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ASYMPTOTIC ANALYSIS OF AN ARRAY OF CLOSELY SPACED ABSOLUTELY CONDUCTIVE INCLUSIONS

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Abstract. We consider the conductivity problem in an array structure with square closely spaced absolutely conductive inclusions of the high concentration, i.e. the concentration of inclusions is assumed to be close to 1. The problem depends on two small parameters: ε, the ratio of the period of the micro-structure to the characteristic macroscopic size, and δ, the ratio of the thickness of the strips of the array structure and the period of the micro-structure. The complete asymptotic expansion of the solution to problem is constructed and justified as both ε and δ tend to zero. This asymptotic expansion is uniform with respect to ε and δ in the area $\{\varepsilon = O(\delta^\alpha),\ \delta = O(\varepsilon^\beta)\}$ for any positive α, β.

1. **Introduction: statement of the problem.** A lot of engineering problems lead to the PDE's stated in some domains of a small measure. One of such examples is the so called array structures (Fig. 1) introduced in [11], [12] and then studied by several authors in [1], [5], [13], [14], [15].

These array structures are presented by domains in $\mathbb{R}^s$ $(s = 2, 3)$ depending on two small parameters ε and δ. Here ε stands for a period of the microstructure (while the macroscopic characteristic size is taken equal to 1), and every periodic cell consists of thin strips (rods in 3-dimensional case) of thickness $\varepsilon\delta$, i.e. δ is the ratio of the thickness to the length of each rod.

As mentioned above, the PDE's modeling the physical field or process, are set in this array structure, and at the boundary Neumann or Dirichlet conditions are prescribed (cf. [11], [12], [1], [5], [13], [14], [15], [9], [10]). In [12] the complete asymptotic expansion was constructed for a solution of the conductivity equation, and in [13] for a solution of the elasticity equations. In particular, in [13] it was

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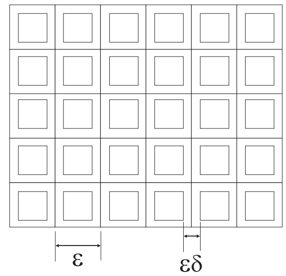

FIGURE 1. Rectangular array

proved that if $\frac{\varepsilon}{\delta} = \text{const}$ or $\frac{\varepsilon}{\delta} \to \infty$ then there is no convergence to the solution of formally homogenized problem; the leading term of the solution was constructed; it may be not bounded. For the details we refer to the book [14].

In the present paper we consider a composite rod which consists of one layer of the array structure $\Omega_{\varepsilon\delta}$ and infinitely conductive inclusions occupying the "holes" G^i of such structure (Fig. 2).

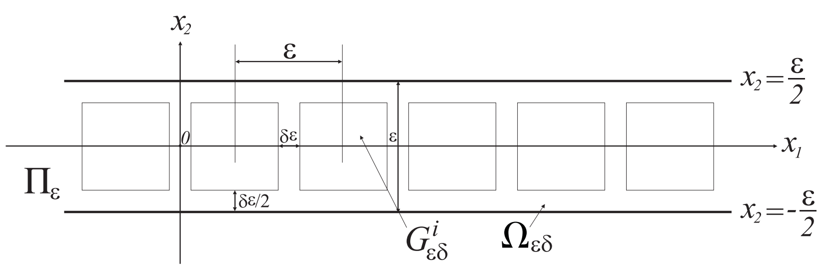

FIGURE 2. Domain $\Omega_{\varepsilon\delta}$

The leading term of the effective conductivity of such structure (as $\varepsilon \to 0$ followed by $\delta \to 0$) was obtained in [6] by the network approximation technique (see [2], [3], [4]). Below we construct an asymptotic expansion of the solution independently of the order of passage to the limit as $\varepsilon \to 0$, $\delta \to 0$.

More precisely, we consider a domain $\Omega_{\varepsilon\delta} = \Pi_\varepsilon \setminus \bigcup_{i \in \mathbb{Z}} G^i_{\varepsilon\delta}$, where $\Pi_\varepsilon = \left\{ \boldsymbol{x} \in \mathbb{R}^2 : |x_2| < \frac{\varepsilon}{2} \right\}$,

$$G^i_{\varepsilon\delta} = \left\{ \boldsymbol{x} \in \mathbb{R}^2 : i\varepsilon + \frac{\varepsilon\delta}{2} < x_1 < i\varepsilon + \left(1 - \frac{\delta}{2}\right)\varepsilon,\ |x_2| < \frac{\varepsilon\,(1-\delta)}{2} \right\},$$

in which the Laplace equation is set

$$-\Delta u_{\varepsilon\delta} = f\,(x_1)\,, \quad \boldsymbol{x} \in \Omega_{\varepsilon\delta} \tag{1}$$

with the Neumann boundary condition at the boundary $\partial\Pi_\varepsilon$:

$$\frac{\partial u_{\varepsilon\delta}}{\partial x_2} = 0, \quad \boldsymbol{x} \in \partial\Pi_\varepsilon \tag{2}$$

and with conditions of infinitely conductive inclusions at the boundary of each $G^i_{\varepsilon\delta}$, that is

$$\begin{aligned} &u_{\varepsilon\delta} = C_i, \\ &\int_{\Gamma^i_{\varepsilon\delta}} \frac{\partial u_{\varepsilon\delta}}{\partial n}\, dS = 0, \qquad \Gamma^i_{\varepsilon\delta} = \partial G^i_{\varepsilon\delta} \end{aligned} \tag{3}$$

where C_i is unknown constant, $f \in C^\infty(\mathbb{R})$ is a T-periodic function of x_1 independent on ε (the number T is divisible by ε), such that

$$\int_{\Omega_{\varepsilon\delta}\bigcap\{x_1\in(0,T)\}} f(x_1)\,d\boldsymbol{x}=0,\ \int_0^T f(x_1)\,dx_1=0,\ u_{\varepsilon\delta} \text{ is } T\text{-periodic function of } x_1. \tag{4}$$

Our goal is to study an asymptotic of the solution $u_{\varepsilon\delta}$ to the problem (1)÷(3) as $\varepsilon\to 0^+$ and $\delta\to 0^+$. Extend $u_{\varepsilon\delta}$ by constant C_i on every $G^i_{\varepsilon\delta}$. The existence and the uniqueness of solution $u_{\varepsilon\delta}$ of this problem, such that

$$\int_{(0,T)\times\left(-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right)} u_{\varepsilon\delta}\left(\boldsymbol{x}\right)\,d\boldsymbol{x}=0 \tag{5}$$

is proved in Appendix 1.

Remark 1. Assume that f and $f(x_1-\frac{T}{2})$ are odd functions, $\frac{T}{2}$ is divisible by ε. Then there exists a unique solution of problem (1)÷(3), (5) such that

$$u_{\varepsilon\delta|_{x_1=0}}=0 \quad\text{and}\quad u_{\varepsilon\delta}|_{x_1=\frac{T}{2}}=0.$$

Indeed, we observe that in our assumption $u_{\varepsilon\delta}\left(\cdot,x_2\right)$ is odd, too. In fact if we consider $-u_{\varepsilon\delta}\left(-x_1,x_2\right)$, we have

$$\begin{aligned}\Delta_{x_1x_2}\left(-u_{\varepsilon\delta}\left(-x_1,x_2\right)\right)&=-\Delta_{y_1y_2}\left(u_{\varepsilon\delta}\left(y_1,y_2\right)\right)|_{y_1=-x_1,y_2=x_2}\\&=-f\left(y_1\right)|_{y_1=-x_1}=-f\left(-x_1\right)=f\left(x_1\right).\end{aligned}$$

Moreover $-u_{\varepsilon\delta}\left(-x_1,x_2\right)$ satisfies conditions (2), (3) and (5). Then it is also a solution of problem (1)÷(3), (5). By uniqueness of solution we have

$$u_{\varepsilon\delta}\left(x_1,x_2\right)=-u_{\varepsilon\delta}\left(-x_1,x_2\right).$$

If $x_1=0$, the last equality is true if and only if $u_{\varepsilon\delta}\left(x_1,x_2\right)=0$. So

$$u_{\varepsilon\delta}\left(x_1,x_2\right)=0 \ \text{ when }\ x_1=0. \tag{6}$$

Since $f(x_1-\frac{T}{2})$ is odd and by periodicity we see that

$$u_{\varepsilon\delta}\left(x_1,x_2\right)=0\ \text{ if }\ x_1=\frac{T}{2}. \tag{7}$$

So, the T-periodic in x_1 solution of problem (1)÷(3), (5) is also a solution of more usual boundary value problem (1)÷(3), (6), (7). All results of Appendix 1 are still valid for solution of this problem (Fig. 3).

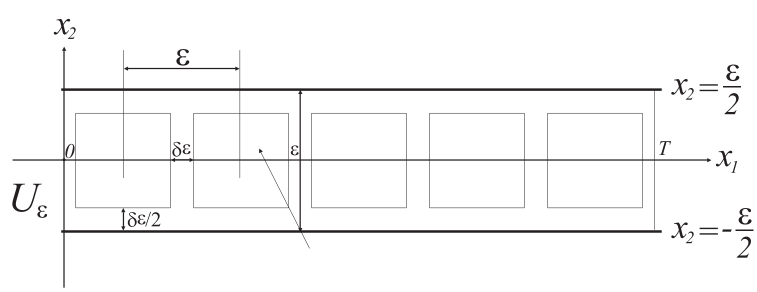

FIGURE 3. Finite bar

The paper is organized as follows. First we recall the asymptotic expansion technique in the case of finitely conductive inclusions (we follow [14] section 2.2.2). Then in section 2.1, we consider the simplified one-dimensional problem with absolutely conductive inclusions. This auxiliary problem helps to understand the behavior of the solutions of cell problems inside the inclusions and in the vertical strips of the domain $\Omega_{\varepsilon\delta}$. These solutions are used further in section 2.2 for the construction of the complete asymptotic expansion of the solution of problem (1)÷(3), (5). In particular, the solutions of one-dimensional cell problems of section 2.1 are modified in the neighborhoods of the corners of the periodic cell outside inclusions, as well as in the horizontal strips of $\Omega_{\varepsilon\delta}$. The "corner correctors" are functions of the boundary layer type: they are exponentially decaying as the distance from the corner divided by $\varepsilon\delta$ tends to infinity.

Finally a priori estimate for (1)÷(3), (5) is applied to prove the estimate of order $(\varepsilon^{\mathcal{K}-1}+\delta^{\mathcal{K}-1})\sqrt{\varepsilon}$ in H^1-norm for the $\mathcal{K}-$th partial sum of the truncated asymptotic expansion (it is uniform with respect to ε and δ such that $\varepsilon = O(\delta^\alpha)$ and $\delta = O(\varepsilon^\beta)$ for any positive α, β). This a priori estimate (as well as the existence and uniqueness of solution of problem (1)÷(3), (5)) is proved in Appendix 1. The exponential decaying of the solutions of the corner boundary layer problems (58), (59) is proved in Appendix 2: for each infinite branch the problem is periodically extended and reduced to the case [8].

Recall the asymptotic expansion method for the case of finite conductivity of inclusions described in [14] section 2.2.2. That is, consider the problem, which is similar to (1)÷(3), (5) but with finite conductivity $\mathcal{X}(\frac{\boldsymbol{x}}{\varepsilon})$ (since there is no dependence on the parameter δ here we drop such a subscript):

$$\operatorname{div}\left(\mathcal{X}\left(\frac{\boldsymbol{x}}{\varepsilon}\right)\nabla u_\varepsilon\right) = f\left(x_1\right), \quad \boldsymbol{x}\in\Pi_\varepsilon \tag{8}$$

$$\frac{\partial u_\varepsilon}{\partial x_2} = 0, \quad \boldsymbol{x}\in\partial\Pi_\varepsilon \tag{9}$$

where $\mathcal{X}\left(\xi_1,\xi_2\right)$ is a 1-periodic differentiable function of ξ_1,ξ_2, $\mathcal{X}\left(\xi_1,\xi_2\right) > 0$ on $\overline{\Pi}_1 = [0,1]\times[0,1]$ (the differentiability condition here is not important: see [14]). Suppose that $f\left(x_1\right)$ is T-periodic in x_1 and $\int_0^T f\left(x_1\right)dx_1 = 0$; we seek for a T-periodic in x_1 solution u_ε of problem (8), (9) with vanishing average on $U_\varepsilon = [0,T]\times[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}]$. Assume that $\mathcal{X}\left(\xi_1,\xi_2\right) = \mathcal{X}\left(\xi_2,\xi_1\right)$.

We look for the asymptotic solution u_ε in the following form:

$$u_\varepsilon^{(\mathcal{K})} = \sum_{\ell=0}^{\mathcal{K}+1}\varepsilon^\ell\mathcal{N}_\ell\left(\frac{\boldsymbol{x}}{\varepsilon}\right)D_1^\ell v_\varepsilon^{(\mathcal{K})}(x_1), \tag{10}$$

for some $\mathcal{K}>0$, where $\mathcal{N}_\ell(\boldsymbol{\xi})$, with $\boldsymbol{\xi} = \frac{\boldsymbol{x}}{\varepsilon}$, is 1-periodic function of ξ_1, $\mathcal{N}_0 = 1$,

$$v_\varepsilon^{(\mathcal{K})} = \sum_{j=0}^{\mathcal{K}}\varepsilon^j v_j(x_1), \quad v_j\in C^\infty(\mathbb{R}), \tag{11}$$

and $D_1^\ell = \frac{\partial^\ell}{\partial x_1^\ell}$. Denote $A_{kj} = \mathcal{X}\left(\boldsymbol{\xi}\right)\delta_{kj}$, where δ_{kj} is the Kronecker symbol.

Substituting (10) into (8) yields:

$$-\sum_{\ell=2}^{\mathcal{K}+1}\varepsilon^{\ell-2}\mathcal{H}_\ell\left(\frac{\boldsymbol{x}}{\varepsilon}\right)D_1^\ell v_\varepsilon^{(\mathcal{K})}(x_1) + \varepsilon^{\mathcal{K}}r_\varepsilon(\boldsymbol{x}) = f(x_1), \quad \boldsymbol{x}\in\Omega_{\varepsilon\delta}, \tag{12}$$

this relation is supposed to be true up to the terms of order $\varepsilon^{\mathcal{K}}$, here the discrepancy function

$$r_\varepsilon = \left\{ \sum_{k=1}^{2} \frac{\partial}{\partial \xi_k} \left(A_{k1} \mathcal{N}_{\mathcal{K}+1} (\boldsymbol{\xi}) \right) + \sum_{j=1}^{2} A_{1j} \frac{\partial \mathcal{N}_{\mathcal{K}+1} (\boldsymbol{\xi})}{\partial \xi_j} + A_{11} \mathcal{N}_{\mathcal{K}} (\boldsymbol{\xi}) \right\} \Bigg|_{\boldsymbol{\xi} = \frac{\boldsymbol{x}}{\varepsilon}} \cdot$$

$$D_1^{\mathcal{K}+2} v_\varepsilon^{(\mathcal{K})} (x_1) + \varepsilon A_{11} \mathcal{N}_{\mathcal{K}+1} \left(\frac{\boldsymbol{x}}{\varepsilon} \right) D_1^{\mathcal{K}+3} v_\varepsilon^{(\mathcal{K})} (x_1)$$

can be estimated by

$$\|r_\varepsilon\|_{L_2,C} \leq \mathcal{C},$$

(see below) and

$$\begin{aligned} \mathcal{H}_\ell(\boldsymbol{\xi}) &= \mathcal{L}_{\xi\xi} \mathcal{N}_\ell + \sum_{k=1}^{2} \frac{\partial}{\partial \xi_k} \left(A_{k1} \mathcal{N}_{\ell-1} \right) + \sum_{j=1}^{2} A_{1j} \frac{\partial \mathcal{N}_{\ell-1}}{\partial \xi_j} + A_{11} \mathcal{N}_{\ell-2} \\ &= \mathcal{L}_{\xi\xi} \mathcal{N}_\ell + \frac{\partial}{\partial \xi_1} \left(\mathcal{X} \mathcal{N}_{\ell-1} \right) + \mathcal{X} \frac{\partial \mathcal{N}_{\ell-1}}{\partial \xi_1} + \mathcal{X} \mathcal{N}_{\ell-2} \end{aligned} \tag{13}$$

with operator $\mathcal{L}_{\xi\xi} = \operatorname{div}_\xi \left(\mathcal{X} (\boldsymbol{\xi}) \nabla_\xi \right)$. Hereafter we set $\mathcal{N}_m = 0$ for $m < 0$.

Substituting (10) into the boundary condition (9) gives us:

$$\frac{\partial u_\varepsilon^{(\mathcal{K})}}{\partial n} = \sum_{\ell=1}^{\mathcal{K}+1} \varepsilon^{\ell-1} \frac{\partial \mathcal{N}_\ell}{\partial \xi_2} D_1^\ell v_\varepsilon^{(\mathcal{K})} (x_1) = 0. \tag{14}$$

We require that

$$\begin{aligned} &(a) \qquad \mathcal{H}_\ell(\boldsymbol{\xi}) = h_\ell, \quad \ell > 0, \\ &(b) \qquad \frac{\partial \mathcal{N}_\ell}{\partial n_\xi} = 0, \qquad \xi_2 = \pm \frac{1}{2}, \end{aligned} \tag{15}$$

where h_ℓ is a constant, defined below in Remark 2.

Note that for $\mathcal{L}_{\xi\xi} = \triangle$ (that is, $A_{kj} = \delta_{kj}$) the equality (13) becomes:

$$\mathcal{H}_\ell(\boldsymbol{\xi}) = \triangle \mathcal{N}_\ell + 2 \frac{\partial \mathcal{N}_{\ell-1}}{\partial \xi_1} + \mathcal{N}_{\ell-2}. \tag{16}$$

Remark 2. **(Solvability of the problem** (15)**)** There exists (up to a constant) a solution to problem (15) if and only if $h_\ell = \left\langle \sum_{j=1}^2 A_{1j} \frac{\partial \mathcal{N}_{\ell-1}}{\partial \xi_j} + A_{11} \mathcal{N}_{\ell-2} \right\rangle$, where $\langle \cdot \rangle$ is an average over Π_1. In particular, if $\mathcal{L}_{\xi\xi} = \triangle$, one has $h_\ell = \left\langle \frac{\partial \mathcal{N}_{\ell-1}}{\partial \xi_1} + \mathcal{N}_{\ell-2} \right\rangle$.

For $\ell = 0, 1, 2$ we have $h_0 = 0$, $h_1 = 0$, $h_2 = \left\langle \sum_{j=1}^2 A_{1j} \frac{\partial \mathcal{N}_1}{\partial \xi_j} + A_{11} \right\rangle = 0$, respectively, and when $\ell = 1$ the problem (15) is the standard cell problem:

$$\begin{aligned} &\mathcal{L}_{\xi\xi} \mathcal{N}_1 + \sum_{k=1}^{2} \frac{\partial}{\partial \xi_k} A_{k1} = 0, \qquad \boldsymbol{\xi} \in Y, \\ &\frac{\partial \mathcal{N}_1}{\partial n_\xi} = 0, \qquad \xi_2 = \pm \tfrac{1}{2}, \end{aligned} \tag{17}$$

where $Y = \left(-\frac{1}{2}, \frac{1}{2} \right) \times \left(-\frac{1}{2}, \frac{1}{2} \right)$. (For $\mathcal{L}_{\xi\xi} = \triangle$ the equation is $\triangle \mathcal{N}_1 = 0$). From (12) it follows:

$$-\sum_{\ell=2}^{\mathcal{K}+1} \varepsilon^{\ell-2} \, h_\ell \, D_1^\ell \, v_\varepsilon^{(\mathcal{K})}(x_1) + \varepsilon^{\mathcal{K}} r_\varepsilon(\boldsymbol{x}) = f(x_1), \tag{18}$$

which is called the *higher order homogenized equation.*

Next we find coefficients of the expansion (11) by substituting it into (18):

$$-\sum_{\ell=2}^{\mathcal{K}+1}\sum_{j=0}^{\mathcal{K}}\varepsilon^{\ell+j-2}\,h_\ell\,D_1^\ell\,v_j(x_1)+\varepsilon^{\mathcal{K}}r_\varepsilon(\boldsymbol{x})+\varepsilon^{\mathcal{K}+1}r_\varepsilon^1(x_1)=f(x_1),$$

or

$$-\sum_{m=0}^{2\mathcal{K}-1}\varepsilon^m\sum_{0\le j\le\min(m,\mathcal{K})}h_{m-j+2}\,D_1^{m-j+2}\,v_j(x_1)+\varepsilon^{\mathcal{K}}r_\varepsilon(\boldsymbol{x})+\varepsilon^{\mathcal{K}+1}r_\varepsilon^1(x_1)=f(x_1). \quad (19)$$

This relation is supposed to be true up to the terms of order $\varepsilon^{\mathcal{K}}$, and the second remainder is

$$\varepsilon^{\mathcal{K}+1}r_\varepsilon^1(x_1)=\sum_{m=\mathcal{K}+1}^{2\mathcal{K}-1}\varepsilon^m\sum_{0\le j\le\mathcal{K}}h_{m-j+2}\,D_1^{m-j+2}\,v_j(x_1)$$

such that

$$\left\|\varepsilon^{\mathcal{K}+1}r_\varepsilon^1(x_1)\right\|_{L^\infty(\mathbb{R})}\le c\varepsilon^{\mathcal{K}+1}.$$

Thus, for any $m=0,\dots,\mathcal{K}$ the function $v_j(x_1)$ satisfies the following equation:

$$-\sum_{j=0}^{m}h_{m-j+2}\,D_1^{m-j+2}\,v_j(x_1)=f(x_1)\,\delta_{m0}, \quad (20)$$

with periodic boundary conditions. In particular, for $m=0$:

$$\begin{array}{l}-h_2\,D_1^2\,v_0=f(x_1),\\ v_0 \text{ is } 1-\text{periodic in } x_1,\end{array} \quad (21)$$

for $m=1$:

$$\begin{array}{l}-h_2\,D_1^2\,v_1-h_3\,D_1^3\,v_0=0,\\ v_1 \text{ is } 1-\text{periodic in } x_1,\end{array} \quad (22)$$

for $1<m\le\mathcal{K}$:

$$\begin{array}{l}-h_2\,D_1^2\,v_m-\sum\limits_{j=0}^{m-1}h_{m-j+2}\,D_1^{m-j+2}\,v_j=0,\\ v_m \text{ is } 1-\text{periodic in } x_1.\end{array} \quad (23)$$

The solvability condition

$$\text{for } m=0: \quad \int_0^T f(x_1)\,dx_1=0$$

is satisfied due to the assumption made above, while

$$\int_0^T h_3\,D_1^3\,v_0\,dx_1=\left(h_3\,D_1^2\,v_0\right)\Big|_0^T=0, \qquad \text{for } m=1,$$

as well as

$$\int_0^T\sum_{j=0}^{m-1}h_{m-j+2}\,D_1^{m-j+2}\,v_j\,dx_1=0, \quad \text{for } m>1 \quad (24)$$

is satisfied automatically due to the periodicity of the function v_j.

After this step, the equation (8) is satisfied up to a remainder $\varepsilon^{\mathcal{K}}r_\varepsilon(\boldsymbol{x})+\varepsilon^{\mathcal{K}+1}r_\varepsilon^1(x_1)$. Applying the standard a priori estimate, we obtain

$$\left\|u_\varepsilon^{(\mathcal{K})}-u_\varepsilon\right\|_{H^1\left((0,T)\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})\right)}\le c\varepsilon^{\mathcal{K}}\sqrt{\varepsilon}, \quad (25)$$

which justifies the above procedure.

In the next section we apply the same methods to the problem (1)÷(3), (5). It should be modified in order to take into account the new boundary conditions (3) and the dependency of the domain $\Omega_{\varepsilon\delta}$ on the second small parameter δ.

Remark 3. Extending problem (1)÷(3), (5) $\varepsilon-$periodically with respect to x_2, one can see that problem (8), (9) set in $\mathbb{R}^2$ has a solution which tends to v_0, and v_0 satisfies equation (21) equivalent to the equation $-h_2\Delta v_0 = f(x_1)$ (because v_0 depends only on x_1). It means that h_2 is the effective conductivity of the periodic medium in x_1 direction. Here the effective conductivity is defined as the conductivity of an homogeneous medium mechanically equivalent to the heterogeneous one. It means that for any smooth right hand side f the solution of the conductivity equation for the heterogeneous medium is close to the solution of the conductivity problem of the effective homogeneous medium. We recall that if $\mathcal{X} = +\infty$ in the inclusions and $\mathcal{X} = 1$ out of inclusions, then it is proved in [14], p. 316 that $h_2 = \frac{1}{\delta} + o\left(\frac{1}{\delta}\right)$ as $\delta \to 0$. Let us mention here that if we consider a perforated medium and the right hand side has a support out of the holes then the effective conductivity is defined as h_2 multiplied by the measure of the periodic cell without the hole in dilated $\xi-$variables, that is, one minus the volume concentration of the holes. This factor can be explained by the following reason: in the mechanically equivalent homogeneous medium the macroscopically equivalent right hand side is "diffused" everywhere, even inside the holes. It means that it is equal to the original f multiplied by the mentioned above factor.

2. **Asymptotic expansion of the problem for a strip with infinitely conductive inclusions.** We apply now the technique presented in the previous section to the case of the infinitely conductive inclusions, that is, to problem (1)÷(3), (5).

For $\varepsilon \to 0^+$ and $\delta \to 0^+$ we are looking for the solution of (1)÷(3), (5) in the form of an asymptotic expansion:

$$u_{\varepsilon\delta}^{(\mathcal{K})} = \sum_{\ell=0}^{\mathcal{K}+1} \varepsilon^{\ell} \mathcal{N}_{\ell}\left(\frac{\boldsymbol{x}}{\varepsilon}\right) D_1^{\ell} v_{\varepsilon\delta}^{(\mathcal{K})}(x_1), \tag{26}$$

for some $\mathcal{K} > 0$, where

$$v_{\varepsilon\delta}^{(\mathcal{K})}(x_1) = \sum_{j,r=0}^{\mathcal{K}} \varepsilon^j \delta^r v_{jr}(x_1), \tag{27}$$

where, as before, $\mathcal{N}_\ell$ is 1-periodic continuous function of $\xi_1 = \frac{x_1}{\varepsilon}$, $\mathcal{N}_0 = 1$.

2.1. **Simplified one-dimensional problem.** First, we consider a simplified one-dimensional problem for a strip with vertical infinitely conductive inclusions (Fig. 4):

$$\widehat{G}_{\varepsilon\delta}^{i} = \left\{\boldsymbol{x} \in \mathbb{R}^2 : i\varepsilon + \frac{\varepsilon\delta}{2} < x_1 < i\varepsilon + \left(1 - \frac{\delta}{2}\right)\varepsilon,\ |x_2| < \frac{\varepsilon}{2}\right\}$$

Then the solution of (1)÷(3), (5) in such a domain depends on x_1 only and such problem can be rewritten as (x_1 is denoted by x)

$$u_{\varepsilon\delta}'' = f(x), \quad x \in \widehat{\Omega}_{\varepsilon\delta}, \tag{28}$$

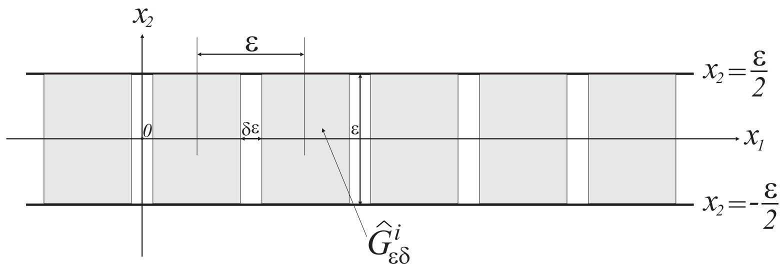

FIGURE 4. Simplified one-dimensional structure

where $\widehat{\Omega}_{\varepsilon\delta} = \bigcup_{i\in\mathbb{Z}} \left\{x \in \mathbb{R} : |x - i\varepsilon| < \frac{\varepsilon\delta}{2}\right\}$ with boundary conditions

$$\begin{aligned} &u_{\varepsilon\delta} = C_i, \quad x \in \left(i\varepsilon + \tfrac{\delta\varepsilon}{2},\, i\varepsilon + (1 - \tfrac{\delta}{2})\varepsilon\right), \\ &-\left.\frac{du_{\varepsilon\delta}}{dx}\right|_{x=i\varepsilon+\frac{\delta\varepsilon}{2}} + \left.\frac{du_{\varepsilon\delta}}{dx}\right|_{x=i\varepsilon+(1-\frac{\delta}{2})\varepsilon} = 0, \end{aligned} \tag{29}$$

or due to periodicity

$$\begin{aligned} &(a) \quad u_{\varepsilon\delta} = C_i, \quad x \in \left(\tfrac{\delta\varepsilon}{2},\, (1 - \tfrac{\delta}{2})\varepsilon\right), \\ &(b) \quad -\left.\frac{du_{\varepsilon\delta}}{dx}\right|_{x=\frac{\delta\varepsilon}{2}} + \left.\frac{du_{\varepsilon\delta}}{dx}\right|_{x=(1-\frac{\delta}{2})\varepsilon} = 0, \end{aligned} \tag{30}$$

where C_i is an unknown constant. As before, we seek for a T-periodic solution $u_{\varepsilon\delta}$.

Substitution of the expansion (26) into equation (28) yields (ξ_1 is denoted by ξ):

$$\frac{d^2\mathcal{N}_\ell}{d\xi^2} + 2\frac{d\mathcal{N}_{\ell-1}}{d\xi} + \mathcal{N}_{\ell-2} = h_\ell, \quad \text{in } \left(0, \frac{\delta}{2}\right) \bigcup \left(1 - \frac{\delta}{2}, 1\right), \quad \ell \geq 1, \tag{31}$$

with $h_\ell = \left\langle \frac{d\mathcal{N}_{\ell-1}}{d\xi} + \mathcal{N}_{\ell-2} \right\rangle_1 + \frac{1}{\delta}\left(\frac{d\mathcal{N}_\ell}{d\xi} + \mathcal{N}_{\ell-1}\right)\Big|_{\xi=-\delta/2+0}^{\xi=\delta/2-0}$ where $\langle\cdot\rangle_1 = \frac{1}{\delta}\int_{-\delta/2}^{\delta/2} \cdot d\xi$. Note that $\mathcal{N}_0 = 1$ and $\mathcal{N}_\ell = 0$ for $\ell < 0$.

We remark that condition (30a) means that $\frac{du_{\varepsilon\delta}}{dx} = 0$ in $\left(\frac{\delta}{2}\varepsilon, (1 - \frac{\delta}{2})\varepsilon\right)$, then after substituting (26) into this equation we have:

$$\sum_{\ell=1}^{\mathcal{K}} \varepsilon^{\ell-1} \left(\frac{d\mathcal{N}_\ell}{d\xi} + \mathcal{N}_{\ell-1}\right) \frac{d^\ell v_{\varepsilon\delta}}{dx^\ell} = 0, \quad \text{in } \left(\frac{\delta}{2}\varepsilon, (1 - \frac{\delta}{2})\varepsilon\right), \tag{32}$$

up to a remainder

$$r^{(3)}_{i,\varepsilon\delta}(x) = \varepsilon^{\mathcal{K}} \mathcal{N}_{\mathcal{K}}(\xi) \frac{d^{\mathcal{K}+1} v_{\varepsilon\delta}(x)}{dx^{\mathcal{K}+1}}$$

from which we obtain the following equations for $\mathcal{N}_\ell$:

$$\frac{d\mathcal{N}_\ell}{d\xi} + \mathcal{N}_{\ell-1} = 0, \quad \text{in } \left(\frac{\delta}{2}, 1 - \frac{\delta}{2}\right). \tag{33}$$

Remark 4. The remainder $r^{(3)}_{i,\varepsilon\delta}$ shows that the solution does not belong to the subspace of functions equal to constant on the inclusions. So we will have to think about the construction of a special corrector, equal to $-\int r^{(3)}_{i,\varepsilon\delta}\, dx_1$ on inclusions placing the asymptotic expansion into the space of functions, constant on inclusions.

Condition (30b) implies:

$$
\begin{aligned}
&-\sum_{\ell=1}^{\mathcal{K}} \varepsilon^{\ell-1}\left(\frac{d\mathcal{N}_\ell}{d\xi}+\mathcal{N}_{\ell-1}\right)\frac{d^\ell v_{\varepsilon\delta}}{dx^\ell}\Bigg|_{\substack{x=\frac{\delta}{2}\varepsilon\\ \xi=\frac{\delta}{2}}}\\
&+\sum_{\ell=1}^{\mathcal{K}} \varepsilon^{\ell-1}\left(\frac{d\mathcal{N}_\ell}{d\xi}+\mathcal{N}_{\ell-1}\right)\frac{d^\ell v_{\varepsilon\delta}}{dx^\ell}\Bigg|_{\substack{x=(1-\frac{\delta}{2})\varepsilon\\ \xi=-\frac{\delta}{2}}}=0,
\end{aligned}
\tag{34}
$$

up to the remainder

$$
r_{\mathcal{K}}^{(2)}(x)=-\varepsilon^{\mathcal{K}}\mathcal{N}_{\mathcal{K}}(\xi)\,\frac{d^{\mathcal{K}+1}v_{\varepsilon\delta}}{dx^{\mathcal{K}+1}}\Bigg|_{\substack{x=\frac{\delta}{2}\varepsilon\\ \xi=\frac{\delta}{2}}}+\varepsilon^{\mathcal{K}}\mathcal{N}_{\mathcal{K}}(\xi)\,\frac{d^{\mathcal{K}+1}v_{\varepsilon\delta}}{dx^{\mathcal{K}+1}}\Bigg|_{\substack{x=(1-\frac{\delta}{2})\varepsilon\\ \xi=-\frac{\delta}{2}}}.
$$

The function $v_{\varepsilon\delta}$ is expanded into the Taylor series around a point $x_0\in\left(\frac{\delta\varepsilon}{2},\,(1-\frac{\delta}{2})\varepsilon\right)$:

$$
v_{\varepsilon\delta}(x)=\sum_{j=0}^{\mathcal{M}}\frac{v_{\varepsilon\delta}^{(j)}(x_0)}{j!}(x-x_0)^j+\frac{1}{(\mathcal{M}+1)!}v_{\varepsilon\delta}^{(\mathcal{M}+1)}(y)(x-x_0)^{\mathcal{M}+1},\tag{35}
$$

for some $y\in(x_0,x)$. For the point $x_0=\frac{\varepsilon}{2}$ we have the following:

$$
\begin{aligned}
(x-x_0)|_{x=\frac{\delta}{2}\varepsilon}&=\varepsilon\left(\frac{\delta-1}{2}\right),\\
(x-x_0)|_{x=(1-\frac{\delta}{2})\varepsilon}&=\varepsilon\left(\frac{1-\delta}{2}\right).
\end{aligned}
$$

Hence,

$$
\frac{d^\ell v_\varepsilon\left(\frac{\delta}{2}\varepsilon\right)}{dx^\ell}=\sum_{j=0}^{\mathcal{M}}\frac{d^{\ell+j}v_{\varepsilon\delta}(x_0)}{dx^{\ell+j}}\left(\frac{\delta-1}{2}\right)^j\frac{\varepsilon^j}{j!}+R^+_{\varepsilon,\mathcal{M},l},\tag{36}
$$

$$
\frac{d^\ell v_\varepsilon\left((1-\frac{\delta}{2})\varepsilon\right)}{dx^\ell}=\sum_{j=0}^{\mathcal{M}}\frac{d^{\ell+j}v_{\varepsilon\delta}(x_0)}{dx^{\ell+j}}\left(\frac{1-\delta}{2}\right)^j\frac{\varepsilon^j}{j!}+R^-_{\varepsilon,\mathcal{M},l}.\tag{37}
$$

where

$$
\left|R^{\pm}_{\varepsilon,\mathcal{M},l}\right|\le\frac{1}{(\mathcal{M}+1)!}\sup_{0\le j\le\mathcal{K}+\mathcal{M}}\left(\sup_{[0,1]}\left|\frac{d^j v_{\varepsilon\delta}(x)}{dx^j}\right|\right)\varepsilon^{\mathcal{M}+1}
$$

After substituting (35), (36), (37) into (34) we obtain:

$$
\begin{aligned}
&\sum_{\ell=1}^{\mathcal{K}}\sum_{j=0}^{\mathcal{M}}\varepsilon^{\ell-1}\left(\frac{d\mathcal{N}_\ell}{d\xi}+\mathcal{N}_{\ell-1}\right)\left(\frac{\delta}{2}\right)\frac{d^{\ell+j}v_\varepsilon(x_0)}{dx^{\ell+j}}\left(\frac{\delta-1}{2}\right)^j\frac{\varepsilon^j}{j!}+\\
&+\sum_{\ell=1}^{\mathcal{K}}\sum_{j=0}^{\mathcal{M}}\varepsilon^{\ell-1}\left(\frac{d\mathcal{N}_\ell}{d\xi}+\mathcal{N}_{\ell-1}\right)\left(1-\frac{\delta}{2}\right)\frac{d^{\ell+j}v_\varepsilon(x_0)}{dx^{\ell+j}}\left(\frac{1-\delta}{2}\right)^j\frac{\varepsilon^j}{j!}=0,
\end{aligned}
\tag{38}
$$

up to remainders

$$
r_{\mathcal{K}}^{(2)}=\sum_{\ell=1}^{\mathcal{K}}R^+_{\varepsilon,\mathcal{M},\ell}+\sum_{\ell=1}^{\mathcal{K}}R^-_{\varepsilon,\mathcal{M},\ell},
$$

where $x_0=\frac{\varepsilon}{2}$. Thus, due to periodicity of $\mathcal{N}_\ell$ (38) can be rewritten as follows:

$$
\sum_{\pm}\mp\sum_{r\ge1}\varepsilon^{r-1}\frac{d^r v_\varepsilon(x_0)}{dx^r}\sum_{j=0}^{r}\frac{1}{j!}\left(\frac{d\mathcal{N}_{r-j}}{d\xi}+\mathcal{N}_{r-j-1}\right)\left(\pm\frac{\delta}{2}\right)\left(\pm\frac{\delta-1}{2}\right)^j=0,\tag{39}
$$

(also up to above remainders) from what we obtain

$$\sum_{\pm} \mp \sum_{j=0}^{r} \frac{1}{j!} \left(\frac{d\mathcal{N}_{r-j}}{d\xi} + \mathcal{N}_{r-j-1} \right) \left(\pm\frac{\delta}{2} \right) \left(\pm\frac{\delta-1}{2} \right)^j = 0, \quad \text{for } r = 1, 2, \ldots \tag{40}$$

Now slightly move coordinate system so that its origin is in the middle of the inclusion. Taking into account the periodicity of the function $\mathcal{N}_\ell$ and (39) we obtain the following problem for $\mathcal{N}_\ell$, $\ell \geq 2$:

$$\begin{aligned}
&(a)\ \frac{d^2\mathcal{N}_\ell}{d\xi^2} + 2\frac{d\mathcal{N}_{\ell-1}}{d\xi} + \mathcal{N}_{\ell-2} = h_\ell, \quad \text{in } \left(-\frac{1}{2}, \frac{\delta-1}{2}\right) \bigcup \left(\frac{1-\delta}{2}, \frac{1}{2}\right), \\
&\quad \text{where} \\
&h_\ell = \left\langle \frac{d\mathcal{N}_{\ell-1}}{d\xi} + \mathcal{N}_{\ell-2} \right\rangle_1 + \delta^{-1} \sum_{\pm} \mp \sum_{j=1}^{r} \frac{1}{j!} \left(\frac{d\mathcal{N}_{r-j}}{d\xi} + \mathcal{N}_{r-j-1} \right) \left(\pm\frac{\delta}{2} \right) \left(\pm\frac{\delta-1}{2} \right)^j, \\
&(b)\ \frac{d\mathcal{N}_\ell}{d\xi} + \mathcal{N}_{\ell-1} = 0, \quad \text{in } \left(\frac{\delta-1}{2}, \frac{1-\delta}{2}\right), \\
&(c)\ \sum_{\pm} \mp \sum_{j=0}^{\ell} \frac{1}{j!} \left(\frac{d\mathcal{N}_{\ell-j}}{d\xi} + \mathcal{N}_{\ell-j-1} \right) \left(\pm\frac{\delta}{2} \right) \left(\pm\frac{\delta-1}{2} \right)^j = 0, \\
&(d)\ \mathcal{N}_\ell \text{ is 1-periodic and continuous.}
\end{aligned} \tag{41}$$

In particular, the first three problems for $\mathcal{N}_0$, $\mathcal{N}_1$ and $\mathcal{N}_2$ are as follows.

$$\begin{aligned}
&(a)\ \mathcal{N}_0 = 1, \quad \text{in } \left(-\frac{1}{2}, \frac{\delta-1}{2}\right) \bigcup \left(\frac{1-\delta}{2}, \frac{1}{2}\right), \\
&(b)\ \frac{d\mathcal{N}_0}{d\xi} = 0, \quad \text{in } \left(-\frac{1-\delta}{2}, \frac{1-\delta}{2}\right), \\
&(c)\ -\frac{d\mathcal{N}_0}{d\xi}\bigg|_{\xi=-\frac{1-\delta}{2}} + \frac{d\mathcal{N}_0}{d\xi}\bigg|_{\xi=\frac{1-\delta}{2}} = 0, \\
&(d)\ \mathcal{N}_0 \text{ is 1-periodic and continuous} \quad \text{in } \left(-\frac{1}{2}, \frac{1}{2}\right)
\end{aligned} \tag{42}$$

hence, $\mathcal{N}_0 \equiv 1$ in $(-\frac{1}{2}, \frac{1}{2})$. Also

$$\begin{aligned}
&(a)\ \frac{d^2\mathcal{N}_1}{d\xi^2} = h_1 = 0, \quad \text{in } \left(-\frac{1}{2}, -\frac{1-\delta}{2}\right) \bigcup \left(\frac{1-\delta}{2}, \frac{1}{2}\right), \\
&(b)\ \frac{d\mathcal{N}_1}{d\xi} + 1 = 0, \quad \text{in } \left(-\frac{1-\delta}{2}, \frac{1-\delta}{2}\right), \\
&(c)\ -\left(\frac{d\mathcal{N}_1}{d\xi} + 1\right)\bigg|_{\xi=-\frac{1-\delta}{2}} + \left(\frac{d\mathcal{N}_1}{d\xi} + 1\right)\bigg|_{\xi=\frac{1-\delta}{2}} = 0, \\
&(d)\ \mathcal{N}_1 \text{ is 1-periodic and continuous in } \left(-\frac{1}{2}, \frac{1}{2}\right)
\end{aligned} \tag{43}$$

hence (see Fig. 5),

$$\mathcal{N}_1(\xi) = \begin{cases} \frac{1-\delta}{\delta}\xi + \frac{1-\delta}{2\delta}, & \text{in } \left(-\frac{1}{2}, -\frac{1-\delta}{2}\right) \\ -\xi, & \text{in } \left(-\frac{1-\delta}{2}, \frac{1-\delta}{2}\right) \\ \frac{1-\delta}{\delta}\xi - \frac{1-\delta}{2\delta}, & \text{in } \left(\frac{1-\delta}{2}, \frac{1}{2}\right). \end{cases} \tag{44}$$

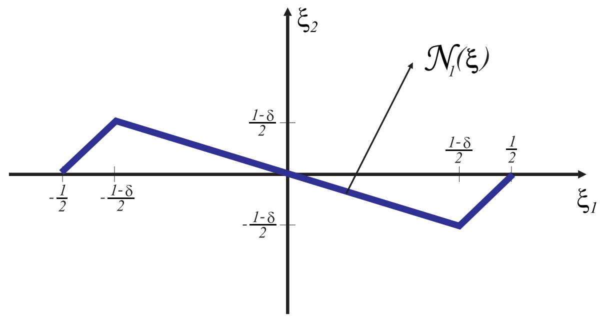

FIGURE 5. Function $\mathcal{N}_1(\xi)$

Also consider problem for $\mathcal{N}_2$:

$$
\begin{aligned}
&(a)\ \frac{d^2\mathcal{N}_2}{d\xi^2}+2\frac{d\mathcal{N}_1}{d\xi}+1=h_2, \quad \text{in } \left(-\frac{1}{2},-\frac{1-\delta}{2}\right)\bigcup\left(\frac{1-\delta}{2},\frac{1}{2}\right),\\
&\qquad \text{with } h_2=\left\langle\frac{d\mathcal{N}_1}{d\xi}+1\right\rangle_1+\frac{1-\delta}{\delta}\left(\frac{d\mathcal{N}_1}{d\xi}+1\right)\Bigg|_{\xi=\pm\frac{1-\delta}{2}}=\frac{1}{\delta^2},\\
&(b)\ \frac{d\mathcal{N}_2}{d\xi}+\mathcal{N}_1=0, \quad \text{in } \left(-\frac{1-\delta}{2},\frac{1-\delta}{2}\right),\\
&(c)\ \sum_{\pm}\mp\left(\frac{d\mathcal{N}_2}{d\xi}+\mathcal{N}_1\right)\Bigg|_{\xi=\mp\frac{1-\delta}{2}}\mp\left(\mp\frac{1-\delta}{2}\right)\left(\frac{d\mathcal{N}_1}{d\xi}+1\right)\Bigg|_{\xi=\pm\frac{1-\delta}{2}}=0,\\
&(d)\ \mathcal{N}_2 \text{ is 1-periodic and continuous in } \left(-\frac{1}{2},\frac{1}{2}\right)
\end{aligned}
\tag{45}
$$

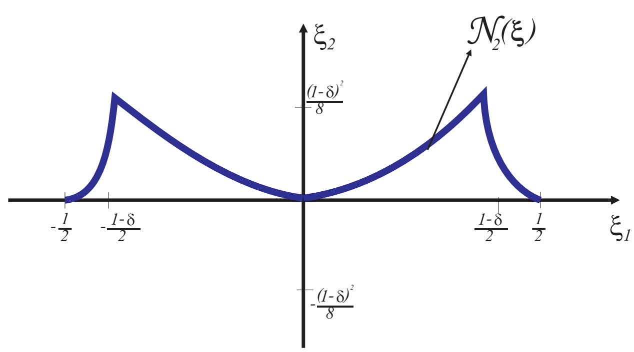

FIGURE 6. Function $\mathcal{N}_2(\xi)$

Hence,

$$
\mathcal{N}_2(\xi)=\begin{cases}
\frac{1}{2}\left(\frac{1-\delta}{\delta}\right)^2\left(\xi+\frac{1}{2}\right)^2, & \text{in } \left(-\frac{1}{2},-\frac{1-\delta}{2}\right)\\
\frac{\xi^2}{2}, & \text{in } \left(-\frac{1-\delta}{2},\frac{1-\delta}{2}\right)\\
\frac{1}{2}\left(\frac{1-\delta}{\delta}\right)^2\left(\xi-\frac{1}{2}\right)^2, & \text{in } \left(\frac{1-\delta}{2},\frac{1}{2}\right)
\end{cases}
\tag{46}
$$

For simplicity we move the system of coordinates so that perfectly conducting inclusion occupies the interval $(-\frac{1}{2}, -\frac{\delta}{2})\bigcup(\frac{\delta}{2}, \frac{1}{2})$. Thus, two functions $\mathcal{N}_1(\xi)$ and $\mathcal{N}_2(\xi)$ would be

$$\mathcal{N}_1(\xi) = \begin{cases} -\xi - \dfrac{1}{2}, & \text{in } \left(-\dfrac{1}{2}, -\dfrac{\delta}{2}\right) \\ \dfrac{1-\delta}{\delta}\xi, & \text{in } \left(-\dfrac{\delta}{2}, \dfrac{\delta}{2}\right) \\ -\xi + \dfrac{1}{2}, & \text{in } \left(\dfrac{\delta}{2}, \dfrac{1}{2}\right) \end{cases} \tag{47}$$

$$\mathcal{N}_2(\xi) = \begin{cases} \dfrac{(-\xi - 1/2)^2}{2}, & \text{in } \left(-\dfrac{1}{2}, -\dfrac{\delta}{2}\right) \\ \dfrac{1}{2}\left(\dfrac{1-\delta}{\delta}\right)^2 \xi^2, & \text{in } \left(-\dfrac{\delta}{2}, \dfrac{\delta}{2}\right) \\ \dfrac{(-\xi + 1/2)^2}{2}, & \text{in } \left(\dfrac{\delta}{2}, \dfrac{1}{2}\right) \end{cases} \tag{48}$$

(see Fig. 7, 8).

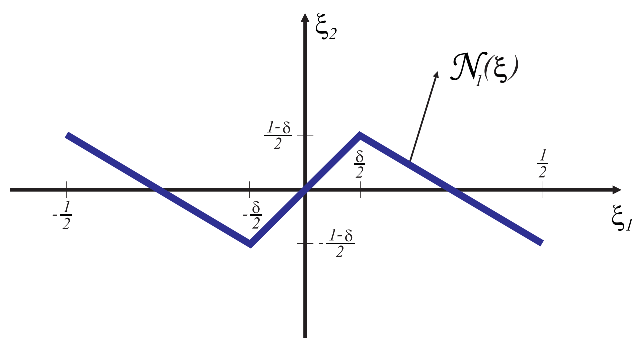

FIGURE 7. Function $\mathcal{N}_1(\xi)$

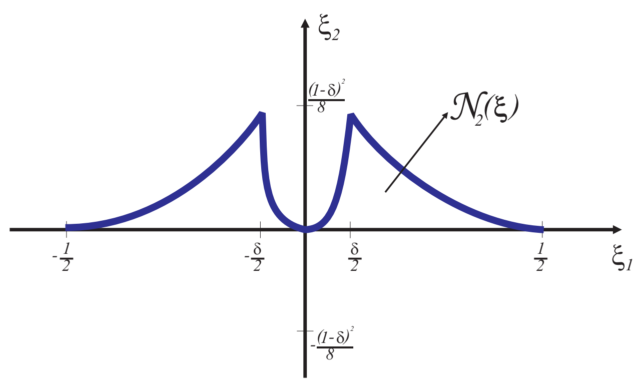

FIGURE 8. Function $\mathcal{N}_2(\xi)$

The goal of this section was to obtain problem (41) for the functions $\mathcal{N}_\ell$. They can be constructed as some piecewise polynomial functions. The coefficients h_l can be expanded in powers of δ:

$$h_l = \frac{1}{\delta^2}\sum_{j=0}^{\mathcal{K}+2}\delta^j h_{l_j} + O\left(\delta^{\mathcal{K}}\right).$$

Then we seek for the solution $v_{\varepsilon\delta}$ in the form of series (27) and obtain the chain of problems for v_{jr} in the same way as in Introduction: $v''_{lj} = f_{lj}\left(x_1\right)$, where are the right hand sides defined by $v''_{l_1 j_1}$ such that $l_1 \le l$ and $j_1 < j$ or $l_1 < l$ and $j_1 \le j$; $f_{00} = f_{01} = 0$, $f_{02} = f$.

2.2. **Asymptotic expansion of solution to problem (1)÷(3), (5).** In this subsection we modify functions $\mathcal{N}_\ell$ constructed for the one-dimensional case in order to obtain solutions of cell problems for two-dimensions. For this end we construct some special correctors in horizontal strips of $\Omega_{\varepsilon\delta}$ and some exponentially decaying boundary layer type correctors in the neighborhoods of the corners of $\Omega_{\varepsilon\delta}$. The values of constants h_ℓ will be also modified because the measure of the periodic cell is greater in the 2-dimensional case, and the functions $\mathcal{N}_\ell$ are not the same.

For the correspondent two-dimensional problem we work with the solution $u_{\varepsilon\delta}^{(\mathcal{K})}$ of the form (26) where $v_{\varepsilon\delta}^{(\mathcal{K})}$ are sought in the form (27), satisfying (12), with $\mathcal{H}_\ell$, defined by (15) and (16).

Denote

$$\boldsymbol{\mathcal{S}}_\delta = \left\{\boldsymbol{\xi} = \frac{\boldsymbol{x}}{\varepsilon},\ \boldsymbol{x} \in \Omega_{\varepsilon\delta}\right\}, \quad \Gamma_i = \left\{\boldsymbol{\xi} = \frac{\boldsymbol{x}}{\varepsilon},\ \boldsymbol{x} \in \partial G^i_{\varepsilon\delta}\right\},$$
$$\square_i = \left\{\boldsymbol{\xi} = \frac{\boldsymbol{x}}{\varepsilon},\ \boldsymbol{x} \in G^i_{\varepsilon\delta}\right\}, \quad \boldsymbol{\mathcal{S}}^i_\delta = (i, i+1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \diagdown \square_i.$$

Also $\Gamma_i = \Gamma_1^+ \cup \Gamma_1^- \cup \Gamma_2^+ \cup \Gamma_2^-$ as shown in Fig. 9, that is, $\Gamma_q^\pm = \left\{x_q = i + \frac{1}{2} \pm \frac{1-\delta}{2}\right\} \cap \Gamma_i$, $q = 1, 2$. Since $\mathcal{N}_\ell$ is 1-periodic in ξ_1 we drop the index i and work with the periodicity cell $\boldsymbol{\mathcal{S}}^i_\delta$.

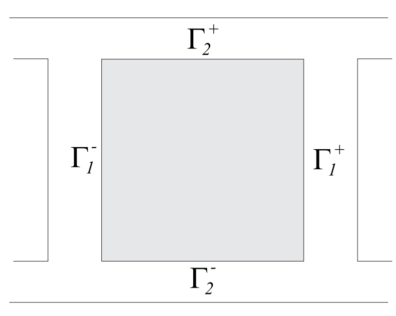

Figure 9. The boundary of an inclusion

Substituting $u_{\varepsilon\delta}^{(\mathcal{K})}$ of the form (26) into (1)÷(3), (5) we obtain, as in the previous subsections, the following chain of cell problems for $\mathcal{N}_\ell$ (recalling that $\mathcal{L}_{\xi\xi} = \triangle$):

$$\begin{cases} \triangle_\xi \mathcal{N}_\ell + 2\dfrac{\partial \mathcal{N}_{\ell-1}}{\partial \xi_1} + \mathcal{N}_{\ell-2} = h_\ell, \quad \boldsymbol{\xi} \in \boldsymbol{\mathcal{S}}_\delta \\ \dfrac{\partial \mathcal{N}_\ell}{\partial \xi_2} = 0, \quad \xi_2 = \pm\dfrac{1}{2} \\ \mathcal{N}_\ell(\boldsymbol{\xi}) = \mathcal{N}_\ell^{in}(\xi_1), \quad \boldsymbol{\xi} \in \Gamma \\ \displaystyle\sum_{\pm}\int_{\Gamma_1^\pm} \pm \sum_{j=0}^{r} \frac{1}{j!}\left(\pm\frac{1-\delta}{2}\right)^j \left(\frac{\partial \mathcal{N}_{r-j}}{\partial \xi_1} + \mathcal{N}_{r-j-1}\right)\bigg|_{\xi_1 = \mp\frac{\delta}{2}} d\xi_2 + \\ \quad + \displaystyle\sum_{\pm}\int_{\Gamma_2^\pm} \pm \frac{\partial \mathcal{N}_r}{\partial \xi_2}\bigg|_{\xi_2 = \pm\frac{1-\delta}{2}} d\xi_1 = 0, \\ \mathcal{N}_\ell \text{ is 1 periodic in } \xi_1, \end{cases} \tag{49}$$

where $\mathcal{N}_\ell^{in}(\xi_1)$ is defined in the inclusion $\square$ by relations (33) with $\mathcal{N}_0^{in}(\xi_1) = 1$, and

$$\begin{aligned} h_\ell = & \left\langle \triangle \mathcal{N}_\ell + \frac{\partial \mathcal{N}_{\ell-1}}{\partial \xi_1} + \frac{\partial \mathcal{N}_{\ell-1}}{\partial \xi_1} + \mathcal{N}_{\ell-2} \right\rangle_2 \\ = & \frac{1}{|\boldsymbol{S}_\delta^0|}\left[\int_{\boldsymbol{S}_\delta^0}\left(\frac{\partial \mathcal{N}_{\ell-1}}{\partial \xi_1} + \mathcal{N}_{\ell-2}\right) d\boldsymbol{\xi} + \int_\Gamma \left(\frac{\partial \mathcal{N}_\ell}{\partial \boldsymbol{\nu}_\xi} + \mathcal{N}_{\ell-1}\cos(\boldsymbol{\nu}, \xi_1)\right) ds\right] \end{aligned} \tag{50}$$

with $|\boldsymbol{S}_\delta^0| = 1 - |\square| = 2\delta - \delta^2$, $\langle\cdot\rangle_2 = \frac{1}{|\boldsymbol{S}_\delta^0|}\int_{\boldsymbol{S}_\delta^0}(\cdot)\, d\boldsymbol{\xi}$ and the normal vector $\boldsymbol{\nu}$ is directed inside the inclusion $\square$. We find the surface integral of the right hand side of (50) from the following integral condition:

$$\begin{aligned} &\sum_{\pm}\int_{\Gamma_1^\pm} \pm \sum_{j=0}^{r} \frac{1}{j!}\left(\pm\frac{1-\delta}{2}\right)^j \left(\frac{\partial \mathcal{N}_{r-j}}{\partial \xi_1} + \mathcal{N}_{\ell-j-1}\right)\bigg|_{\xi_1 = \mp\frac{\delta}{2}} d\xi_2 \\ &+ \sum_{\pm}\int_{\Gamma_2^\pm} \left(\frac{\partial \mathcal{N}_r}{\partial \xi_2}\right)\bigg|_{\xi_2 = \pm\frac{1-\delta}{2}} d\xi_1 = 0, \end{aligned} \tag{51}$$

for $r = \ell$. The surface integral of the right hand side of (50) corresponds to $j = 0$ in (51) taking into account the direction of $\boldsymbol{\nu}$.

Therefore,

$$\begin{aligned} h_\ell = \ & \frac{1}{2\delta - \delta^2}\left[\int_{\boldsymbol{S}_\delta^0}\left(\frac{\partial \mathcal{N}_{\ell-1}}{\partial \xi_1} + \mathcal{N}_{\ell-2}\right) d\boldsymbol{\xi} \right. \\ & \left. + \sum_{\pm}\int_{\Gamma_1^\pm} \pm \sum_{j=1}^{\ell} \frac{1}{j!}\left(\pm\frac{1-\delta}{2}\right)^j \left(\frac{\partial \mathcal{N}_{\ell-j}}{\partial \xi_1} + \mathcal{N}_{\ell-j-1}\right)\bigg|_{\xi_1 = \mp\frac{\delta}{2}} d\xi_2 \right]. \end{aligned} \tag{52}$$

Note that

$$\triangle_\xi \mathcal{N}_\ell + 2\frac{\partial \mathcal{N}_{\ell-1}}{\partial \xi_1} + \mathcal{N}_{\ell-2} = \frac{\partial}{\partial \xi_1}\left(\frac{\partial \mathcal{N}_\ell}{\partial \xi_1} + \mathcal{N}_{\ell-1}\right) + \left(\frac{\partial \mathcal{N}_{\ell-1}}{\partial \xi_1} + \mathcal{N}_{\ell-2}\right) = h_\ell, \tag{53}$$

where two last parentheses in (53) are equal zero on the inclusions.

We decompose the solution $\mathcal{N}_\ell$ of ℓ^{th}-cell problem (53) as follows:

$$\mathcal{N}_\ell(\boldsymbol{\xi}) = \mathcal{N}_\ell^v(\xi_1) + \mathcal{N}_\ell^h(\xi_1, \xi_2) + \mathcal{N}_\ell^c(\xi_1, \xi_2),$$

where $\mathcal{N}_\ell^v(\xi_1)$ is defined in the vertical strip V, $\mathcal{N}_\ell^h(\xi_1,\xi_2)$ in the horizontal strips H_1 and H_2, $\mathcal{N}_\ell^c(\xi_1,\xi_2)$ is in the half-crosses $C_1\cup V\cup H_1$, $C_2\cup V\cup H_2$, where $H_1=((-\frac{1-\delta}{2},-\frac{\delta}{2})\cup(\frac{\delta}{2},\frac{1-\delta}{2}))\times(-\frac{1}{2},-\frac{1}{2}+\frac{\delta}{2})$, $H_2=((-\frac{1-\delta}{2},-\frac{\delta}{2})\cup(\frac{\delta}{2},\frac{1-\delta}{2}))\times(\frac{1}{2}-\frac{\delta}{2},\frac{1}{2})$, $V=(-\frac{\delta}{2},\frac{\delta}{2})\times(-\frac{1-\delta}{2},\frac{1-\delta}{2})$, $C_1=(-\frac{\delta}{2},\frac{\delta}{2})\times(-\frac{1}{2},-\frac{1-\delta}{2})$, $C_2=(-\frac{\delta}{2},\frac{\delta}{2})\times(\frac{1-\delta}{2},\frac{1}{2})$ (see Fig. 10, 11).

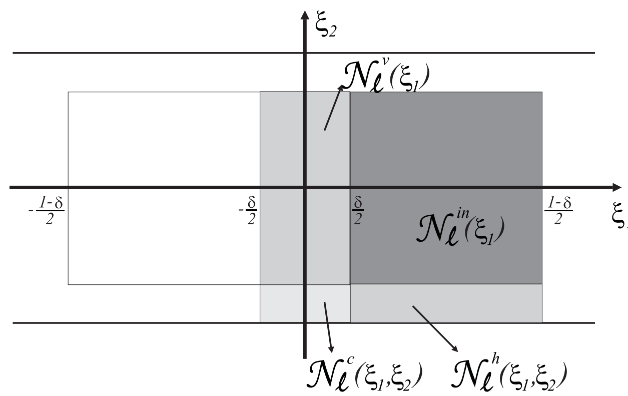

FIGURE 10. Decomposition of the function $\mathcal{N}_\ell(\xi)$

For each region H_1,H_2,V,C_1,C_2 shown in Fig. 11 the corresponding solutions $\mathcal{N}_\ell^h$, $\mathcal{N}_\ell^v$, $\mathcal{N}_\ell^c$ are split into

$$\mathcal{N}_\ell^{\cdot\cdot} = \overline{\mathcal{N}}_\ell^{\cdot\cdot} + h_\ell\mathcal{M}_\ell^{\cdot\cdot}$$

where h_ℓ is defined by relation (52) and $\overline{\mathcal{N}}_\ell^{\cdot\cdot}$ satisfies the first equation of (49) with the zero right-hand side and all boundary conditions of this problem except integral condition. The function $\mathcal{M}_\ell^{\cdot\cdot}$ in each region would be constructed separately. So, constant h_ℓ is fixed by the integral condition.

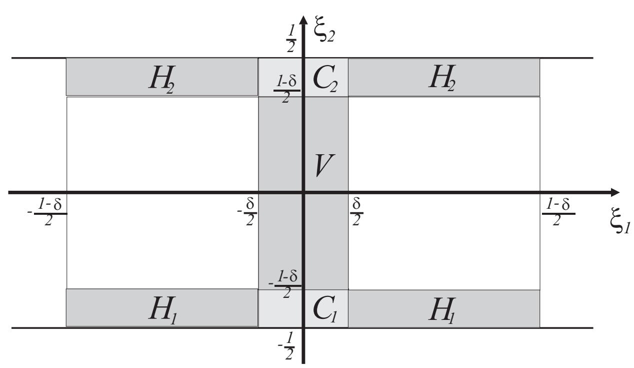

FIGURE 11. Decomposition of a cell.

First, consider the vertical strip V. Then as mentioned above:

$$\mathcal{N}_\ell^v(\xi_1) = \overline{\mathcal{N}}_\ell^v + h_\ell\mathcal{M}_\ell^v.$$

We choose the function $\mathcal{M}_\ell^v$ to satisfy the following problem:

$$\begin{aligned} &\frac{\partial^2 \mathcal{M}_\ell^v}{\partial \xi_1^2} = 1, \quad \boldsymbol{\xi} \in V \\ &\mathcal{M}_\ell^v = 0, \qquad \xi_1 = \pm\frac{\delta}{2} \end{aligned} \tag{54}$$

hence,

$$\mathcal{M}_\ell^v(\xi_1) = \frac{1}{2}\left(\xi_1 - \frac{\delta}{2}\right)\left(\xi_1 + \frac{\delta}{2}\right) \quad \text{in } V.$$

Since the constructed function $\mathcal{M}_\ell^v$ does not depend on ℓ we drop this subscript hereafter.

Thus, for function $\overline{\mathcal{N}}_\ell^v(\xi_1)$ we have the following problem

$$\begin{aligned} &\frac{\partial^2 \overline{\mathcal{N}}_\ell^v}{\partial \xi_1^2} + 2\frac{\partial \mathcal{N}_{\ell-1}^v}{\partial \xi_1} + \overline{\mathcal{N}}_{\ell-2}^v + h_{\ell-2}\mathcal{M}^v = 0, \quad \text{in } V \\ &\overline{\mathcal{N}}_\ell^v = \mathcal{N}_\ell^{in}, \quad \xi_1 = \pm\frac{\delta}{2} \end{aligned} \tag{55}$$

where $\mathcal{N}_\ell^{in}$ is taken to be equal to the solution of the corresponding one-dimensional ℓ^{th}-cell problem considered above in (33).

In horizontal strips H_1 and H_2 we decompose the function $\mathcal{N}_\ell^h(\xi_1, \xi_2)$ as follows:

$$\mathcal{N}_\ell^h(\xi_1, \xi_2) = \overline{\mathcal{N}}_\ell^h + h_\ell \mathcal{M}_\ell^h,$$

where the function $\mathcal{M}_\ell^h(\xi_2)$ is chosen to satisfy:

$$\begin{aligned} &\frac{\partial^2 \mathcal{M}_\ell^h}{\partial \xi_2^2} = 1, \quad \boldsymbol{\xi} \in H_1 \cup H_2 \\ &\mathcal{M}_\ell^h = 0, \qquad \xi_2 = \pm\frac{1-\delta}{2} \\ &\frac{\partial \mathcal{M}_\ell^h}{\partial \xi_2} = 0, \quad \xi_2 = \pm\frac{1}{2} \end{aligned} \tag{56}$$

hence,

$$\mathcal{M}_\ell^h(\xi_2) = \begin{cases} \frac{1}{2}\left(\xi_2 - \frac{1-\delta}{2}\right)\left(\xi_2 - \frac{1+\delta}{2}\right), & \xi_2 \in \left(\frac{1-\delta}{2}, \frac{1}{2}\right) \\ \frac{1}{2}\left(\xi_2 + \frac{1-\delta}{2}\right)\left(\xi_2 + \frac{1+\delta}{2}\right), & \xi_2 \in \left(-\frac{1}{2}, -\frac{1-\delta}{2}\right) \end{cases}$$

Since the constructed function $\mathcal{M}_\ell^h$ does not depend on ℓ we drop this subscript hereafter.

We choose $\overline{\mathcal{N}}_\ell^h$ in the form:

$$\overline{\mathcal{N}}_\ell^h(\xi_1, \xi_2) = \mathcal{N}_\ell^{in}(\xi_1) + \widetilde{\mathcal{N}}_\ell^h(\xi_2),$$

where $\widetilde{\mathcal{N}}_\ell^h(\xi_2)$ satisfies

$$\begin{aligned} &\frac{\partial^2 \widetilde{\mathcal{N}}_\ell^h}{\partial \xi_2^2} + \widetilde{\mathcal{N}}_{\ell-2}^h + h_{\ell-2}\mathcal{M}^h = 0, \quad \text{in } H_1 \cup H_2 \\ &\widetilde{\mathcal{N}}_\ell^h = 0, \quad \xi_2 = \pm\frac{1-\delta}{2} \\ &\frac{\partial \widetilde{\mathcal{N}}_\ell^h}{\partial \xi_2} = 0, \quad \xi_2 = \pm\frac{1}{2} \end{aligned} \tag{57}$$

We extend both $\mathcal{N}_\ell^v$ and $\mathcal{N}_\ell^h$ in the periodic cell with zero where they are not defined.

For the corresponding solution $\mathcal{N}_\ell^c(\xi_1,\xi_2)$ in the half-cross $C_1 \cup V \cup H_1$ we consider similar decomposition (the other half-cross $C_2 \cup V \cup H_2$ is treated similarly):

$$\mathcal{N}_\ell^c(\xi_1,\xi_2) = \overline{\mathcal{N}}_\ell^c + h_\ell \mathcal{M}_\ell^c,$$

where the function $\overline{\mathcal{N}}_\ell^c$ satisfies the following problem:

$$\begin{aligned}
&\triangle_\xi \overline{\mathcal{N}}_\ell^c + 2\frac{\partial \mathcal{N}_{\ell-1}^c}{\partial \xi_1} + \overline{\mathcal{N}}_{\ell-2}^c + h_{\ell-2}\mathcal{M}_{\ell-2}^c = 0, \quad \text{in } C_1 \cup V \cup H_1\\
&\overline{\mathcal{N}}_\ell^c = 0, \quad \text{if } \xi_1 = \pm\frac{\delta}{2}, \quad \xi_2 \in \left(-\frac{1-\delta}{2}, \frac{1-\delta}{2}\right) \ \text{ or } \ \xi_2 = -\frac{1-\delta}{2}, \quad |\xi_1| > \frac{\delta}{2}\\
&\frac{\partial \overline{\mathcal{N}}_\ell^c}{\partial \xi_2} = 0, \quad \text{if } \xi_2 = -\frac{1}{2}\\
&[\overline{\mathcal{N}}_\ell^c] = -\overline{\mathcal{N}}_\ell^v, \quad \text{on } \Sigma_2\\
&[\overline{\mathcal{N}}_\ell^c] = \mp\overline{\mathcal{N}}_\ell^h, \quad \text{on } \Sigma_1^\pm
\end{aligned} \tag{58}$$

and $\mathcal{M}_\ell^c$ satisfies

$$\begin{aligned}
&\triangle \mathcal{M}_\ell^c = \frac{\partial^2 \mathcal{M}_\ell^c}{\partial \xi_1^2} + \frac{\partial^2 \mathcal{M}_\ell^c}{\partial \xi_2^2} = \begin{cases} 1, & \xi_2 \in \left(-\frac{1}{2}, -\frac{1}{2}+\frac{\delta}{2}\right) \text{ and } \xi_1 \in \left(-\frac{\delta}{2}, \frac{\delta}{2}\right)\\ 0, & \text{otherwise}\end{cases}\\
&\mathcal{M}_\ell^c = 0, \quad \text{if } \xi_1 = \pm\frac{\delta}{2}, \quad \xi_2 \in \left(-\frac{1-\delta}{2}, \frac{1-\delta}{2}\right) \ \text{ or } \ \xi_2 = -\frac{1-\delta}{2}, \quad |\xi_1| > \frac{\delta}{2}\\
&\frac{\partial \mathcal{M}_\ell^c}{\partial \xi_2} = 0, \quad \text{if } \xi_2 = -\frac{1}{2}\\
&[\mathcal{M}_\ell^c] = -\mathcal{M}^v, \quad \text{on } \Sigma_2\\
&[\mathcal{M}_\ell^c]_{\Sigma_1^\pm} = \mp\mathcal{M}^h, \quad \text{on } \Sigma_1^\pm
\end{aligned} \tag{59}$$

where $\Sigma_i^\pm$, $i = 1, 2$ are shown in Fig. 12. Note that the problem for $\mathcal{M}_\ell^c$ also does not depend on ℓ so we drop this subscript. Let us translate the origin of the coordinates to point $(0, -1/2)$ and extend this problem to the infinite half-cross $(-\infty, +\infty) \times (0, \frac{\delta}{2}) \cup (-\frac{\delta}{2}, \frac{\delta}{2}) \times (0, +\infty)$.

Here we remark that if one denotes

$$\mathcal{M}^c(\xi_1,\xi_2) = \delta^2 \widetilde{\mathcal{M}}^c\left(\frac{\xi_1}{\delta}, \frac{\xi_2 + \frac{1}{2}}{\delta}\right) =: \delta^2 \widetilde{\mathcal{M}}^c(\eta_1,\eta_2),$$

then $\widetilde{\mathcal{M}}^c$ does not depend on δ and satisfies:

$$\triangle_\eta \widetilde{\mathcal{M}}^c = \begin{cases} 1, & \eta_2 \in \left(0, \frac{1}{2}\right) \text{ and } \eta_1 \in \left(-\frac{1}{2}, \frac{1}{2}\right)\\ 0, & \text{otherwise}\end{cases}$$

with the boundary conditions generated by the above boundary conditions for $\mathcal{M}^c$. For such a function there exists an unique solution and it satisfies the following estimate:

$$\left|\widetilde{\mathcal{M}}^c(\boldsymbol{\eta})\right| \le c_1 e^{-c_2|\boldsymbol{\eta}|}$$

with some positive constants c_1, c_2 due to Phrägmen-Lindelöf type theorem (see Appendices 1 and 2).

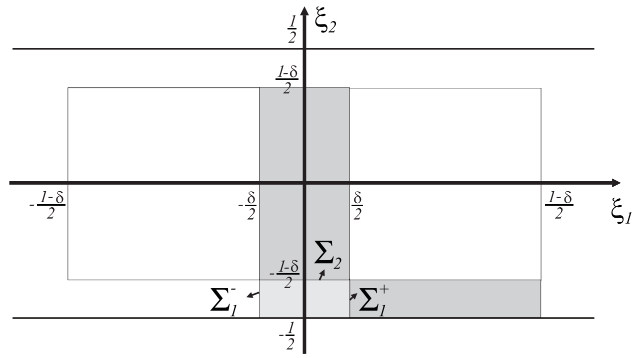

FIGURE 12. The jump surfaces $\sum_1^{\pm}$ and $\sum_2$

The functions $\bar{\mathcal{N}}_\ell^c$ and $\mathcal{N}_\ell^c$ also decay exponentially as translated $\frac{\xi}{\delta} \to +\infty$ (cf. Appendix 2). Therefore, we multiply these functions by a cutting function that vanishes at the distance $|\boldsymbol{\xi}| \geq \frac{2}{3}$ and that is equal to 1 if $|\boldsymbol{\xi}| \leq \frac{1}{3}$. This multiplication will produce an error of order $O\left(e^{-\frac{c}{\delta}}\right)$ that is negligible in comparison with the desired error estimate $O\left(\varepsilon^{\mathcal{K}-1} + \delta^{\mathcal{K}-1}\right)\sqrt{\varepsilon}$ if δ and ε are related by some bounds

$$\delta = O\left(\varepsilon^{\alpha}\right) \quad \text{and} \quad \varepsilon = O\left(\delta^{\beta}\right)$$

with some α and β from $(0, +\infty)$.

Thus, we have to add one more remainder in equation (1), that is, $r_{\varepsilon\delta}^{(4)}(x)$ such that $\left\| r_{\varepsilon\delta}^{(4)} \right\|_{L^\infty(\Omega_{\varepsilon\delta})} = O\left(e^{-\frac{c}{\delta}}\right)$.

Consider now a periodicity cell $\left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ and denote by $\boldsymbol{a}_j$ $(j = 1, 2)$ "the corner points" $\boldsymbol{a}_1 = \left(0, -\frac{1}{2}\right)$ and $\boldsymbol{a}_2 = \left(0, \frac{1}{2}\right)$; denote by $\overline{\mathcal{N}}_\ell^c(\boldsymbol{a}_j; \boldsymbol{\xi})$ and $\mathcal{M}^c(\boldsymbol{a}_j; \boldsymbol{\xi})$ the solutions of the above boundary layer problems corresponding to the half-crosses containing point $\boldsymbol{a}_j$. Set

$$\overline{\mathcal{N}}_\ell = \overline{\mathcal{N}}_\ell^v + \overline{\mathcal{N}}_\ell^h + \sum_{j=1}^{2} \overline{\mathcal{N}}_\ell^c(\boldsymbol{a}_j; \boldsymbol{\xi}) \chi(-\boldsymbol{a}_j + \boldsymbol{\xi}),$$

and

$$\mathcal{N}_\ell = \overline{\mathcal{N}}_\ell + h_\ell \left(\mathcal{M}^v + \mathcal{M}^h + \sum_{j=1}^{2} \mathcal{M}^c(\boldsymbol{a}_j; \boldsymbol{\xi}) \chi(-\boldsymbol{a}_j + \boldsymbol{\xi}) \right),$$

where $\boldsymbol{a}_j$ $(j = 1, .., 4)$ is the corner of the unit square and $\chi(\boldsymbol{\xi})$ is defined by:

$$\chi(\boldsymbol{\xi}) = \chi(|\boldsymbol{\xi}|) = \begin{cases} 1, & |\boldsymbol{\xi}| < \dfrac{1}{3} \\ \sin \dfrac{3\pi|\boldsymbol{\xi}|}{2}, & \dfrac{1}{3} \leq |\boldsymbol{\xi}| < \dfrac{2}{3} \\ 0, & |\boldsymbol{\xi}| \geq \dfrac{2}{3} \end{cases} \tag{60}$$

Note that functions $\mathcal{N}_\ell$ satisfy the cell problem (49) up to a remainder $r_{\varepsilon\delta}^{(4)}(x)$ such that $\left\| r_{\varepsilon\delta}^{(4)} \right\|_{L^\infty(\Omega_{\varepsilon\delta})} = O\left(e^{-\frac{c}{\delta}}\right)$.

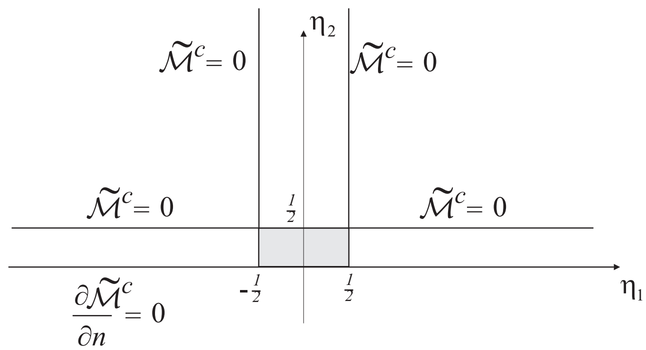

FIGURE 13. The half-cross domain of definition of $\widetilde{\mathcal{M}}^c$

By induction it can be seen that

$$\overline{\mathcal{N}}^v_\ell(\xi_1) = \sum_{j=0}^{\mathcal{K}} \delta^j \overline{\mathcal{N}}^v_{\ell j}\left(\frac{\xi_1}{\delta}\right) + \delta^{\mathcal{K}+1} r^v_{\ell\,\mathcal{K}}, \quad \widetilde{\mathcal{N}}^h_\ell(\xi_2) = \sum_{j=0}^{\mathcal{K}} \delta^j \widetilde{\mathcal{N}}^h_{\ell j}\left(\frac{\xi_2 \pm 1/2}{\delta}\right) + \delta^{\mathcal{K}+1} r^h_{\ell\,\mathcal{K}}$$

with $\overline{\mathcal{N}}^v_{\ell j}(\eta), \widetilde{\mathcal{N}}^h_{\ell j}(\eta)$ independent of δ, the sign $\pm$ is taken with respect to the domain H_1 or H_2 where the solution is sought; and $r^v_{\ell\,\mathcal{K}}$, $r^h_{\ell\,\mathcal{K}}$ are bounded in the same sense, and that

$$\overline{\mathcal{N}}^c_\ell(\boldsymbol{a}_j;\boldsymbol{\xi}) = \sum_{j=0}^{\mathcal{K}} \delta^j \overline{\mathcal{N}}^c_{\ell j}\left(\frac{-\boldsymbol{a}_j + \boldsymbol{\xi}}{\delta}\right) + \delta^{\mathcal{K}+1} r^c_{\delta\ell\,\mathcal{K}}(\boldsymbol{\xi})$$

with $\overline{\mathcal{N}}^c_{\ell j}(\boldsymbol{\eta})$ independent of δ and exponentially decaying, and $r^c_{\delta\ell\,\mathcal{K}}$ is exponentially decaying and bounded in L^∞ norm.

Then (52) yields:

$$h_\ell = \frac{1}{\delta^2}\sum_{j=0}^{\mathcal{K}+2} \delta^j h_{\ell j} + \delta^{\mathcal{K}} r_{\delta\mathcal{K}}, \qquad \text{with } |r_{\delta\mathcal{K}}| \le C_{\mathcal{K}},$$

where $h_{\ell j}$ are independent of parameters. In particular

$$h_2 = \frac{1}{2\delta^2} + O\left(\frac{1}{\delta}\right).$$

Taking into account that the right hand side support is the domain $\Omega_{\varepsilon\delta}$, we can calculate the leading term of the effective conductivity of the strip multiplying this value h_2 by the measure of the periodic cell in the extended variables ξ (see Remark 3). We get then that the leading term for the effective conductivity is $\frac{\varepsilon}{\delta}$. If we extend the problem (1)÷(3), (5) $\varepsilon-$periodically with respect to x_2, then it will model the conductivity of the periodic medium with absolutely conductive inclusions. Its effective conductivity has the leading term $\frac{1}{\delta}$. It corresponds to the asymptotic analysis of [14], p. 316, Theorem 4.10.1. Substituting now expansion (27) into (18) allows us to obtain the set of equations for v_{jr}:

$$v''_{jr} = f_{jr}(x_1)$$

where f_{jr} are the right hand sides defined by $v_{j_1 r_1}$, such that $j_1 \le j$ and $r_1 < r$ or $j_1 < j$ and $r_1 \le r$; $f_{00} = f_{01} = 0$, $f_{02} = f$. These equations can be solved successfully by induction in j, r with an additional condition

$$\int_0^T v_{jr}(x_1)\, dx_1 = 0.$$

Finally, we obtain the equation (1) satisfied up to the remainders which are:

- the remainder analogous to $\varepsilon^{\mathcal{K}} r_\varepsilon$ of Introduction, that is,

$$\varepsilon^{\mathcal{K}} \widehat{r}_{\varepsilon\delta} = \varepsilon^{\mathcal{K}} \left\{ 2\frac{\partial}{\partial \xi_1} \mathcal{N}_{\mathcal{K}+1}(\boldsymbol{\xi}) + \mathcal{N}_{\mathcal{K}}(\boldsymbol{\xi}) \right\}\Big|_{\boldsymbol{\xi}=\frac{\boldsymbol{x}}{\varepsilon}} D_1^{\mathcal{K}+2} v_{\varepsilon\delta}^{(\mathcal{K})}(x_1) + \varepsilon^{\mathcal{K}+1} \mathcal{N}_{\mathcal{K}+1}\left(\frac{\boldsymbol{x}}{\varepsilon}\right) D_1^{\mathcal{K}+3} v_{\varepsilon\delta}^{(\mathcal{K})}(x_1);$$

- the remainder analogous to $\varepsilon^{\mathcal{K}+1} r_\varepsilon^{(1)}(x_1)$ of Introduction, that is,

$$\varepsilon^{\mathcal{K}} \widehat{r}_{\varepsilon\delta}^{(1)} = \sum_{m=\mathcal{K}+1}^{2\mathcal{K}-1} \varepsilon^m \sum_{0 \le j \le \mathcal{K}} h_{m-j+2}\, D_1^{m-j+2}\, v_{j\delta}(x_1),$$

  where $v_{j\delta}(x_1) = \sum_{r=0}^{\mathcal{K}} \delta^r v_{jr}(x_1)$;
- the remainder of Taylor formula:

$$r_{\mathcal{K}}^{(2)} = \sum_{\ell=1}^{\mathcal{K}} R_{\varepsilon,\mathcal{M},\ell}^{+} + \sum_{\ell=1}^{\mathcal{K}} R_{\varepsilon,\mathcal{M},\ell}^{-},$$

  of relations (34), (38);
- the remainder $r_{\mathcal{K}}^{(4)}$ related to the multiplication of boundary layer functions $\mathcal{N}_\ell^c$ and $\mathcal{M}^c$ by the function χ given by (60);
- the remainder related to the truncation of the expansions in δ of $\widetilde{\mathcal{N}}_\ell^v$, $\widetilde{\mathcal{N}}_\ell^h$, $\widetilde{\mathcal{N}}_\ell^c$ and h_ℓ at the terms of order $\delta^{\mathcal{K}}$;
- the remainder in (32):

$$r_{i,\varepsilon\delta}^{(3)}(x_1) = \varepsilon^{\mathcal{K}} \mathcal{N}_{\mathcal{K}}\left(\frac{x_1}{\varepsilon}\right) \frac{d^{\mathcal{K}+1} v_{\varepsilon\delta}(x_1)}{dx_1^{\mathcal{K}+1}} \tag{61}$$

  that should be compensated by a corrector equal to a primitive of the function $-r_{i,\varepsilon\delta}^{(3)}(x_1)$ extended by 0 outside the domain $\left(i\varepsilon + \frac{\delta}{2}\varepsilon, i\varepsilon + \left(1 - \frac{\delta}{2}\right)\varepsilon\right) \times \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$. This corrector will place an asymptotic solution into the space of functions equal to constants on the inclusions (Remark 4).

More precisely, this means that to compensate the last remainder $r_{i,\varepsilon\delta}^{(3)}$ given by (61), in the equation $\frac{\partial u}{\partial x_1} = 0$ on the infinitely conductive inclusion, we add a corrector $\Phi_{\varepsilon\delta}(x_1)$ to the asymptotic solution $u_{\varepsilon\delta}^{(\mathcal{K})}$. This corrector is a primitive in x_1 of $-r_{i,\varepsilon\delta}^{(3)}(x_1)$, constant for all segments $x_1 \in \left\{\left[-\frac{\sqrt{\varepsilon}}{2}, \frac{\sqrt{\varepsilon}}{2}\right] + \varepsilon\mathbb{Z}\right\}$ such that $\Phi_{\varepsilon\delta}(0) = 0$. This corrector places the asymptotic solution to the space $H^1_{per,\varepsilon}(U_\varepsilon)$ where an a priori estimate is proved (cf. Appendix 1), but it generates a new remainder in the right hand side of the Laplace equation, which is the Laplacian of the corrector $\Phi_{\varepsilon\delta}$ or simply the derivative of the remainder $r_{i,\varepsilon\delta}^{(3)}$; it is of order $O\left(\varepsilon^{\mathcal{K}-1}\right)$ in L^∞ norm. This adds a complementary term of such an order to the error estimate.

Finally, taking into account estimates for all the remainders and applying a priori estimate of Appendix 1, we obtain

$$\left\| u_{\varepsilon\delta}^{(\mathcal{K})} - u_{\varepsilon\delta} \right\|_{H^1\left(\Omega_{\varepsilon\delta}\cap(0,T)\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})\right)} \le C\left(\varepsilon^{\mathcal{K}-1}+\delta^{\mathcal{K}-1}\right)\sqrt{\varepsilon}. \tag{62}$$

So we have proved the following result:

Theorem 1. *Let for any $\alpha, \beta > 0$, $\varepsilon = O\left(\delta^{\alpha}\right)$ and $\delta = O\left(\varepsilon^{\beta}\right)$. Then estimate (62) holds.*

Theorem 4 justifies the asymptotic expansion of the solution of problem (1)÷(3), (5) constructed in section 2. It gives the complete analysis of the conductivity of the periodic strip with infinitely conductive inclusions of the high concentration **uniformly** with respect to small parameters ε and δ such that $\varepsilon = O\left(\delta^{\alpha}\right)$ and $\delta = O\left(\varepsilon^{\beta}\right)$ for any $\alpha, \beta > 0$. It justifies the existence of the effective conductivity of the strip, that is not evident for array structures (cf. [13], where it is not the case for the elasticity equations). The leading term of this macroscopic conductivity coincides with calculated in [6].

3. Appendix 1. Existence and uniqueness of solution of the problem in a thin strip with infinitely conductive periodic inclusions.

Theorem 2. *There exists a unique solution of problem (63).*

Proof. Recall the notation $U_\varepsilon = [0,T]\times[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}]$ (where T is assumed to be divisible by ε) and consider the following space

$$H^1_{per,\varepsilon}(U_\varepsilon) = \left\{u \in H^1_{per}(U_\varepsilon) : \ u = C_i \text{ on } \partial G^i_{\delta\varepsilon}\right\}, \quad \text{with } C_i \text{ is an arbitrary constant,}$$

where $H^1_{per}(U_\varepsilon)$ is the completion of the space of C^∞-function defined in $\overline{\Pi}_\varepsilon$ in the norm of $H^1(U_\varepsilon)$. Recall that $f(x_1)$ is T-periodic such that $\int_{U_\varepsilon} f(x_1)d\boldsymbol{x} = 0$. Define $\widetilde{f}\left(\boldsymbol{x}\right) = f\left(x_1\right)\chi_{\Omega_{\varepsilon\delta}}\left(\boldsymbol{x}\right)$, where $\chi_{\Omega_{\varepsilon\delta}}\left(\boldsymbol{x}\right)$ is the characteristic function of $\Omega_{\varepsilon\delta}$. Note that $\int_{U_\varepsilon}\widetilde{f}(\boldsymbol{x})d\boldsymbol{x} = \int_{\Omega_{\varepsilon\delta}\cap\{x_1\in(0,T)\}} f(x_1)d\boldsymbol{x} = 0$. Define the subspace

$$\widetilde{H}^1_{per,\varepsilon}(U_\varepsilon) = \left\{u \in H^1_{per,\varepsilon}(U_\varepsilon) : \int_{U_\varepsilon} u d\boldsymbol{x} = 0\right\}.$$

Variational formulation of problem (1)÷(4) is as follows:

$$\text{Find } u_{\varepsilon\delta} \in \widetilde{H}^1_{per,\varepsilon}(U_\varepsilon) \ \text{ such that : } \int_{U_\varepsilon}\nabla u_{\varepsilon\delta}\nabla\varphi d\boldsymbol{x} = \int_{U_\varepsilon}\widetilde{f}(\boldsymbol{x})\varphi d\boldsymbol{x}, \ \ \forall\varphi \in H^1_{per,\varepsilon}(U_\varepsilon) \tag{63}$$

By Lax-Milgram lemma there exists a unique $u_{\varepsilon\delta} \in \widetilde{H}^1_{per,\varepsilon}(U_\varepsilon)$ such that (63) holds for every $\varphi \in \widetilde{H}^1_{per,\varepsilon}(U_\varepsilon)$. Let us show that (63) holds for every $\varphi \in H^1_{per,\varepsilon}(U_\varepsilon)$. For this take $\varphi \in H^1_{per,\varepsilon}$ and consider

$$\varphi = \langle\varphi\rangle + (\varphi - \langle\varphi\rangle),$$

where

$$\langle\cdot\rangle = \frac{1}{|U_\varepsilon|}\int_{U_\varepsilon}\cdot d\boldsymbol{x},$$

thus, $\varphi - \langle\varphi\rangle \in \widetilde{H}^1_{per,\varepsilon}$. We apply (63) for this function:

$$\int_{U_\varepsilon}\nabla u_{\varepsilon\delta}\nabla(\varphi - \langle\varphi\rangle)d\boldsymbol{x} = \int_{U_\varepsilon}\widetilde{f}(\boldsymbol{x})(\varphi - \langle\varphi\rangle)d\boldsymbol{x},$$

hence,

$$-\int_{U_\varepsilon} \nabla u_{\varepsilon\delta} \nabla \varphi d\boldsymbol{x} = \int_{U_\varepsilon} \widetilde{f}(\boldsymbol{x}) \varphi d\boldsymbol{x},$$

since $\nabla\langle\varphi\rangle = 0$ and $\int_{U_\varepsilon} \widetilde{f}(\boldsymbol{x}) d\boldsymbol{x} = 0$. Therefore, $\forall \varphi \in H^1_{per,\varepsilon}(U_\varepsilon)$ we have

$$-\int_{U_\varepsilon} \nabla u_{\varepsilon\delta} \nabla \varphi d\boldsymbol{x} = \int_{U_\varepsilon} \widetilde{f}(\boldsymbol{x}) \varphi d\boldsymbol{x}. \tag{64}$$

So the existence is proved. And the uniqueness of solution $\widetilde{u}_{\varepsilon\delta}$ of (63) $\forall \varphi \in H^1_{per,\varepsilon}(U_\varepsilon)$ is consequence of the uniqueness of the same formulation $\forall \varphi \in \widetilde{H}^1_{per,\varepsilon}(U_\varepsilon)$. □

Proposition 1. *Theorem 2 holds for function of two variables $\widetilde{f}$, such that $\widetilde{f}(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in G^i_{\varepsilon\delta}$ and satisfying $\int_{U_\varepsilon} \widetilde{f}(\boldsymbol{x}) d\boldsymbol{x} = 0$.*

Proof. Applying the Poincaré inequality in $U_\varepsilon = [0, T] \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ we have (cf. Lemma 4.A2.6 of [14])

$$\|u_{\varepsilon\delta}\|^2_{L^2(U_\varepsilon)} \leq 8T^2 \|\nabla u_{\varepsilon\delta}\|^2_{L^2(U_\varepsilon)}.$$

Then, from (64) we obtain that

$$\|\nabla u_{\varepsilon\delta}\|^2_{L^2(U_\varepsilon)} \leq \left\|\widetilde{f}\right\|_{L^2(U_\varepsilon)} \|u_{\varepsilon\delta}\|_{L^2(U_\varepsilon)} \leq 2\sqrt{2}T \left\|\widetilde{f}\right\|_{L^2(\Omega_{\varepsilon\delta}\cap U_\varepsilon)} \|\nabla u_{\varepsilon\delta}\|_{L^2(U_\varepsilon)}.$$

Hence,

$$\|\nabla u_{\varepsilon\delta}\|_{L^2(U_\varepsilon)} \leq 2\sqrt{2}T \left\|\widetilde{f}\right\|_{L^2(\Omega_{\varepsilon\delta}\cap U_\varepsilon)},$$

and

$$\|u_{\varepsilon\delta}\|_{H^1(U_\varepsilon)} \leq 2\sqrt{2}T\sqrt{1+8T^2} \left\|\widetilde{f}\right\|_{L^2(\Omega_{\varepsilon\delta}\cap U_\varepsilon)}. \tag{65}$$

□

4. Appendix 2. Existence and uniqueness of solution of the boundary layer problems and theorems of Phrägmen-Lindelöf type.

4.1. Now consider the problem

$$\triangle u = f(\boldsymbol{\xi}) \tag{66}$$

with boundary conditions shown in Fig. 14(a) and 14(b) such that

$$|f(\boldsymbol{\xi})| \leq c_1 e^{-c_2|\boldsymbol{\xi}|}. \tag{67}$$

Here c_1, c_2 are two positive constants.

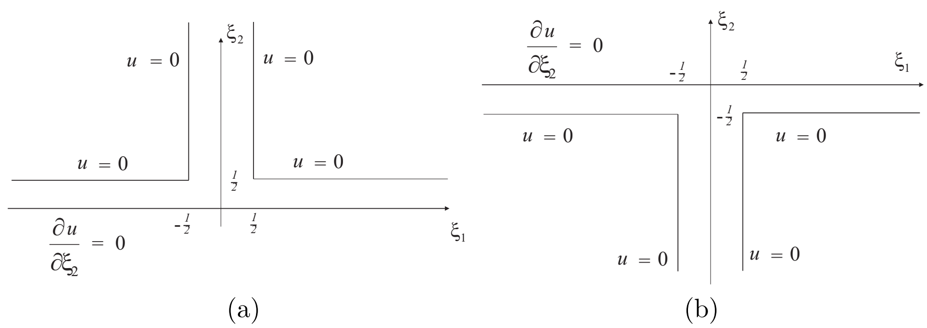

FIGURE 14. Domain in which problem (66) is set and the boundary conditions

We can extend the domain given in Fig. 14(a) by reflection to obtain the domain shown in Fig. 15 and extend the right hand side as an even function with respect

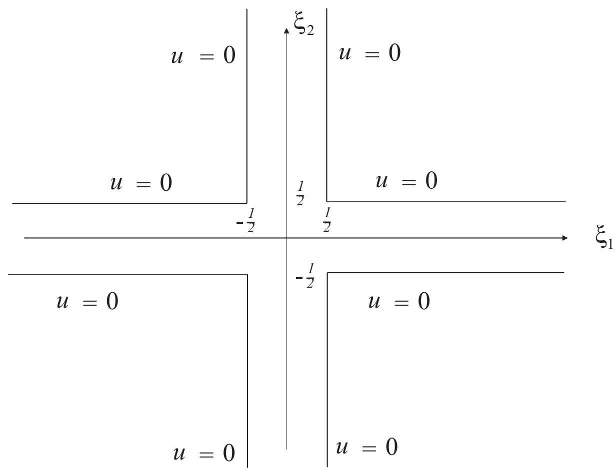

FIGURE 15. New domain obtained from two half-crosses given by Fig. 14

to the ξ_1-axis. The existence and uniqueness of solution in the domain shown in Fig. 15 follows from the Lax-Milgram theorem applied in the H_0^1 Sobolev space.

4.2. **Theorems of Phrägmen-Lindelöf type.** Consider the following problem

$$\begin{aligned} &\Delta u = f(\boldsymbol{\xi}) \quad \text{for } \boldsymbol{\xi} \in (0,+\infty) \times \left(-\frac{1}{2}, \frac{1}{2}\right), \\ &u = 0 \qquad\quad \text{for } \xi_2 = \pm\frac{1}{2}, \end{aligned} \tag{68}$$

such that $|f(\boldsymbol{\xi})| \leq c_1 e^{-c_2|\boldsymbol{\xi}|}$.

We reduce problem (68) to a problem with periodicity condition at the lateral boundary. To this end let us extend the domain $(0,+\infty) \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ by reflection to $(0,+\infty) \times \left(-\frac{1}{2}, \frac{3}{2}\right)$ and then periodically in ξ_2. Moreover, we extend the right hand side f as an odd function with respect to the line $\xi_2 = \frac{1}{2}$. Then we obtain the equivalent problem

$$\Delta \widetilde{u} = \widetilde{f}(\boldsymbol{\xi}), \quad \xi_1 > 0, \ \xi_2 \in \mathbb{R}, \tag{69}$$

where $\widetilde{f}(\boldsymbol{\xi})$ is 2-periodic in ξ_2 and

$$\widetilde{f}(\boldsymbol{\xi}) = \begin{cases} f(\boldsymbol{\xi}) & \text{for } \xi_2 \in \left(-\frac{1}{2}, \frac{1}{2}\right), \\ -f(\xi_1, -\xi_2) & \text{for } \xi_2 \in \left(\frac{1}{2}, \frac{3}{2}\right). \end{cases}$$

We can apply now the result of [8] that every 2-periodic in ξ_2 solution of equation (69) set in half-space $(0,+\infty) \times \mathbb{R}$ can either have a linear or an exponential growth as $\xi_1 \to +\infty$, or it stabilizes to some constant. Theorem 2 in [8] leaves only the last possibility. Moreover, such a constant is zero because $\widetilde{u} = 0$ for $\xi_2 = \pm\frac{1}{2}$. Applying this analysis to each branch of the cross (Fig. 15) we obtain that the solution of the Dirichlet problem for the Laplace equation (66) with exponentially decaying right hand side exponentially tends to zero as $|\xi| \to \infty$: there exist two positive constants $\bar{c_1}, \bar{c_2}$ such that,

$$|u(\boldsymbol{\xi})| \leq \bar{c}_1 e^{-\bar{c}_2|\boldsymbol{\xi}|}. \tag{70}$$

Note that the same result can be obtained easily directly from (68) by the Fourier expansion of f and u in ξ_2.

Note that the Agmon-Duglas-Nirenberg theory gives estimate (70) with some constants for the derivatives of u if the derivatives of the right hand side f decay exponentially.

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