int_{(\partial\Pi)_\delta^- \setminus \bigcup_{i=1}^{N_\delta} Q_\delta(x_i)} I(x, t) \leq C_{n,s}^2 \mathcal{H}^{n-1} \left(\left(\bigcup_{j=1}^J \Sigma_{\delta,j} \right) \setminus \bigcup_{i=1}^{N_\delta} Q_\delta(x_i) \right) \leq C_{n,s}^2 \varepsilon.$$

Case three: $x \in \Pi^- \cap \bigcup Q_\delta(x_i)$. In this case, we write

$$I(x, t) = \int_{\Pi^c \cap \{y: |x-y| \geq \delta^2\}} P^s(x-y, t) dy + \int_{\Pi^c \cap \{y: |x-y| < \delta^2\}} P^s(x-y, t) dy.$$

For the first integral, arguing as in (4.21), we have

$$\int_{\Pi^c \cap \{y: |x-y| \geq \delta^2\}} P^s(x-y, t) dy \leq \frac{\delta^2 t}{2s(\delta^4 + t^2)^s}.$$

Therefore, for each $1 \leq i \leq N_\delta$ we have

$$\begin{aligned}
 \int_{Q_\delta(x_i) \cap \Pi} \int_{\Pi^c \cap \{y: |x-y| < \delta^2\}} P^s(x-y, t) dy &= \int_{Q_\delta(x_i) \cap \Pi} \int_{Q_{\delta+\delta^2}(x_i) \cap \Pi^c} P^s(x-y, t) dy \\
 &\leq \int_{Q_{\delta+\delta^2}(x_i) \cap \Pi} \int_{Q_{\delta+\delta^2}(x_i) \cap \Pi^c} P^s(x-y, t) dy \\
 &\leq \Gamma^{n,s}(g_s(t) + o(g_s(t))) \mathcal{H}^{n-1}(Q_{\delta+\delta^2}).
 \end{aligned} \tag{4.23}$$

Summing (4.23) for $1 \leq i \leq N_\delta$, we get

$$\frac{1}{g_s(t)} \int_{\bigcup_i Q_\delta(x_i)} \int_{\Pi^c} P^s(x-y, t) dx dy \leq \Gamma^{n,s} N_\delta (\delta + \delta^2)^{n-1} \leq \Gamma^{n,s} \mathcal{H}^{n-1}(\partial \Pi) + \varepsilon. \quad (4.24)$$

Collecting (4.21), (4.22), (4.24), (4.23) and sending $t \rightarrow 0$ first and $\varepsilon \rightarrow 0$ secondly, we get

$$\limsup_{t \rightarrow 0} \frac{1}{g_s(t)} \int_{\Pi} \int_{\Pi^c} P^s(x-y, t) dx dy \leq \Gamma^{n,s} \mathcal{P}(\Pi).$$

The argument for a generic set E of finite perimeter follows by density of the polyhedral sets in the class of the sets with finite perimeter. $\square$

4.4 Characterization of sets of finite perimeter

As a consequence of our Γ -convergence analysis, we prove that a set $E \in \mathcal{M}(\mathbb{R}^n)$ has finite perimeter if and only if for all $s \in [\frac{1}{2}, 1)$

$$\limsup_{t \rightarrow 0} \frac{\mathcal{E}_t^s(E)}{g_s(t)} < +\infty.$$

Theorem 4.10. *Let $E \in \mathcal{M}(\mathbb{R}^n)$.*

- (i) *If $\limsup_{t \rightarrow 0} \frac{\mathcal{E}_t^s(E)}{g_s(t)} < +\infty$ for some $s \in [\frac{1}{2}, 1)$, then E is a set of finite perimeter.*
- (ii) *If E is a set of finite perimeter, then*

$$\limsup_{t \rightarrow 0} \frac{\mathcal{E}_t^s(E)}{g_s(t)} < +\infty \quad \text{for every } s \in \left[\frac{1}{2}, 1\right). \quad (4.25)$$

Proof. To prove (i), we just need to apply Theorem 2.1 to the constant sequence $E_i = E$ for every $i \in \mathbb{N}$.

We prove (ii). First, we observe that a simple approximation argument provides

$$\int_{\mathbb{R}^n} |\chi_E(x+h) - \chi_E(x)| dx \leq |h| \mathcal{P}(E). \quad (4.26)$$

Let $s \in (\frac{1}{2}, 1)$. By the very definition of $\mathcal{E}_t^s(E)$ and (4.26), we have

$$\begin{aligned} \frac{\mathcal{E}_t^s(E)}{g_s(t)} &= \frac{1}{g_s(t)} \int_E \int_{E^c} P^s(h, t) dh dx \\ &= \frac{1}{2g_s(t)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\chi_E(x+h) - \chi_E(x)| P^s(h, t) dh dx \\ &\leq \frac{\mathcal{P}(E)}{2g_s(t)} \int_{\mathbb{R}^n} |h| P^s(h, t) dh. \end{aligned} \quad (4.27)$$

By (4.27) and the decay estimate for $P^s(h, t)$ in (1.7), we obtain

$$\limsup_{t \rightarrow 0^+} \frac{\mathcal{E}_t^s(E)}{g_s(t)} \leq \limsup_{t \rightarrow 0^+} \frac{\mathcal{P}(E)}{2} \left[\frac{1}{g_s(t)} \int_{B_{t^{\frac{1}{2s}}}} |h| t^{-\frac{n}{2s}} dh + \frac{1}{g_s(t)} \int_{B_{t^{\frac{1}{2s}}}^c} |h| \frac{t}{|h|^{n+2s}} dh \right] = \frac{\mathcal{P}(E)}{2} \left[\frac{n\omega_n}{n-1} + \frac{n\omega_n}{2s-1} \right].$$

Hence, we have that (4.25) is true for all $s \in (\frac{1}{2}, 1)$.

Let now $s = \frac{1}{2}$. Using Fubini Theorem and (4.26), we get

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\chi_E(x+h) - \chi_E(x)| P^{\frac{1}{2}}(h, t) dh dx &= \int_{B_1} P^{\frac{1}{2}}(h, t) \int_{\mathbb{R}^n} |\chi_E(x+h) - \chi_E(x)| dx dh \\ &\quad + \int_{B_1^c} P^{\frac{1}{2}}(h, t) \int_{\mathbb{R}^n} |\chi_E(x+h) - \chi_E(x)| dx dh \\ &\leq \mathcal{P}(E) \int_{B_1} |h| P^{\frac{1}{2}}(h, t) dh + 2|E| \int_{B_1^c} P^{\frac{1}{2}}(h, t) dh. \end{aligned}$$

Hence, using the very definition of $P^{\frac{1}{2}}$ (see (1.6)), we have

$$\begin{aligned}
 \limsup_{t \rightarrow 0^+} \frac{\mathcal{E}_t^{\frac{1}{2}}(E)}{g_s(t)} &= \frac{1}{g_s(t)} \int_E \int_{E^c} P^{\frac{1}{2}}(x-y, t) dy dx \\
 &= \limsup_{t \rightarrow 0^+} \frac{1}{2g_s(t)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\chi_E(x+h) - \chi_E(x)| P^{\frac{1}{2}}(h, t) dh dx \\
 &\leq \limsup_{t \rightarrow 0^+} \frac{1}{2t|\log(t)|} \left[P(E) \int_{B_1} \frac{t|h|}{(|h|^2 + t^2)^{\frac{n+1}{2}}} dh + 2|E| \int_{B_1^c} \frac{t}{(|h|^2 + t^2)^{\frac{n+1}{2}}} dh \right] \\
 &= \limsup_{t \rightarrow 0^+} \frac{1}{2|\log(t)|} \left[P(E) \int_{B_{\frac{1}{t}}} \frac{|\eta|}{(|\eta|^2 + 1)^{\frac{n+1}{2}}} d\eta + 2 \frac{|E|}{t} \int_{B_{\frac{1}{t}}^c} \frac{1}{(|\eta|^2 + 1)^{\frac{n+1}{2}}} d\eta \right] \\
 &= \limsup_{t \rightarrow 0^+} \frac{n\omega_n}{2|\log(t)|} \left[P(E) \int_0^{\frac{1}{t}} \frac{\rho^n}{(\rho^2 + 1)^{\frac{n+1}{2}}} d\rho + \frac{2|E|}{t} \int_{\frac{1}{t}}^{+\infty} \frac{\rho^{n-1}}{(\rho^2 + 1)^{\frac{n+1}{2}}} d\rho \right] \\
 &= C(n)P(E),
 \end{aligned} \tag{4.28}$$

where in the last step we have used the L'Hôpital's rule and the fact that

$$\frac{1}{t} \int_{\frac{1}{t}}^{+\infty} \frac{\rho^{n-1}}{(\rho^2 + 1)^{\frac{n+1}{2}}} d\rho \leq \frac{1}{t} \int_{\frac{1}{t}}^{+\infty} \frac{\rho}{(\rho^2 + 1)^{\frac{3}{2}}} d\rho \leq C.$$

Hence, by (4.28), we obtain (4.25) for $s = \frac{1}{2}$. □

5 Application: The local and the non-local isoperimetric inequality

As a consequence of the Γ -convergence result, we have a short proof of the isoperimetric inequality. Recall that $\int_{\mathbb{R}^n} P^s(z, t) dz = 1$ and $P^s(|z|, t)$ is a decreasing function of the modulus of z . Given a measurable set E of finite measure and any $s \in (0, 1)$, it is a straightforward application of Riesz rearrangement inequality that

$$\begin{aligned}
 \int_E \int_{E^c} P^s(x-y, t) dx dy &= |E| - \int_E \int_E P^s(x-y, t) dx dy \\
 &\geq |E^*| - \int_{E^*} \int_{E^*} P^s(x-y) dx dy \\
 &= \int_{E^*} \int_{(E^*)^c} P^s(x-y, t) dx dy,
 \end{aligned}$$

where E^* denotes the ball centered at the origin such that $|E^*| = |E|$. Therefore, if $\Pi \subset \mathbb{R}^n$ is a polyhedron, we have

$$\frac{1}{g_s(t)} \left(\int_{\Pi} \int_{\Pi^c} P^s(x-y, t) dx dy - \int_B \int_{B^c} P^s(x-y, t) dx dy \right) \geq 0,$$

where $|B| = |\Pi|$. Recalling (4.20), (4.15) and the fact $\Gamma^{n,s} = \Gamma_{n,s}$, we take the $\limsup$ as $t \rightarrow 0$ in the above inequality for $s \in [\frac{1}{2}, 1)$ to infer

$$P(\Pi) \geq P(B) = n\omega_n^{\frac{1}{n}} |\Pi|^{\frac{n-1}{n}}$$

and by density of the polyhedral sets in the class of the sets with finite perimeter we recover the isoperimetric inequality for all sets of finite perimeter. For $s \in (0, \frac{1}{2})$ we even have pointwise convergence, hence it is immediate to get that any set E with finite $2s$ -perimeter it holds

$$P_{2s}(E) - P_{2s}(B) \geq 0,$$

which is the fractional isoperimetric inequality.

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References

- [1] G. Alberti and G. Bellettini, A non-local anisotropic model for phase transitions: asymptotic behaviour of rescaled energies, *European J. Appl. Math.* **9** (1998), no. 3, 261–284.
- [2] L. Ambrosio, G. De Philippis and L. Martinazzi, Gamma-convergence of nonlocal perimeter functionals, *Manuscripta Math.* **134** (2011), no. 3–4, 377–403.
- [3] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Math. Monogr., Oxford University, New York, 2000.
- [4] J. Berendsen and V. Pagliari, On the asymptotic behaviour of nonlocal perimeters, *ESAIM Control Optim. Calc. Var.* **25** (2019), Paper No. 48.
- [5] J. Bourgain, H. Brezis and P. Mironescu, Another look at Sobolev spaces, in: *Optimal Control and Partial Differential Equations*, IOS, Amsterdam (2001), 439–455.
- [6] X. Cabré, Calibrations and null-Lagrangians for nonlocal perimeters and an application to the viscosity theory, *Ann. Mat. Pura Appl.* (4) **199** (2020), no. 5, 1979–1995.
- [7] L. A. Caffarelli and P. E. Souganidis, Convergence of nonlocal threshold dynamics approximations to front propagation, *Arch. Ration. Mech. Anal.* **195** (2010), no. 1, 1–23.
- [8] A. Carbonaro and G. Mauceri, A note on bounded variation and heat semigroup on Riemannian manifolds, *Bull. Aust. Math. Soc.* **76** (2007), no. 1, 155–160.
- [9] A. Carbotti, S. Cito, D. A. La Manna and D. Pallara, Gamma-convergence of Gaussian fractional perimeter, *Adv. Calc. Var.* **16** (2023), no. 3, 571–595.
- [10] V. Crismale, L. De Luca, A. Kubin, A. Ninno and M. Ponsiglione, The variational approach to s-fractional heat flows and the limit cases $s \rightarrow 0^+$ and $s \rightarrow 1^-$, *J. Funct. Anal.* **284** (2023), no. 8, Article ID 109851.
- [11] E. De Giorgi, Su una teoria generale della misura $(r - 1)$ -dimensionale in uno spazio ad r dimensioni, *Ann. Mat. Pura Appl.* (4) **36** (1954), 191–213.
- [12] L. De Luca, A. Kubin and M. Ponsiglione, The core-radius approach to supercritical fractional perimeters, curvatures and geometric flows, *Nonlinear Anal.* **214** (2022), Article ID 112585.
- [13] L. De Luca, M. Novaga and M. Ponsiglione, The 0-fractional perimeter between fractional perimeters and Riesz potentials, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **22** (2021), no. 4, 1559–1596.
- [14] A. De Rosa and D. A. La Manna, A non local approximation of the Gaussian perimeter: Gamma convergence and isoperimetric properties, *Commun. Pure Appl. Anal.* **20** (2021), no. 5, 2101–2116.
- [15] F. Ferrari, M. Miranda, Jr., D. Pallara, A. Pinamonti and Y. Sire, Fractional Laplacians, perimeters and heat semigroups in Carnot groups, *Discrete Contin. Dyn. Syst. Ser. S* **11** (2018), no. 3, 477–491.
- [16] I. Fonseca and S. Müller, Quasi-convex integrands and lower semicontinuity in L^1 , *SIAM J. Math. Anal.* **23** (1992), no. 5, 1081–1098.
- [17] N. Garofalo, Fractional thoughts, in: *New Developments in the Analysis of Nonlocal Operators*, Contemp. Math. 723, American Mathematical Society, Providence (2019), 1–135.
- [18] A. Garroni and S. Müller, A variational model for dislocations in the line tension limit, *Arch. Ration. Mech. Anal.* **181** (2006), no. 3, 535–578.
- [19] L. Gennaioli and G. Stefani, Sharp conditions for the BBM formula and asymptotics of heat content-type energies, preprint (2025), <https://arxiv.org/abs/2502.14655>.
- [20] B. Güneysu and D. Pallara, Functions with bounded variation on a class of Riemannian manifolds with Ricci curvature unbounded from below, *Math. Ann.* **363** (2015), no. 3–4, 1307–1331.
- [21] A. Kubin, V. Pagliari and A. Tribuzio, Second order asymptotic of fractional Gagliardo seminorms as $s \rightarrow 1^-$ and convergence of the associated gradient flows, preprint (2024), <https://arxiv.org/abs/2410.17829>.
- [22] A. Kubin and M. Ponsiglione, Attractive Riesz potentials acting on hard spheres, *Nonlinearity* **34** (2021), no. 1, 350–371.
- [23] T. Laux and F. Otto, Convergence of the thresholding scheme for multi-phase mean-curvature flow, *Calc. Var. Partial Differential Equations* **55** (2016), no. 5, 1–74.
- [24] M. Ledoux, Semigroup proofs of the isoperimetric inequality in Euclidean and Gauss space, *Bull. Sci. Math.* **118** (1994), no. 6, 485–510.
- [25] G. Leoni, *A First Course in Fractional Sobolev Spaces*, Grad. Stud. Math. 229, American Mathematical Society, Providence, 2023.
- [26] B. Merriman, J. K. Bence and S. J. Osher, Diffusion generated motion by mean curvature, CAM Report 92-18, Department of Mathematics, University of California, Los Angeles, 1992.

- [27] M. Miranda, Jr., D. Pallara, F. Paronetto and M. Preunkert, Heat semigroup and functions of bounded variation on Riemannian manifolds, *J. Reine Angew. Math.* **613** (2007), 99–119.
- [28] V. Pagliari, Halfspaces minimise nonlocal perimeter: a proof *via* calibrations, *Ann. Mat. Pura Appl. (4)* **199** (2020), no. 4, 1685–1696.