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Multilateral Market Power in Input-output Networks

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Matteo Bizzarri*

Abstract

This paper develops a strategic model of large firms connected in an input–output network, where firms exert market power in both input and output markets through double auctions. Markups and markdowns are determined by firms' centrality in a network connecting goods. Many standard models are recovered as special cases in which the ability of firms to affect some prices is restricted. In comparison, multilateral market power increases the final price and modifies the division of surplus among firms. As a result, it is key for applications such as the evaluation of the welfare impact of mergers and the diffusion of productivity shocks.

JEL classification: L13, D43, D44, D57.

Keywords: production networks, oligopoly, double auction, supply function equilibrium.

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also sizable, both in labor (Azar and Marinescu, 2024) and intermediate inputs (Morlacco, 2019; Dhyne et al., 2022; Alviarez et al., 2023). Moreover, input-market power has also received attention from competition authorities, particularly in relation to labor (Kariel et al., 2024). However, with the exceptions noted in the literature, most customarily used models of firm-to-firm trade impose the simplifying assumption that firms are price-takers on the input markets. In this paper, I study a strategic model of large firms connected in an input-output network, where firms have the ability to affect prices both in input and output markets, in an endogenously determined way. I label this feature *multilateral market power*.

The main contributions of the paper are two. First, I show that, in the unique equilibrium of the game, there is an appropriate weighted network connecting the goods of the economy, the *goods network*, such that the markups and markdowns can be seen as a measures of centrality with respect to this network. Second, I explore the implications of multilateral market power. To do so, I compare the equilibrium of the benchmark model with two variations: one where firms are price-takers on inputs (which I label *unilateral* market power), and one where firms take as given all the prices of markets they are not directly involved in (which I label *local* market power). As discussed below in the literature section, these two types of assumptions are widespread in the production networks literature. The key result is that, when firms take some prices as given, both the total surplus and the way it is distributed change: the final price is lower, and more upstream firms absorb more of the surplus. Finally, I illustrate the importance in a few applications.

In the model, firms have a set of input goods and produce each an output. Some outputs are the input of other firms, and these trade relationships, or *input-output links*, are exogenous. Firms trade using a uniform-price double auction for each good, as in models of the financial market (Malamud and Rostek, 2017). More specifically, firms play a simultaneous game in which the available actions are supply and demand schedules, relating quantities of the traded goods to prices. The realized price on every trade relationship is the one where demand and supply cross. The key feature generating market power is that firms, being non-infinitesimal, fully internalize the mechanism and choose their schedules to affect prices in their favor. The classic metaphor for the price-taking general equilibrium behavior is that a "Walrasian" auctioneer proposes prices and collects supply and demand "bids", until all markets clear. The approach followed in this paper takes this metaphor one step further, applying it to non-infinitesimal firms. The auctioneer acts as a market maker in financial markets, collecting firms' conditional schedules. The competition in schedules is meant not as a literal description of the workings of the market (although they are in some cases, e.g., the electricity or financial markets), but as an abstraction of a bargaining procedure, parsimonious but powerful enough for the complexity of the problem. In practice (as Klemperer and Meyer (1989) suggest), the choice of a schedule can be thought of as representing all the organizational or managerial decisions that affect how a firm responds to different market conditions. In some industries, e.g., natural gas (Hubbard and Weiner, 1991),contracts commonly include clauses allowing price/quantity changes conditional on some events,

Tractability is assured by the functional form assumptions: intermediate inputs are perfect complements,[1] and consumers' demand is linear. In the main text, I assume that firms can only choose linear schedules. Under the technology constraint, this assumption boils down to assuming that firms choose a single number, representing the slope of the supply schedule. This considerably reduces the dimensionality of the problem and buys a lot of tractability, because the game is a supermodular potential game. Theorem 1 exploits these properties to show that this game has a unique equilibrium. Uniqueness is due to the potential structure and, to be best of my knowledge, is new. In the Supplementary Material, I show that the same profile of linear schedules remains an equilibrium even when firms are allowed to choose schedules of any functional form. Introducing uncertainty in cost and demand, the linear equilibrium still survives, and the linear schedule is the unique best reply to a profile of linear schedules.[2] Moreover, the only result whose proof depends on the perfect complementarity assumption is the equilibrium uniqueness. In the Supplementary material, I show that all the other results and economic intuitions are still valid when intermediate inputs are imperfect complements or substitutes, provided the production function is such that the functional form for the profits is still linear-quadratic (otherwise tractability is lost).

In this setting, market power cannot be captured by a single index, because firms charge both a markup when selling and potentially heterogeneous markdowns when buying. Markup and markdowns depend on the equilibrium slopes of residual supply and demand, which in turn depend on all direct and indirect connections across markets. The object that summarizes these connections is the *goods network*, an undirected network in which the nodes are the goods, and two goods are linked if some firms trade both goods. The strength of a link between good i and j is high if the price of i is very sensitive to the supply of j, or vice-versa. The equilibrium markup and markdowns are proportional to the Bonacich centrality of each input and output in the goods network (Theorem 2). This is because the network affects the price impact via the pass-through of prices: the number of direct and indirect connections measures the strength of the pass-through effect. In the case of a supply chain with layers, all these effects can be

[1] The technology generalizes slightly the Leontief functional form, because it allows the input requirement for labor to depend quadratically on the output quantity. This simplifies the conditions under which an equilibrium exists.

[2] In the special case without the network, which is Klemperer and Meyer (1989), the linear equilibrium is the unique one surviving the introduction of uncertainty. With the input-output network, it is still possible to show that uncertainty makes the linear schedule the unique best reply to linear schedules, but I am not aware of either equilibrium uniqueness proofs or counterexamples for the general case.

precisely characterized (Proposition 1): we obtain that, in the homogeneous situation where all layers have the same size and number of firms, the markup is larger for the more upstream firm (the farther from the consumer), while the markdown is larger for the more downstream firm (the closer to the consumer).

To single out the effect of multilateral market power, I then generalize the model to the "Generalized SDFE", in which firms still choose schedules, but where the price impact functions are a given primitive. The key property required (beyond continuity) is that the price impact functions must be decreasing (in the positive semidefinite sense) in the slopes of the schedules. For different choices of price impact functions, we obtain as special cases the model of the previous paragraphs and other standard models: the classic Cournot oligopoly (without input-output dimension) and also the sequential monopoly à la Spengler (1950). I study in detail two relevant special cases: (i) firms take input prices as given (which I call unilateral market power) and (ii) firms take as given all prices of markets where they are not directly involved in (local market power). These two sets of assumptions are relevant because they are often used in quantitative models of input-output networks, as discussed in the Literature below.

Theorem 3 proves existence by using the fact that best replies are still increasing. Moreover, it proves that, if the price impact function is larger (in the positive semidefinite sense), the equilibrium slopes are smaller. This is the fundamental tool that allows us to do comparative statics. Theorem 4 shows that both the unilateral and local competition models are Generalized SDFEs for the proper choice of the price impact function. Moreover, in both cases the price impact is lower in the positive semidefinite sense, than with multilateral market power. Corollary 5.1 yields the main economic conclusion: the final price is larger with multilateral market power. The intuition is as follows. Whenever a firm takes some prices throughout the network as given (which is the case both under unilateral and local competition), that firm perceives a larger elasticity of demand and supply and, as a consequence, is able to charge smaller markups and markdowns. This is because, in the S&D equilibrium, the elasticity of demand depends on the schedules chosen by directly connected firms, but also *indirectly* connected firms. The reason is that, in equilibrium, a change in a price triggers a change in all other prices of connected firms: failing to account for some of these pass-through effects means firms perceive a different elasticity of demand.[3]

Moreover, multilateral market power also has consequences for the division of surplus. Proposition 2 shows that, with unilateral market power, the upstream layers have larger markup and profit than the downstream layers, in contrast with the multilateral case, in which all the layers are symmetric. So, the amount of surplus extraction predicted by

[3]The literature on outsourcing and endogenous supply chains provides evidence that firms are aware of their supply chain and take its structure and their position in it into account in their decisions, see e.g. Berlingieri et al. (2021), Alfaro et al. (2019).

the model is very different.

Finally, in the context of a simple supply chain, I explore the role of multilateral market power in connection to two classic supply chain questions: the welfare impact of mergers and the diffusion of shocks. Proposition 3 shows that unilateral market power has very different implications than bilateral market power, respectively for horizontal and vertical mergers. Proposition 4 shows that unilateral market power changes the ranking of which nodes generate more shock propagation.

These considerations suggest that, in supply chains where firms are large and potentially have the ability to affect prices in both input and output markets,[4] assumptions about which prices a firm can affect are not innocuous. They crucially affect the magnitude of distortions and the division of surplus, which are characteristics that directly affect the answer the model gives to policy-related questions. For some specific markets, the details of the competition mechanism can be precisely observed. However, especially when working with large firm networks, presumably the firms involved are very heterogeneous in terms of the products, bargaining procedures, and timing of the offers. In such contexts, assumptions restricting price impact are typically a simplifying modeling device: in this case, the Supply and Demand function equilibrium can provide a useful tool.

Related literature

This paper contributes to three lines of literature: the literature on competition in supply and demand functions, the literature on production networks or networked markets, and the literature on general equilibrium oligopoly.

My contribution to the literature on competition on supply and demand functions is to introduce the technique to the modeling of general equilibrium oligopoly, in particular with firm-to-firm trade. The literature has studied the situation where the demand firms receive comes from a network structure, in Wilson (2008) and Holmberg and Philpott (2018). Holmberg et al. (2025) study the case of multi-product firms. These papers do not consider firm-to-firm trade. Bilateral trade between two firms is studied in Weretka (2011) and Hendricks and McAfee (2010), but they do not consider a general production network, and do not focus on the effect of multilateral market power. In the finance literature, the model is used to study the simultaneous demand and supply of heterogeneous assets: Malamud and Rostek (2017) show how the strategic complementarity property extends to the network setting, and characterizes an equilibrium in a general network. The model has a different purpose (studying centralization in financial markets) and also two important technical differences: in my paper, the functional form is different,

[4] Firms that have a large dimension with respect to their own sector or the whole economy are often called *superstar firms* since Autor et al. (2020), and are the subject of a large literature.

because the Leontief technology gives a different best reply equation: the difference is important, because the proof of the uniqueness of the equilibrium relies essentially on it. Moreover, I study the Generalized SDFE version with general price impacts. Rostek and Yoon (2021a) and Rostek and Yoon (2021b) also analyze similar models and share the same differences with my work. Ausubel et al. (2014) and Wittwer (2021) also study uniform price auctions, in different settings. Vives (2011) studies market power arising from asymmetric information, rather than network position.

My contribution to the production networks literature is to provide a model of competition in an input-output network in which all firms have market power on both input and output markets and are fully strategic, internalizing their position in the supply chain. Many models explicitly assume that firms have the power to decide/affect prices only on one side of the market. To this class belong the workhorse sequential oligopoly games in Salinger (1988), Ordover et al. (1990), Hart and Tirole (1990).In another class of models, authors assume that output prices are equal to the marginal cost times a markup. The concept of the marginal cost itself implicitly implies price-taking in the input market: indeed, it arises from the price-taking cost minimization problem of the firm. Hence, it implicitly assumes unilateral market power. Examples in this category are Grassi (2017), Bernard et al. (2022), Baqaee and Farhi (2019), Baqaee and Farhi (2020), Magerman et al. (2020), Dhyne et al. (2022), Bizzarri and Vega-Redondo (2024). A third class of models is those in which vertically connected firms share surplus through some form of Nash bargaining. Toxvaerd (2024) reviews the recent work in the area, in the context of a vertical chain. Acemoglu and Tahbaz-Salehi (2025) and Alviarez et al. (2023) apply this idea to general networks. My results complements theirs, providing a model that does not rely on the choice of exogenously specified bargaining weights.[5] More in general, many models of networked markets have studied the network defined by the demand: Galeotti et al. (2024), Pellegrino (2025), Bimpikis et al. (2019), but they do not focus on input-output connections.

Except for Acemoglu and Tahbaz-Salehi (2025), all these papers also feature the implicit or explicit assumption that firms do not internalize the effect of their decisions on sectors/firms further downstream besides the direct customers. Sometimes this is a consequence of the assumption of a continuum of firms in each sector (so that sector-level aggregates are taken as given by every individual firm, as in Baqaee (2018)), other times it is explicitly assumed (e.g., in Grassi (2017), Dhyne et al. (2022)).

I contribute to the literature on general equilibrium with market power by providing a fully strategic model of the production side with endogenous market power and firm-to-firm trade. The recent literature on "general oligopolistic competition" (Azar and

[5]The papers also differ from mine in other dimensions: Alviarez et al. (2023) study buyer-seller, rather than input-output connections; Acemoglu and Tahbaz-Salehi (2025) is a model of endogenous exit: in the benchmark with no exit, the equilibrium is efficient, unlike in my model.

Vives, 2021; Flynn et al., 2025) did not so far consider firm-to-firm trade.

The rest of the paper is organized as follows. Section 1 illustrates the model through a simple example of a vertical economy. Section 2 defines the benchmark model, the Supply and Demand Function Equilibrium (SDFE). Section 3 describes the solution and the existence theorem. Section 4 shows the characterization of markups and the connection with the goods network. Section 5 introduces the Generalized SDFE and explores the effect of multilateral market power. Section 6 concludes. The main proofs are in the Appendix.

1 A simple example

In this section, I illustrate the model and the main take-aways in the simplest network where the concept of multilateral market power is non-trivial: a supply chain consisting of one intermediate producer, U, and a final good producer D. This is represented in Figure 1. The intermediate good producer U produces good U using only labor, and sells it to both the final producer D and consumers. In turn, the final producer D uses good U to produce the final output D. The consumers consume both goods U and D,[6] with linear demands: $D_{c,D}(p_D) = A - p_D$, $D_{c,U}(p_U) = A - p_U$.

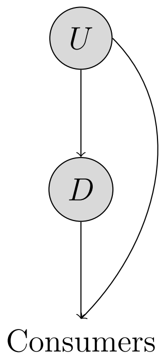


Figure 1: A simple vertical economy.

Firms have a standard linear technology: $F_U(q_U) = q_U$ and $F_D(q_D) = q_D$, so that profits are: $\pi_U = p_U q_U$, $\pi_D = (p_D - p_U)q_D$. The firms play a simultaneous game in which the strategic variables are the (slopes of the) linear schedules connecting prices and quantities. Formally:

1. firm U submits a supply function $S_U(p_U) = B_U p_U$, where B_U is any positive real number;
2. firm D submits a function $S_D(p_U, p_D) = B_D(p_D - p_U)$ indicating both its supply of output, and its demand for the input, where B_D is, again, any positive real number.

Whichever choice of the firms, the prices p_U, p_D and quantities q_U, q_D must satisfy the market clearing conditions:

$$
\begin{aligned}
q_D &= A - p_D = B_D(p_D - p_U) \\
q_U &= A - p_U + B_D(p_D - p_U) = B_U p_U \qquad (1)
\end{aligned}
$$

[6] The fact that consumers also consume good U is necessary for the model to have a non-trivial equilibrium. This is a well-known technical feature of competition in schedules: a non-trivial linear equilibrium exists if goods are traded by at least 3 agents (Malamud and Rostek, 2017).

We look for the Nash equilibrium of this game.

Focus on firm U. For each fixed quantity of output q_U, we can solve the system (1) for the inverse demand:

$$p_{U,U}(q_U) = \left(1 + \frac{B_D}{B_D + 1}\right)^{-1} (A - q_U)$$

and similarly for the inverse demands faced by firm D, that we call $p_{D,U}(q_D)$ and $p_{D,D}(q_D)$. When taking the FOC for firm U, we get:

$$\frac{\partial}{\partial B_U}\pi_U = \frac{\partial q_U}{\partial B_U}\left(p_U + q_U \frac{\partial p_{U,U}}{\partial q_U}\right) = 0$$

From the market-clearing conditions, it is easy to conclude that $\frac{\partial q_U}{\partial B_U} > 0$, and so the FOC are equivalent to: $p_U + q_U \frac{\partial p_{U,U}}{\partial q_U} = 0$. Doing the analogue for firm D, we obtain the equilibrium equations:

$$p_U + q_U \frac{\partial p_{U,U}}{\partial q_U} = 0 \tag{2a}$$

$$p_D - p_U + q_D \left(\frac{\partial p_{D,D}}{\partial q_D} - \frac{\partial p_{D,U}}{\partial q_D}\right) = 0 \tag{2b}$$

Since schedules are linear, the derivatives are just constants: so, it is immediate to write the best response schedules as:

$$S_U(p_U) = \left(-\frac{\partial p_{U,U}}{\partial q_U}\right)^{-1} p_U$$

$$S_D(p_D, p_U) = \left(\frac{\partial p_{D,D}}{\partial q_D} - \frac{\partial p_{D,U}}{\partial q_D}\right)^{-1} (p_D - p_U)$$

So, the slopes B_D^*, B_U^* that constitute an equilibrium of the game must be equal to the slope of the above functions, and satisfy:

$$\begin{aligned} B_U^* &= \left(-\frac{\partial p_{U,U}}{\partial q_U}\right)^{-1} = 1 + \frac{B_D^*}{B_D^* + 1} \\ B_D^* &= \left(\frac{\partial p_{D,D}}{\partial q_D} - \frac{\partial p_{D,U}}{\partial q_D}\right)^{-1} = \left(1 + \frac{1}{B_U^*}\right)^{-1}. \end{aligned} \tag{3}$$

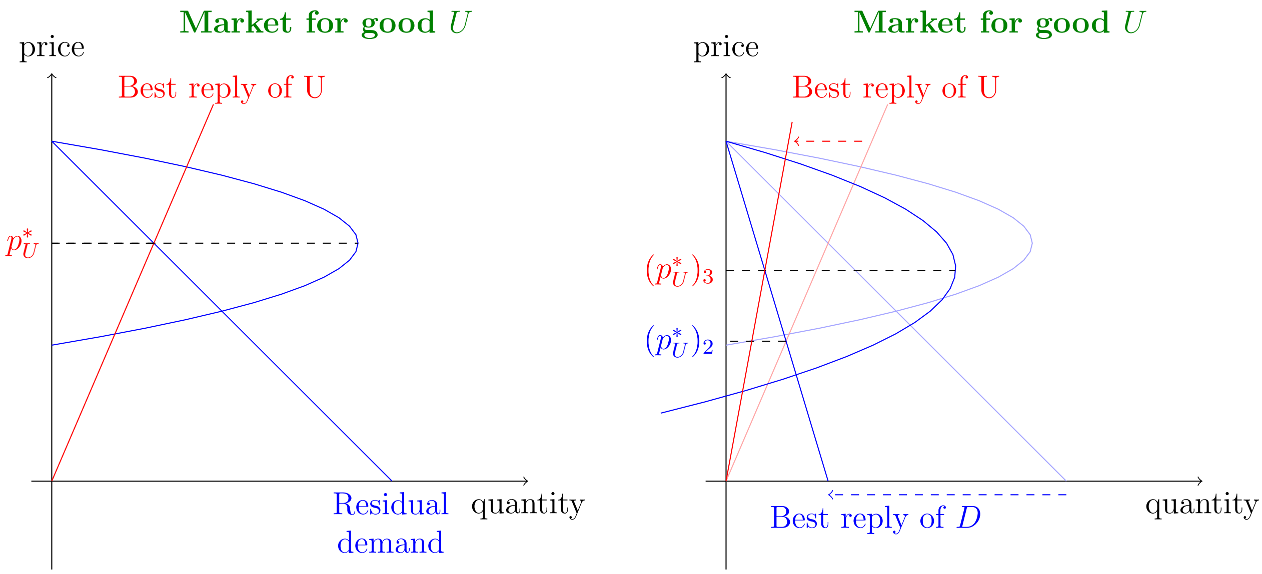
The expression highlights the role of the price impacts (the derivatives of the inverse demand/supply), and in particular, the fact that firm D has a price impact on both the input and the output market. The equations can be solved analytically, and it can be checked that the solution is: $B_D^* = 1/\sqrt{2}$, $B_U^* = \sqrt{2}$.

In a sense, both the buyer and the seller of good U set their "optimal price". This seems a contradiction, since sellers would want to raise p_U while buyers would want to decrease it. The tension is resolved by the fact that firms "implement" a price by modifying the slope of their schedule that, in turn, *changes other firms' incentives to raise prices*. The situation is represented graphically in Figure 2: firm U faces a residual demand $D_U^r(p_U)$, that is the blue line in the graph, depending on the slope of consumers and the slope chosen by D. This residual demand induces a profit as a function of the price p_U. Firm U wants to charge p_U^*, the monopoly price for this residual demand, and so sets a slope that achieves that price: this is the red line in Figure 2a. But, in doing so, it affects the slope of the *residual supply* that firm D faces. As a consequence, firm D changes its choice of schedule, changing the transaction price to $(p_U^*)_2$, the optimal *monopsony* price for firm D. This, in turn, leads to a new residual demand and a new profit function for firm U (as in Figure 2b): as a consequence, the previous optimal price p_U^* is not optimal anymore, and firm U adjusts its slope again. This adjustment process continues until the slopes are such that the optimal price sellers want to charge is equal to the optimal price for the buyers.

In this model both firms have multilateral market power. What happens if instead we adopt the more standard assumption that firm D is a price-taker on the input market? This means that, from D's perspective, its choice does not affect the input price, so $\frac{\partial p_{D,U}}{\partial q_D} = 0$. The equilibrium equations (3) become:

$$B_U^{**} = 1 + \frac{B_D^{**}}{B_D^{**} + 1}$$
$$B_D^{**} = 1.$$

Moreover, this solution is the same we would get solving the model as a standard sequential monopoly, as shown in Example 5.3.1. The solution in this case is $B_D^{**} = 1$, $B_U^{**} = 3/2$. They are both higher than in the case of multilateral market power. Because welfare is increasing in both slopes, we can immediately conclude that consumer welfare is higher in this case. The reason is again illustrated in Figure 2: the slopes are strategic complements. This means that, when firm D increases its markdown, exploiting the input-market power, firm U responds by increasing its own markup, decreasing quantity even further. So, in this context, market power reinforces market power. This same mechanism operates in any network, as Theorem 4 shows below, and has many consequences for applications, as Section 5.3 illustrates.



(a)

(b)

Figure 2: Graphical representation of the choice of schedule by firm U. On the left (a): the best reply for firm U to the residual demand given by the blue line. On the right (b): the optimal choice of firm U leads other firms to adjust, modifying firm U residual demand and optimal price: so firm U further adjusts its best reply.

2 The model

2.1 Setting

Firms and Production Network There are n firms and m goods: their sets are respectively denoted $\mathcal{N}$ and $\mathcal{M}$. Each good might be produced by more firms, but each firm produces only one good. Each firm produces using labor and a set of inputs produced by other firms, which I denote as $\mathcal{N}^{in}(i)$. Denote the set of all goods traded by firm i as $\mathcal{N}(i) = \mathcal{N}(i)^{in} \cup \{i\} \subseteq \mathcal{M}$. The number of non-labor goods traded by i is $d_i = |\mathcal{N}(i)|$, so the number of intermediate inputs is $d_i - 1$. The consumers' utility depends potentially only on a subset of goods, denoted $\mathcal{C} \subseteq \mathcal{M}$. Firms, goods and the connections $(\mathcal{N}(i))_{i\in\mathcal{N}}$ defined above define the *input-output network* of this economy. The network must be *connected*, in the sense that for each good g there is at least a buyer, either another firm i or the consumer.

Notation Bold symbols are used to denote vectors: $\boldsymbol{p}$ is the vector of all prices (always in labor terms), $\boldsymbol{q}$ the vector of all output quantities. If $\boldsymbol{v}$ is a vector indexed on goods and $\mathcal{A}$ is a subset of goods, $\boldsymbol{v}_{\mathcal{A}} = ((v_g)_{g\in\mathcal{A}})$ denotes the vector that selects only the entries relative to goods belonging to $\mathcal{A}$. Similarly, if M is a matrix, $M_{\mathcal{A}}$ denotes the square submatrix where both row and columns are indexed by goods in $\mathcal{A}$; $M_{\mathcal{A}^c}$ denotes the square submatrix indexed by the complement; and $M_{\mathcal{A},\mathcal{A}^c}$ denotes the off-diagonal block. If $\mathcal{A} = \mathcal{N}(i)^{in}$ or $\mathcal{A} = \mathcal{N}(i)$, for brevity, I use the notation $\boldsymbol{p}_i := \boldsymbol{p}_{\mathcal{N}(i)}$, $\boldsymbol{p}_i^{in} := \boldsymbol{p}_{\mathcal{N}^{in}(i)}$

(the prices of all input goods of firm i), and p_i^{out} for the price of the output, so that $\boldsymbol{p}'_{\mathcal{N}(i)} = (p_i^{out}, (\boldsymbol{p}_i^{in})')$. Similarly, $\boldsymbol{p}_c := (p_g)_{g\in\mathcal{C}}$ is the vector of prices of goods consumed by the consumer. If $B_i \in \mathbb{R}^{d_i\times d_i}$ is a matrix, the notation $\hat{B}_i$ denotes the *lifting* of the matrix B_i, namely the matrix in $\mathbb{R}^{m\times m}$ such that the nonzero elements are exactly those corresponding to the elements of matrix B_i, and the rest is filled with zeros. Analogously, if $\boldsymbol{v}_i \in \mathbb{R}^{d_i}$, the notation $\hat{\boldsymbol{v}}_i$ denotes the vector in $\mathbb{R}^m$ in which the additional elements are filled with zeros.

For quantities, it is understood that positive quantities represent outputs and negative quantities represent inputs. So, the vector of input and output quantities traded by firm i is $\boldsymbol{q}_i = (q_i^{out}, -\boldsymbol{q}_i^{in})$, where $\boldsymbol{q}_i^{in} = (q_{ig})_{g\in\mathcal{N}^{in}}$ is the vector of input quantities. The signs follow the standard convention that negative quantities represent inputs, positive represent outputs. The quantity of labor used by firm i is ℓ_i.

Consumers The utility function of the consumers is quadratic in consumption and (quasi-)linear in the disutility of labor L:

$$U(\boldsymbol{c}, L) = \boldsymbol{A}' B_c^{-1}\boldsymbol{c} - \frac{1}{2}\boldsymbol{c}' B_c^{-1}\boldsymbol{c} - L \tag{4}$$

where $\boldsymbol{c} = (c_g)_{g\in\mathcal{C}}$ is the vector of quantities consumed, $\boldsymbol{A}$ is a positive vector, and B_c is a symmetric positive definite matrix. This means that the consumer demand has the form: $D_c = \boldsymbol{A} - B_c\boldsymbol{p}_c$.

Technology Intermediate inputs are perfect complements, so that to produce a quantity of output q_i firm i needs $f_{ih}q_i$ units of input h. We adopt the standard assumption that the technology has to be *viable*, namely that there must exist a positive quantity vector $\boldsymbol{q}$ such that $q_i > \sum_h f_{hi}q_h$ for each i. Denote $F \in \mathbb{R}^{n\times m}$ the matrix with entries f_{ih}. To produce q_i^{out} units of output, the firm also needs $\ell_i = f_{i,L}q_i^{out} + \dfrac{1}{2k_i}(q_i^{out})^2$ labor units. This slightly generalizes the Leontief technology to allow decreasing returns: as illustrated below, this facilitates the existence of an equilibrium. Formally, the production function is: $q_i^{out} = \min\{f_i(\ell_i), q_{ij}/f_{ij}\}$. If $k_i \to \infty$, the technology becomes the standard Leontief. Denote the vector of labor requirements $f_{i,L}$ as $\boldsymbol{f}_L$. The relation between ℓ_i and q_i^{out} can be inverted as $q_i^{out} = f_i(\ell_i)$, defining $f_i(\ell_i) := f_{i,L}k_i\left(\sqrt{2\ell_i/(f_{i,L}^2k_i)+1} - 1\right)$. So, we can write the technology constraints of firm i as:

$$\begin{aligned} q_{ij} &= f_{ij}f_i(\ell_i) \quad \forall j \in \mathcal{N}_i^{in} \\ q_i^{out} &= f_i(\ell_i) \end{aligned} \tag{5}$$

It is going to be convenient to define the vector $\boldsymbol{v}_i = (1, -f_{i1}, \ldots, -f_{1N})$, so that the constraints become $\boldsymbol{q}_i = f_i(\ell_i)\boldsymbol{v}_i$.

2.2 The game

Schedules The competition among firms takes the form of a game in which firms compete by choosing a supply function for the output, and demand functions for intermediate inputs and labor, respecting the technology constraint (5). The players of the game are the firms: $i = 1, \ldots, N$, and the actions available to each firm i are *linear schedules*, one for the output $\mathcal{S}_i^{out}$, and others for intermediate inputs $\mathcal{S}_i^{in}$, and labor $\mathcal{S}_{\ell,i}$. Denote the schedule of intermediate input trades of firm i as: $\mathcal{S}_i = (\mathcal{S}_i^{out}, -\mathcal{S}_i^{in})$.[7] The assumption of linearity means that there exists a matrix of coefficients $B_i \in \mathbb{R}^{d_i \times d_i}$ and a vector $B_{i,f} \in \mathbb{R}^{d_i}$, such that the schedule is linear:

$$\mathcal{S}_i(\boldsymbol{p}_i) = B_i \boldsymbol{p}_i - B_{i,f} \tag{6}$$

The technology constraints (5) imply that the supply function $\mathcal{S}_i^{out}$ determines the whole input schedule, including labor, as inputs are bought in constant proportion. So, it follows that: $\mathcal{S}_i = \mathcal{S}_i^{out} \boldsymbol{v}_i$ and $\mathcal{S}_{\ell,i} = f_{i,L} \mathcal{S}_i^{out} + \frac{1}{2k_i} (\mathcal{S}_i^{out})^2$. Moreover, it turns out (proven in Lemma 3.2 below) that for the schedule (6) to be a best reply, it must be that for some $\overline{B}_i \in [0, k_i]$:

$$B_i = \overline{B}_i \boldsymbol{v}_i \boldsymbol{v}_i', \quad B_{i,f} = \overline{B}_i \boldsymbol{v}_i f_{i,L}. \tag{7}$$

In particular, it turns out that B_i is positive semidefinite and symmetric. So, it is sufficient to restrict firms' choice to the choice of a single coefficient $\overline{B}_i$ in $[0, k_i]$. In the main text, I set up the game directly as a choice of this single coefficient, because it simplifies the analysis. Sometimes, for convenience of notation, I still use the B_i, $B_{i,f}$ matrices defined in (7). Denote $B = (B_i)_{i \in \mathcal{N}}$ a profile of coefficient matrices, $B_f = (B_{i,f})_{i \in \mathcal{N}}$ a profile of intercept vectors, and $\overline{B} = (\overline{B}_i)_{i \in \mathcal{N}}$ a profile of slope coefficients. When needed, we index the matrices B_i directly with the relevant goods, for simplicity. So, $B_{i,gh}$ means the entries of the schedule relative to the effect of the price of h on demand for the input goods g.

Most proofs also only use properties of B_i directly, and so they generalize to the case of imperfect complements and substitutes, explored in the Supplementary Material.

Prices The market prices are, by assumption, those satisfying the market-clearing conditions. The market-clearing equations for intermediate inputs are:

$$\sum_{i\,:\, g \in \mathcal{N}_i} \mathcal{S}_{ig}(\boldsymbol{p}_i) = D_{cg}(\boldsymbol{p}_c) \quad \forall g \in \mathcal{M} \tag{8}$$

[7]We denote the quantities as $\boldsymbol{q}_i$ when they are simply variables, with $\mathcal{S}_i$ when they are explicit functions of prices.

Since the demand derived by (4) satisfies Walras's law, it is standard that one of the market-clearing conditions is redundant: indeed, we leave out the labor market-clearing equation $\sum_i \mathcal{S}_{\ell,i}(\boldsymbol{p}_i) = L(\boldsymbol{p}_c)$. Lemma 3.1 below shows that the system has a unique solution, a *pricing function* mapping coefficient matrices to prices: $\boldsymbol{p} : (\overline{B}) \to \boldsymbol{p}(\overline{B})$. This function is crucial: it embeds the information about competition and network interconnections.

Payoffs To complete the definition of the game, it remains to define the payoffs. These are, in short, the profits, calculated in the prices that satisfy the market-clearing conditions (8):

$$\begin{aligned}\pi_i(\overline{B}) &:= \boldsymbol{p}_i'(\overline{B})\mathcal{S}_i(\boldsymbol{p}_i(\overline{B})) - \mathcal{S}_{\ell,i}(\boldsymbol{p}_i(\overline{B})) \\ &= \overline{B}_i \left(1 - \frac{1}{2k_i}\overline{B}_i\right) (\hat{\boldsymbol{v}}_i'\boldsymbol{p}(\overline{B}) - f_{iL})^2 \end{aligned} \tag{9}$$

In particular, since $\overline{B}_i \leq k_i$, we get that $\pi_i(\overline{B})$ is always nonnegative, so we do not need to worry about firm exit.

Definition 2.1.
A Supply and Demand Function Equilibrium (SDFE) is a Nash equilibrium of the game $G = (\mathcal{N}, ([0, k_i], \pi_i)_{i\in\mathcal{N}})$, where the players are the firms, actions are slopes, and the payoffs are defined in (9).

Example 1. Horizontal economy/Standard Supply Function Equilibrium

Consider the case of n firms, producing the same output good, without input-output connections (producing using only labor): $\boldsymbol{v}_i = 1$ for $i = 1, 2, \ldots, n$. The demand function in this case is $D_c = A_c - B_c p_c$, where A_c, $B_c \in \mathbb{R}_+$. This is an instance of the Supply Function competition by Klemperer and Meyer (1989) (in the parametric case of the quadratic cost function).

Example 2. The vertical economy The vertical economy with an upstream and a downstream firm illustrated in the Section 1 is an example where there are two goods, U and D, and two firms with the same indices. The network is defined by: $\mathcal{N}(U) = \{D\}$, $\mathcal{N}(D) = \{U, D\}$ and $\mathcal{C} = \{D\}$. The schedules are: $\mathcal{S}_U = \overline{B}_U(p_U - f_{U,L})$ and $\mathcal{S}_D = \overline{B}_D(p_D - f_{DU}p_U - f_{D,L})$. The section explores the special case with $n = 2$ firms and $m = 2$ goods in which $f_{i,L} = 0$ and $k_i \to \infty$, so that the marginal costs are zero. The parameters satisfy: $\boldsymbol{v}_U = 1, \boldsymbol{v}_D = (1, -1)$, $B_c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

We can generalize the vertical economy above to a supply chain with layers. This example is very tractable, so I am going to use it intensively to illustrate the model in the next sections.

(a) homogeneous case, with $n_1 = n_2 = 2$. (b) Asymmetric layers, with $n_1 = 1$, $n_2 = 2$.

Figure 3: A supply chain with 2 layers and 1 consumer good: $\mathcal{C} = \{1\}$.

Example 3 (Supply chain with layers). A layered supply chain is a production structure in which firms are divided in N *layers*, and each layer produces one of the goods, as in Figure 3. There are n_i firms in layer i. Layers are indexed from 1 to N moving upstream (we can consider the consumers as layer 0). Firms in layer i use as intermediate input the good produced in layer $i+1$, and sell to firms in layer $i-1$ and possibly to the consumer. Firms in layer 1 sell only to the consumer, firms in layer N only use labor for production. So, if firm h is in layer $i < N$ then $\mathcal{N}(h) = \{i+1, i\}$, while for a firm h in layer N $\mathcal{N}(h) = \{N\}$. The consumer may or may not consume all the goods: $\mathcal{C} \subseteq \{1, \ldots, N\}$, but a particularly useful example will be with $\mathcal{C} = \{1\}$. If $N = 1$, we obtain the standard Supply Function equilibrium as in Klemperer and Meyer (1989), and the example above. For simplicity, assume that firms in each layer share the same parameters: so $f_{i,L}$, k_i and $f_{i,i+1}$ only depend on the layer i in which the firm is. So, $\boldsymbol{v}_i = (1, -f_{i,i+1})$ for each layer $i < N$, and $\boldsymbol{v}_N = 1$ for the last layer. In this case, firms in each layer also have identical schedules, so we can index schedules directly by the layer index: $\mathcal{S}_i = (\mathcal{S}_i^{out}, -\mathcal{S}_i^{in})$ for $i < N$. The coefficient matrix $B_i \in \mathbb{R}^{2\times 2}$ satisfies:

$$B_i = \overline{B}_i \boldsymbol{v}_i \boldsymbol{v}_i' = \overline{B}_i \begin{pmatrix} 1 & -f_{i,i+1} \\ -f_{i,i+1} & f_{i,i+1}^2 \end{pmatrix}$$

or, equivalently: $\mathcal{S}_i^{out} = \overline{B}_i(p_i - f_{i,i+1}p_{i+1} - f_{i,L})$ and $\mathcal{S}_i^{in} = f_{i,i+1}\overline{B}_i(p_i - f_{i,i+1}p_{i+1} - f_{i,L})$

3 Solution: existence and uniqueness

3.1 Residual schedule and price impact

First of all, the next Lemma makes sure that the pricing function and the payoffs defined in 2.2 is well-defined: the payoffs are indeed uniquely defined as a function of the slope coefficients. The proof is in the Appendix A.1.

Lemma 3.1. Define the matrices $M := \sum_j \hat{B}_j + \hat{B}_c$ and $M_f := \sum_j \hat{B}_{j,f} f_{j,L}$. Then, the market-clearing conditions (8) are the linear system:

$$M\boldsymbol{p} = \overline{\boldsymbol{A}} \tag{10}$$

where $\overline{\boldsymbol{A}} := \hat{\boldsymbol{A}} + M_f$.

Under our assumptions, $M > \hat{B}_i$ in the positive semidefinite order. In particular, M is invertible, so this system has a unique solution.

As in Section 1, it is convenient to express the optimization in terms of a monopolist optimizing against a (inverse) residual demand and supply. I use the term *residual schedule* to indicate both the demand and supplies: the formal definition is below. This formulation is useful for clarifying the role of multilateral market power, both intuitively and in the generalization of Section 5. Moreover, it is also useful for the generalizations in which firms are allowed to choose general coefficient matrices and general schedules (in the Supplementary Material).

Definition 3.1.
Define the (inverse) residual schedule *as the function:*

$$\boldsymbol{p}_i^r(\boldsymbol{q}_i; \overline{B}_{-i}) = \Lambda_i(\overline{B}_{-i})(\tilde{\boldsymbol{A}}_i(\overline{B}_{-i}) - \boldsymbol{q}_i) \tag{11}$$

where $\Lambda_i(\overline{B}_{-i}) : -\partial_{\boldsymbol{q}_i}\boldsymbol{p}_i^r = [(M - \hat{B}_i)^{-1}]_{\mathcal{N}(i)}$ *and* $\tilde{\boldsymbol{A}}_i$ *is a vector that is also a function of* $\overline{B}_{-i}$.

The coefficient matrix Λ_i is called the *price impact* because it collects the slope coefficients of the (inverse) supply and demand schedules, describing the effect of firm i quantity decision on its input and output prices. It is a measure of market power: the larger the price impact, the larger the surplus that the firm can extract from that buyer or seller.

Lemma 3.2. 1. Λ_i is positive definite, and is decreasing in the positive semi-definite order in each $\overline{B}_j$ for $j \neq i$.

2. Consider the monopoly problem with the inverse demand $\boldsymbol{p}_i^r$:

$$\max_{\boldsymbol{q}_i, \ell_i} \boldsymbol{q}_i' \boldsymbol{p}_i^r(\boldsymbol{q}_i; \overline{B}_{-i}) - \ell_i \tag{12}$$

subject to the technology constraints (5):

$$\boldsymbol{q}_i = f_i(\ell_i)\boldsymbol{v}_i.$$

The slope $\overline{B}_i$ optimizes (9) if and only if the solution $\boldsymbol{q}_i^*$ of (12) is such that the pair $(\boldsymbol{q}_i^*, \boldsymbol{p}_i^r(\boldsymbol{q}_i^*; \overline{B}_{-i}))$ satisfies Equation (14): $\boldsymbol{q}_i^* = \overline{B}_i(\boldsymbol{v}_i'\boldsymbol{p}_i^r(\boldsymbol{q}_i^*; \overline{B}_{-i}) - f_{i,L})$. In other words, the two formulations of the best reply problem are equivalent.

Moreover, the optimal coefficient $\overline{B}_i$ satisfies:

$$\overline{B}_i = \left(\boldsymbol{v}_i' \Lambda_i \boldsymbol{v}_i + \frac{1}{k_i}\right)^{-1} \tag{13}$$

Note that the "monopolist" optimization (12) does not impose directly any structure on the functional form of the price-quantity schedule. Indeed, in principle, this optimization is much more general, allowing for general linear functions as in (14). However, solving it we get that the optimal schedule is linear, and has the expression:

$$q_i^{out} = \mathcal{S}_i^{out}(\boldsymbol{p}_i) = \overline{B}_i \left(\boldsymbol{v}_i' \boldsymbol{p}_i^r - f_{i,L}\right), \tag{14}$$

for the coefficient $\overline{B}_i$ in (13). In particular, this proves that considering only coefficient matrices with the form in (7) is without loss of generality.[8]

3.2 The price impact and the pass-through

The price impact Λ_i is the key object containing the information on multilateral market power: how much each firm can affect input and output prices. The price impact in each market depends on the slope of the residual schedule. This naturally depends on the number of competitors, but not only. Another determinant is the pass-through effect of a price change to indirectly connected markets. This depends on all directly and indirectly connected customers and suppliers, and is the key mechanism that connects the price impact to the network position.

To understand, focus on the vertical economy of Section 1. Firm D is the only firm trading both goods, so the price impact matrix of D is diagonal, and the diagonal elements are simply the inverse of the demand slope and supply slope. Firm U trades only one good, so the price impact is a number:

$$\Lambda_U = \left(B_{c,U} + \left(\frac{1}{B_{c,D}} + \frac{1}{B_D}\right)^{-1}\right)^{-1} \tag{15}$$

$$\Lambda_D = \begin{pmatrix} 1/B_{c,D} & 0 \\ 0 & 1/(B_U + B_{c,U}) \end{pmatrix} \tag{16}$$

The more interesting term is the price impact of U. This is composed of two elements:

[8] It is important to observe that this optimization only constrains the schedule in the optimal point $\boldsymbol{q}_i^*$, $\boldsymbol{p}_i^r(\boldsymbol{q}_i^*)$. So, among general, possibly nonlinear, schedules there are (a continuum of) infinite best replies (and equilibria): this is the insight of Klemperer and Meyer (1989). As specified before, in the Supplementary Material I show that, following Klemperer and Meyer (1989), introducing uncertainty in marginal cost and utility, the linear schedule (14) is the unique surviving best reply. In the main text I restrict the game directly to the linear schedule for ease of exposition.

the one coming from the vertical connection, $\left(B_{c,D}^{-1} + B_D^{-1}\right)^{-1}$ and the consumer demand $B_{c,U}$. It is easier to understand the intuition focusing on the direct demand, which is: $A - B_{c,U}p_U + \overline{B}_D(p_D - p_U)$. The price impact is the inverse of the derivative of this with respect to p_U. However, in equilibrium, p_D is determined by the market-clearing conditions in the downstream market, so it is itself a function of p_U: a change in price passes through to connected markets. So, the effective slope of the demand is not $B_{c,U} + \overline{B}_D$, but it also needs to include the *pass-through* term: $-B_D \dfrac{\partial p_D}{\partial p_U}$, Since the pass-through increases the downstream price, as a result, the demand slope is lower and the price impact higher. The calculation yields the expression in (16). In particular, the harmonic average (or, more correctly, sum) $\left(\dfrac{1}{B_{c,D}} + \dfrac{1}{B_D}\right)^{-1}$ represents the demand from the downstream layer, adjusted for the pass-through effect, and is smaller than $B_{c,D}$. So, the overall ranking of price impacts is ambiguous: the upstream price impact Λ_U can be both larger or smaller than Λ_D, depending on whether the horizontal dimension $B_{c,U}$ dominates the vertical.

In a general supply chain with N layers, with $f_{i,i+1} = 1$, for simplicity, the price impacts are:

$$\Lambda_i = \begin{pmatrix} (\overline{\Lambda}_i^{out})^{-1} + (n_i - 1)\overline{B}_i + B_{c,i} & -(n_i - 1)\overline{B}_i \\ -(n_i - 1)\overline{B}_i & (\overline{\Lambda}_i^{in})^{-1} + (n_i - 1)\overline{B}_i + B_{c,i+1} \end{pmatrix}^{-1} \quad \text{if } i < N$$
$$\Lambda_N = \left((\overline{\Lambda}_N^{out})^{-1} + (n_N - 1)\overline{B}_N + B_{c,N}\right)^{-1}, \tag{17}$$

where the coefficients $\overline{\Lambda}_i^{out}$ and $\overline{\Lambda}_i^{in}$ represent the slope of the "aggregate" residual schedule, and are determined by the recursive relations:

$$\overline{\Lambda}_i^{out} = \left(B_{c,i} + \left(\frac{1}{n_{i-1}\overline{B}_{i-1}} + \overline{\Lambda}_{i-1}^{out}\right)^{-1}\right)^{-1}, \quad \overline{\Lambda}_1^{out} = B_c^{-1}$$
$$\overline{\Lambda}_i^{in} = \frac{1}{n_{i+1}\overline{B}_{i+1} + B_{c,i+1}} + \overline{\Lambda}_{i+1}^{in}, \quad \overline{\Lambda}_N^{in} = 0 \tag{18}$$

These terms measure the vertical dimension connecting layers, with analogous intuition as above. If $n_i > 1$, the slope of the supply of competitors $(n_i - 1)\overline{B}_i$ adds to the horizontal part of the effect.[9]

[9] This has an interesting analogy with the equations describing the electrical resistance: also in that case, when resistances are set in parallel (horizontally related), their total resistance is the sum of individual resistances, whereas when they are in sequence (vertically related), the total resistance is the harmonic sum.

3.3 Existence Theorem

The next Theorem formally proves the existence and uniqueness of the equilibrium (i.e., the fixed point of (13)). The proof is in Appendix A.3.

Theorem 1.
There exists a unique Nash equilibrium of the game $\mathcal{G}$, and is in pure strategies. The game $\mathcal{G}$ is supermodular, so the equilibrium is also the unique rationalizable action profile. The equilibrium coefficients $\overline{B}_1, \ldots, \overline{B}_n \in \mathbb{R}^n_+$ satisfy (13).

Existence follows from the fact that the coefficients $\overline{B}_i$ are bounded above by k_i.[10] Uniqueness follows from considering a modified game $\mathcal{G}'$ where firms choose (logarithm of) the slope coefficients $x_i = \ln \overline{B}_i$. The new game corresponds to a reparameterization of the strategies of the original game, and a monotonic transformation of the payoffs. As such, any Nash equilibrium of the game $\mathcal{G}$ corresponds to one and only one Nash equilibrium of the game $\mathcal{G}'$. The game $\mathcal{G}'$ is a potential game, and the potential is strictly concave: as a consequence, the game has a unique Nash equilibrium.

The key intuition of the fixed-point equation 13 is that the slope depends inversely on the quadratic form $\boldsymbol{v}_i'\Lambda_i\boldsymbol{v}_i$, which is an aggregate index of the strength of the price impact of the firm. For example, in the horizontal economy of Example 1, the price impact is a number, so the aggregation is trivial: $\boldsymbol{v}_i'\Lambda_i\boldsymbol{v}_i = \Lambda_i$. In the vertical economy of Section 1, we have $\boldsymbol{v}'_D\Lambda_D\boldsymbol{v}_D = \Lambda_D^{out} + \Lambda_D^{in}$: the equilibrium slope $\overline{B}_D$ depends inversely on the sum of the price impacts on both the input and the output: multilateral market power appears here.

In general, if the price impact is not diagonal, we have:

$$\boldsymbol{v}_i'\Lambda_i\boldsymbol{v}_i = \Lambda_i^{out} + \boldsymbol{f}_i'\Lambda_i^{in}\boldsymbol{f}_i - 2\boldsymbol{f}_i'\Lambda_i^{out,in}$$

Since $\boldsymbol{v}_i$ has negative elements for the inputs, the links *between* outputs and inputs are negatively weighted (if the off-diagonal elements of Λ_i are positive). This is because, whenever input and output markets are connected, an increase in the output price reflects negatively on the input prices. So, it "subtracts" from the aggregate price impact.

[10] This is the main reason why we introduce the quadratic term in the labor constraint. If we allow $k_i \to \infty$, the equilibrium may still exist, as in the vertical economy example in Section 1, but in general the slopes of the firms may tend to infinity. This happens, in particular, if more than one firm produces the same good. However, from an economic perspective the limit is well defined: those firms simply behave as perfectly competitive. This is not surprising because, as highlighted by Klemperer and Meyer (1989), with constant marginal costs the supply function equilibrium behaves as price competition.

4 Markups, markdowns and the network

This section analyzes what the implications of the model are in terms of firms' market power, particularly in relation to the network of input-output connections.

4.1 Markups and markdowns

Solving the optimization in (12), calling $\boldsymbol{\lambda}_i$ the vector of multipliers of the technology constraints, we get the FOCs:

$$\Lambda_i \boldsymbol{q}_i = \boldsymbol{p}_i - \boldsymbol{\lambda}_i \tag{19}$$

$$1 = \frac{\partial f_i}{\partial \ell_i} \boldsymbol{v}_i' \boldsymbol{\lambda}_i \tag{20}$$

The left-hand side of (19) represents the gap between the price and the shadow value of the output and each input: in the equilibrium, each firm charges *both* a markup on the output and markdowns for each intermediate input. In particular, the value $\mu_i = p_i^{out} - \lambda_i^{out}$ represents the (absolute) *markup*, that is the gap between output price and marginal cost; the other entries represent the (absolute) *markdowns*, the gap between the marginal revenue product of input g and its price.[11] So, the vector of (absolute) markup and markdowns $\boldsymbol{\mu}_i = (\mu_i, -\mu_{ig})$ in equilibrium satisfies:

$$\boldsymbol{\mu}_i = q_i^{out} \Lambda_i \boldsymbol{v}_i, \tag{21}$$

The magnitude of the markup and markdowns depends, not surprisingly, on the price impact: equation (21) is nothing else but the standard equation connecting the slope (or elasticity) of demand to the price charged. Indeed, under perfect competition, we have $\Lambda_i = 0$ and so $\boldsymbol{\mu}_i = \mathbf{0}$. As paragraph 3.2 illustrates, each firm has different price impact in the different markets in which it is involved: the vector $\boldsymbol{\mu}_i$ provides a measure of how much surplus the firm extracts from each market. Once again, the equilibrium slope aggregates these distortions in one index: observe that the aggregate price impact appearing in (13) can also be rewritten as aggregate markup/markdown:

$$\boldsymbol{v}_i' \Lambda_i \boldsymbol{v}_i = (q_i^{out})^{-1} \boldsymbol{v}_i' \boldsymbol{\mu}_i$$

where $\boldsymbol{v}_i' \boldsymbol{\mu}_i = \mu_i + \sum_g f_{i,g} \mu_{i,g}$ aggregates the markup and markdowns: the larger this sum, the smaller the equilibrium slope. How much surplus each firm can extract depends on this aggregation. In the supply chain with layers it is possible to characterize this in detail, and it is the object of the next subsection.

[11] In the Supplementary material, I show that the same values are recovered from the explicit computation of marginal cost and marginal revenue product.

4.2 Equilibrium in the supply chain with layers

Consider the case in which only the final layer sells to consumers: $\mathcal{C} = \{1, \}$, or equivalently $B_{c,i} = 0$ for $i > 1$. This is the most analytically tractable case. In this case, the layer-level slopes in Equation (18) reduce to:

$$\begin{aligned} \overline{\Lambda}_i^{out} &= \frac{1}{B_c} + \sum_{j<i} \frac{1}{n_j \overline{B}_j} \\ \overline{\Lambda}_i^{in} &= \sum_{j>i} \frac{1}{n_j \overline{B}_j} \end{aligned} \tag{22}$$

Using the price impact computed in (17), the first order conditions (13) are:

$$\overline{B}_i = \left(\frac{1}{k_i} + \frac{1}{(n_i - 1)\overline{B}_i + \left(\overline{\Lambda}_i^{in} + \overline{\Lambda}_i^{out} \right)^{-1}} \right)^{-1} \tag{23}$$

One thing we can observe is that as the number of firms in each layer grows, $n_i \to \infty$, the aggregate price impact converges to 0, the perfect competition benchmark. It turns out that, in the homogeneous case of $k_i = k$ and $n_i = n$, the coefficients (22) may be quite different, but their *sum* $\overline{\Lambda}_i^{in} + \overline{\Lambda}_i^{out}$ only depends on the set of slopes $\overline{B}_{-i}$, not on their order. As a consequence, it can be proven that equilibrium slope $\overline{B}_i$ is also homogeneous: this makes it a very tractable case.

The markup-markdown vector is:

$$\boldsymbol{\mu}_i = q_i^{out} \begin{pmatrix} \dfrac{\overline{\Lambda}_i^{out}}{1 + \left(\overline{\Lambda}_i^{in} + \overline{\Lambda}_i^{out} \right) (n_i - 1)\overline{B}_i} \\ -\dfrac{\overline{\Lambda}_i^{in}}{1 + \left(\overline{\Lambda}_i^{in} + \overline{\Lambda}_i^{out} \right) (n_i - 1)\overline{B}_i} \end{pmatrix} \tag{24}$$

In the homogeneous case, $\overline{\Lambda}_i^{in} + \overline{\Lambda}_i^{out}$ is also independent of i. So, only the ranking of $\overline{\Lambda}_i^{out}$ and $\overline{\Lambda}_i^{in}$ matters for the markup and markdown: it immediately follows that markups are increasing upstream, and markdowns are increasing downstream. This remains true even if we study the *relative* markups and markdowns, normalizing by the relative price. The next Proposition makes this formal.

Proposition 1.
Consider a layered supply chain where only layer 1 sells to the consumer, $B_{c,i} = 0$ for $i > 1$. Suppose $N \geq 2$ and $n_i \geq 2$, so the equilibrium slope is nonzero.

If $k_i = k$, then:

1. *If $n_i = n \geq 2$ for all layers i, the markups are larger the more upstream the layer*

is, while markdowns are larger the more downstream a layer is;

2. *If $n_i \geq n_j$ then the aggregate profit of firms in layer j is larger than the aggregate profit of firms in layer i.*

Point 1 characterizes how market power connects to network position, in this simple linear network. The mechanism is the pass-through mechanism illustrated in paragraph 3.2: more upstream firms face a smaller demand elasticity, so are more able to increase the markup. So, the ability to extract surplus is ambiguous: upstream firms can extract more surplus when selling, while downstream firms can extract more surplus when buying. What is the balance? Part 2 answers. If n_i is constant across layers, as in Figure 3a, the situation is completely symmetric, and so the increase in markups and decrease in markdowns exactly offset each other, and the firms all have the same profits. Hence, each layer extracts the same surplus. The proof is in Appendix B.1. Otherwise, if some layer is more competitive (n_i is larger), as in as in Figure 3b, the corresponding firms have lower profits. Not only, but the layer with less firms as a whole also extracts less aggregate surplus than other layers. For example, if in the asymmetric example of Figure 3b $n_2 \to \infty$, the upstream layer behave as price takers, and the only firm in the downstream layer behaves as a monopolist both on the output and the input market.

4.3 The goods network

What can we say about the relation between the network position and market power in general? We summarize the discussion in the following remarks.

Definition 4.1.
Define the goods network $\mathcal{GN}$ *as the network whose nodes are the goods $\mathcal{N}$, and whose adjacency matrix G has elements:*

$$G_{gh} = \frac{-M_{g,h}}{\sum_j \overline{B}_j + B_{c,gh}}$$

where M is the matrix appearing in the market-clearing conditions (10).

The goods network excluding firm i has the same set of nodes but adjacency matrix G_i with elements:

$$G_{i,gh} = \frac{-[M - \hat{B}_i]_{g,h}}{\sum_j \overline{B}_j + B_{c,gh}}$$

A link is present when the price of h directly affects the quantities traded (to be precise, the excess supply) of g. The denominator is a normalization, measuring the effect of the price of g on the own excess supply. Note that the weights of the links are endogenous and determined in equilibrium by the slopes of the schedules chosen.

To better understand the interpretation of the link weights, let us consider the special case in which $\overline{B}_i = 1$ for all i, and the consumer demand satisfies $B_c = I$ (the identity).[12] In this case, the entries of the matrix M are:

$$\begin{aligned} M_{gg} &= |\text{firms trading } g, \text{except } i+1| \\ M_{gh} &= -|\text{firms selling } h, \text{buying } g| - |\text{firms selling } g, \text{buying } h| \\ &\quad + |\text{firms buying both } g, h|, \end{aligned}$$

so, the weight of the link between g and h is high when, among the firms trading h, many transform g and h or vice-versa, but not too many use both as inputs. This highlights that the network effect is strong when h and g have a vertical (input-output) connection: in such a case, when the price of one goes up the other tends to increase too. Instead, a horizontal connection decreases the effect. This is because goods are perfect complements, so an increase in the quantity of one triggers a decrease in the price of the other.

(a) If $n_1 = 2$ (as in the case of Figure 3a). (b) If $n_1 = 1$ (as in the case of Figure 3b).

Figure 4: Goods network in the supply chain with 3 layers.

Example 4 (Supply chain with 3 layers)**.**

The next Theorem characterizes the relation between prices, markups, and the network.

Theorem 2.

Define $D \in \mathbb{R}^{m \times m}$ the diagonal matrix having the same diagonal as M, and D_i the diagonal matrix having the same diagonal as $M - \hat{B}_i$. In the SDF equilibrium:

1. *the prices are proportional (through D) to Bonacich centrality in the goods network:*

$$\boldsymbol{p} = D^{-1}(I - G)^{-1}\overline{\boldsymbol{A}} \tag{25}$$

[12] This in general is not the equilibrium, but it is always possible to find a configuration of k_i such that this is exactly the equilibrium.

2. *the markup and markdowns of each firm i are proportional (through D) to the Bonacich centrality of the respective output and inputs, in the good network excluding firm i:*

$$\boldsymbol{\mu}_i = q_i^{out}[D_i^{-1}(I - G_i)^{-1}\hat{\boldsymbol{v}}_i]_{\mathcal{N}(i)} \tag{26}$$

Prices are high when a good has many direct or indirect connections with goods in high demand $\boldsymbol{A}$, or high cost $\boldsymbol{f}_L$. Equation (25) is also valid in perfect competition (with different equilibrium slopes and so network weights). However, the direct and indirect connections also determine the magnitude of market power distortions through Equation (26).

The centralities summarize the interaction of horizontal and vertical connections described in paragraph 3.2. To understand better Equation (26), notice that $(I - G_i)^{-1}\hat{\boldsymbol{v}}_i$ is the standard vector of Bonacich centralities of each good with respect to adjacency matrix G_i. We only look at centralities of goods in $\mathcal{N}(i)$, so the expssion selects that subvector. The entry g, h of $(I - G_i)^{-1}$ counts the weighted number of direct and indirect connections between g and h, while $\hat{\boldsymbol{v}}_i$ is nonzero only for inputs and output of i. So, what matters in that expression is the only the weighted number of direct and indirect connections with start and endpoint in $\mathcal{N}(i)$, but intermediate steps anywhere in the good network. The diagonal matrix D_i is a normalization.

The expression simplifies a lot if when removing firm i, the goods network becomes disconnected. This is the case, for example, in the supply chain with layers when $n_1 = 1$ in Figure 4b. In this case, the matrix $(I - G_i)^{-1}$ is diagonal, and the diagonal entries count the number of cycles centered in the output good i in the output subnetwork, and the number of cycles of the input $i+1$ in the input subnetwork. In the case of Figure 4b, the centrality is $(I - G_i)^{-1}\hat{\boldsymbol{v}}_i = (1, -(1 - G_{12})^{-1})$. The output subnetwork is composed only of good 1, so the number of cycles is 1. The number of cycles out of 2 in the input subnetwork is $(1 - G_{1,23})^{-1}$, where the weight $G_{1,23}$ is equal to: $n_2\overline{B}_2/(n_2\overline{B}_2 + n_3\overline{B}_3)$. The number of cycles in the input subnetwork is larger than 1, measuring the fact that with a vertical dimension, the price impact is stronger because of the presence of the pass-through effect. In general, we can see the number of cycles is a measure of size of the subnetwork, and the higher the weights, the higher the measure. This number is the general network statistics measuring the strength of the pass-through effect, described in 3.2.

In a more general case, as in Figure 4a the goods network may not become disconnected when removing a firm: this is the case of considering the goods network excluding a firm in layer 1, when $n_1 > 1$. In this case, the connection between good 1 and 2 remains, it simply is weaker than the other (but it depends on equilibrium parameters). In this case, the markup is not only counting the number of cycles in the output good, but also the number of indirect links between the output and input. These are negatively weighted and subtract from the markup, because a decrease in output quantity triggers a price increase, which means the competitor sells and buys more, increasing the input price: this puts a limit to the firm ability to increase the price.

5 The role of multilateral market power

5.1 General price impacts

The key feature of the model studied so far is that firms have multilateral market power: they can affect prices in all the markets they are involved in. What are the implications of multilateral market power? To answer this question, in this section, I introduce a model that simultaneously generalizes the supply and demand function competition and various other classic models of oligopolistic competition, with and without networks.

Definition 5.1.
Consider a profile of functions $\Lambda = (\Lambda_1, \ldots, \Lambda_n)$, where:

$$\Lambda_i : \overline{B}_{-i} \to \Lambda_i(\overline{B}_{-i}) \in \mathbb{R}^{d_i \times d_i}$$

such that, for all i and $\overline{B}_{-i}$, $\Lambda_i(\overline{B}_{-i})$ is positive semidefinite, and the function Λ_i is continuous and decreasing in the positive semidefinite ordering in each $\overline{B}_j$ with $j \neq i$.

A Generalized SDFE a profile of slopes $\overline{B}$ such that for each firm i:

1. *$\overline{B}_i$ satisfies Equation (13) were Λ_i is the above price impact function;*
2. *equilibrium prices and quantities are determined by the market-clearing conditions (10).*

In other words, a Generalized SDFE is a model in which firms choose their slopes optimizing against a modified residual schedule, that satisfies: $\partial_{\boldsymbol{q}_i} \boldsymbol{p}_i^r(\boldsymbol{q}_i, B_{-i}) = -\Lambda_i(B_{-i})$, where Λ_i is now a primitive. The Supply and Demand function competition of the previous sections is a Generalized SDFE, because Λ_i derived in Lemma 3.2 is continuous and decreasing. Many other standard models are also special cases. For example, Walrasian equilibrium is the special case where $\Lambda_i = 0$ for each i. Cournot oligopoly is also a special case.

Example 5. Cournot

In the SDF equilibrium in the horizontal economy of 1 the price impact (18) with $N = 1$ becomes:

$$\Lambda_i = \left(B_c + \sum_{j \neq i} \overline{B}_j \right)^{-1} \tag{27}$$

Consider the Generalized SDFE with:

$$\Lambda_i(\overline{B}_{-i}) = \frac{1}{B_c} \tag{28}$$

This yields the FOC:

$$p^{out} - \frac{1}{B_c}q_i^{out} - \frac{1}{k_i}q_i^{out} = 0$$

The own slope $\overline{B}_i$ disappears, and this is exactly the FOC of Cournot competition. This is not surprising, because the price impact function (28) is exactly the one obtained from (27) by assuming that other firms choose schedules with zero slope $\overline{B}_j = 0$: but a schedule with zero slope is a quantity commitment, as in Cournot competition.

What is perhaps more striking is that some *sequential* models are also special cases of the Generalized SDFE, as shown in Remark 5.2.

The next Theorem proves the existence of a Generalized SDFE, and the fundamental comparative statics result on the price impacts.

Theorem 3. *1. A Generalized SDFE exists. Moreover, it is a game of strategic complements, and as such it always has a maximal and a minimal equilibrium (possibly identical).*

2. Consider two profiles of price impact functions Λ^1 and Λ^2 such that, for each profile $\overline{B}$, we have $\Lambda_i^1(\overline{B}_{-i}) \geq \Lambda_i^2(\overline{B}_{-i})$ for all firms i (in the p.s.d. ordering). Then, in the maximal and the minimal equilibrium, the slope coefficients are lower in the first case: $\overline{B}_i^1 \leq \overline{B}_i^2$ (in the p.s.d. ordering) for each firm i.

The key idea for the proof of both part 1) and 2) comes once again by strategic complementarity: a lower price impact means higher slopes that, in turn, trigger higher best responses, and equilibrium slopes. The proof is in Appendix C.1.

The economic content of the Theorem is in point 2, essentially stating that distortions are lower when the price impact is lower. This shows that countervailing market power actually harms consumers. Out of equilibrium, it is true that buyer power makes the seller less willing to increase the price, but this does not consider the fact that the increase in price is passed through to consumers. As a result, in equilibrium, high buyer power results in a higher final price. Technically, the result refers to equilibrium slopes, because it is the result available in higher generality. Adding more structure, assuming that there is only one final good consumed by consumers, it is possible to show more precise implications on the final price, in the next Corollary.

Corollary 5.1. Suppose for simplicity that $\boldsymbol{f}_L = 0$. Consider two profiles of price impact functions Λ^1 and Λ^2 ordered as in Theorem 3, part 2. Suppose that the consumer only consumes one good, say good 0: $\mathcal{C} = \{0\}$, then the price of the final good p_0 is smaller in the model with smaller price impacts Λ^2.

5.2 Comparison with unilateral and local market power

As discussed in the Literature section, many papers in the production network literature assume as a simplification that input prices are taken as given, and that prices in other markets are taken as given. Let us first define these two assumptions.

Definition 5.2. *1. The SDFE with* unilateral market power *is the model in which firms optimize* (12) *with the additional constraint that they take input prices as given, or the output price as given. For consistency, in the former case, firms also neglect the direct effect of input quantities on output price and, in the latter, they neglect the direct effect of output quantity on input prices.*

2. The SDFE with local market power *is the model in which firms optimize* (12) *with the additional constraint that they take as given the prices of all goods that are not their output or inputs.*

Part 1 deserves a clarification: in the price impact derived in Proposition (3.2) we have two connected features: input prices change as a result of a quantity change, but also the *output* price (potentially) changes as a result to *both* input and output quantity changes, as in the supply chain with layers in Equation (17). This is due to the fact that a change in input demanded changes the equilibrium prices for other firms that also may be connected with the output market: this is the case for example for direct competitors belonging to the same layer in the supply chain. Here different choices are possible, but it seems very artificial to assume that a firm ignores that input prices are affected by the quantity demanded, and at the same time realizes that the quantity demanded has indirect network effects that affect the output price. So, I assume that this is not the case.[13]

The next Theorem is the main result of the Section.

Theorem 4.
The models of Definition 5.2 are special cases of the Generalized SDFE, for different choices of the price impact functions Λ_i:

1. ***Unilateral market power**: for all firms i using intermediate inputs:*

$$\Lambda_i^{unilateral} = \begin{pmatrix} (M - \hat{B}_i)^{-1}_{ii} & \mathbf{0}' \\ \mathbf{0} & O \end{pmatrix}$$

[13]Crucially, the effect is *not* due to the technology constraint of the firm, because the price impact only depends on the residual schedule. So, the effect on the output price we are analyzing does *not* represent considerations of the type "if I buy more, I sell more", which are present in the optimization through the technology constraint, but not in the price impact.

2. ***Local market power:***

$$\Lambda_i^{local}(B_{-i}) = (M_{\mathcal{N}(i)} - B_i)^{-1}$$

Moreover, in both cases, the price impacts are smaller than with multilateral market power: for all $i \in \mathcal{N}$ and all profiles B we have:

$$\Lambda_i^{local}(B_{-i}) \leq \Lambda_i^{multilateral}(B_{-i})$$
$$\Lambda_i^{unilateral}(B_{-i}) \leq \Lambda_i^{multilateral}(B_{-i})$$

In words, the Theorem shows that assuming price-taking on some markets means to use a GSDFE with price-impact function that is lower in the positive semi-definite order. With one input market that is disconnected from the output market, as in Figure 4b, the result would be immediate: it means that the price impact on the input decreases from a positive value to zero. The Theorem shows that this remains true for the relevant matrix order also when we consider multiple inputs.

Using Corollary 5.1, we can also conclude:

Corollary 5.2. For any network such that there is only one final good, with unilateral or local market power, the final price is smaller than in the benchmark SDFE with multilateral market power.

Example 6. Unilateral market power in the vertical economy Consider the setting of the vertical economy of Section 1; and suppose we want to compute the Generalized SDFE with unilateral market power. According to Theorem 4, we simply have to modify the equilibrium equations by changing the price impact of Equation (16) to:

$$\Lambda_D^{unilateral} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

while Λ_U, having no intermediate inputs, is unaffected. Since $\Lambda_D^{\text{unilateral}} \leq \Lambda_D^{\text{multilateral}}$ in the p.s.d order, Theorem 3 implies that in equilibrium the slopes are higher in the unilateral model, consistently with the direct computation in Section 1.

Example 7. Unilateral market power in the layered supply chain

With unilateral market power, using Theorem (4), we get either:

$$\Lambda_i^{unilateral} = \begin{pmatrix} \left((\overline{\Lambda}_i^{out})^{-1} + (n_i - 1)\overline{B}_i\right)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \tag{29}$$

or, if the unilateral market power is on inputs rather than outputs:

$$\Lambda_i^{unilateral} = \begin{pmatrix} 0 & 0 \\ 0 & \left((\overline{\Lambda}_i^{in})^{-1} + (n_i - 1)\overline{B}_i\right)^{-1} \end{pmatrix} \tag{30}$$

In this case, we can characterize precisely markups and profits in equilibrium.

Proposition 2. *1. If firms take the* input *price as given (the price impact is as in* (29)*), markups are still increasing going* upstream*, while there are no markdowns: as a consequence, profits are increasing upstream.*

2. If firms take the output *price as given (the price impact is as in* (30)*), then markdowns are increasing going* downstream*, while there are no markups: as a consequence, profits are increasing downstream.*

The proof is in Appendix C.4. The key mechanisms remain those of paragraph 4.2. However, here the symmetry is broken, because firms consider the effect of network position on the elasticity of demand only on the output side (in case 1) or the input side (in case 2). As a result, we get the asymmetric outcomes described above. Note that the setting is identical: the only thing that changes is the type of competition model.

Remark 5.1 (**The general network interpretation**)**.** In terms of the network interpretation of Section 4, unilateral market power can be understood as the situation where only the subnetwork containing the output good is considered. In Figure 4b, this amounts to canceling the subnetwork containing the input good (the subnetwork containing 2 and 3). Local market power amounts to the situation in which only the subnetwork containing direct inputs and the output is considered.

Remark 5.2 (**Sequential Monopoly is a Generalized SDFE**)**.** The most standard way to model price setting in the context of the vertical economy is perhaps the Sequential Monopoly à la Spengler (1950), that is a sequential game where firms set output prices sequentially, starting upstream with U and then D, and D (by construction) takes the input price p_D as given. It turns out that the first-order conditions of the Sequential Monopoly thus defined imply *exactly* the same equilibrium price and quantity of the Generalized SDFE with unilateral market power. To understand why, let's write the first-order conditions of the sequential model. Since it is a monopoly, setting quantities or prices is exactly equivalent. By backward induction, start from firm D. The inverse demand is as in Section 1: $p_D = A - q_D$. Maximizing the profit of D while taking p_U as given produces the FOCs for the downstream firm:

$$p_D - p_U + \frac{\partial p_D}{\partial q_D} q_D = 0 \tag{31}$$

Notice that this is precisely the same as Equation (2b) when $\frac{\partial p_U}{\partial q_D} = 0$. That Equation represents the FOC of the Generalized SDFE with unilateral market power. The mechanism is the same: since the firm does not internalize the price impact on the input, that term disappears from the FOC.

To understand why the FOC for the upstream firm mimics the SDFE is a bit more subtle. In the sequential monopoly model, the (inverse) demand for firm U, in equilibrium, is given by the equilibrium choice of firm D as a function of p_U. But this means exactly to use equation (31) and the consumer demands, to back up p_U^r: this is exactly the same as solving the market-clearing conditions for a given choice of schedule of firm D. So, we get that the equilibrium demand for firm U is exactly the same in the Sequential Competition, and in the Generalized SDFE with unilateral market power!

In the Supplementary Appendix, I prove that the analogy does not stop at the Sequential Monopoly but it also extends to Sequential Cournot, which falls under the case of unilateral market power, with the additional "Cournot" assumption that firms consider flat the schedules chosen by direct competitors.

5.3 Applications

We conclude the section showing that multilateral market power has implications for the evaluation of the welfare impact of mergers and the diffusion of shocks.

5.3.1 Welfare impact of vertical mergers

In this paragraph, we analyze a vertical merger in the supply chain illustrated in Figure 5. In particular, suppose that there is a merger between the upstream firm and one of the downstream firms.

Proposition 3.
Suppose that $n_2 = 1$. Suppose that the merged firm does not sell its intermediate good to others, but it keeps it all to produce the final output, so that the economy becomes a monopoly. There is an interval (n_*, n^*) such that if $n_1 \in (n_*, n^*)$ the merger is welfare-increasing with multilateral market power, but welfare-decreasing with unilateral market power.

In this setting, a vertical merger features the standard trade-off between decreasing double marginalization and decreasing competition through foreclosure. The former effect dominates for small n_1, while the latter effect dominates for large n_1. If market power is multilateral, the vertical distortion is very strong, and so decreasing double marginalization is more important than in the unilateral case. As a result, for some n, the merger may be welfare-decreasing when evaluated with unilateral market power, and

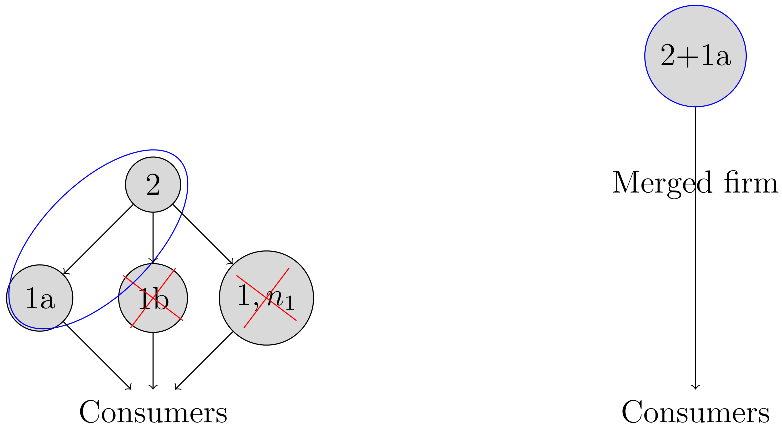


Figure 5: Left: pre-merger economy. The blue circle indicated the merging firms 2 and 1a. Right: the economy after the merger: 1b and 1c are driven out of the market because the merged firm does not sell them the necessary input anymore, and the merged firm becomes a monopolist.

welfare-improving under multilateral market power. The formal proof is in Appendix C.5.

5.3.2 Diffusion of productivity shocks

Consider a "productivity shock" in the form of a change in the cost parameter k_i. A standard question in the production network literature has been to characterize the impact and implications of the diffusion of productivity shocks throughout the network (Acemoglu and Tahbaz-Salehi, 2025; Grassi, 2017; Bizzarri, 2024). In this paragraph, I show how multilateral market power can qualitatively affect the predictions on the diffusion of productivity shocks.

Proposition 4.
Consider a layered supply chain with $N = 2$, $n_i = n$, and $k_i = k$. Call Q the equilibrium quantity of output.

In the standard SDFE, $\frac{dQ}{dk_1}|_{k_1=k} = \frac{dQ}{dk_2}|_{k_2=k}$.

In the unilateral model of Definition (5.2), the effect of a shock downstream is stronger: $\frac{dQ^{unilateral}}{dk_1}|_{k_1=k} > \frac{dQ^{unilateral}}{dk_2}|_{k_2=k}$.

The proof is a direct calculation using the equilibrium conditions and the implicit function theorem, and is in the Supplementary Material. An increase in the cost coefficient k_i reduces the price impact of the targeted firm, because the cost is smaller and the firm reduces the markup. The reason for the result once again depends on the fact that, in the unilateral model, the output price impact is the only relevant one, and it only depends on the downstream firms. As a consequence, a change in k_2 does not affect layer 1 firms' best reply. Instead, a change in k_1 reduces also the price impact of the firms in the upstream layer, and so it has a larger effect on the equilibrium quantity (and welfare). Instead, in the standard SDFE, both changes affect the price impact of the other firm, and so the effect is symmetric.

6 Conclusion

This paper provides a way to model oligopoly in general equilibrium as a game in which firms fully internalize their position in the supply chain and have market power both over inputs and outputs, in an endogenously determined way. I show that such features are desirable in an input-output model with market power: if absent, both the aggregate and the relative ranking of distortions due to imperfect competition is crucially affected. This suggests that, when modeling complex networks of large firms with market power, simplifying assumptions might affect the results in a sizable way.

Appendix

A Proofs of Section 3

A.1 Proof of Lemma 3.1

We prove that it is always possible to invert the market clearing conditions (10).

Consider the quadratic form $\boldsymbol{x}'M\boldsymbol{x}$. This is equal to:

$$\boldsymbol{x}'M\boldsymbol{x} = \sum_i \boldsymbol{x}'\hat{B}_i\boldsymbol{x} + \boldsymbol{x}'\hat{B}_c\boldsymbol{x}.$$

Restrict attention to a subset of m firms, chosen such that each firm produces a distinct good: for each good g, denote i_g the firm producing g that is chosen. Define $\overline{F}$ the square matrix with elements $f_{i_g h}$. By the assumption of viability, $I - \overline{F}'$ is an M-matrix, and in particular is invertible (Horn et al., 1994). Moreover, the columns of $I - \overline{F}'$ are the $\hat{\boldsymbol{v}}_{i_g}$ vectors: so, they must be linearly independent. So, there are at least m linearly independent $\hat{\boldsymbol{v}}_i$ vectors.

We want to prove that $\boldsymbol{x}'\left(M - \hat{B}_i\right)\boldsymbol{x} = 0$ implies $\boldsymbol{x} = 0$: so $M - \hat{B}_i$ is positive definite, hence invertible. Now, $\boldsymbol{x}'\left(M - \hat{B}_i\right)\boldsymbol{x} = \boldsymbol{x}_c'B_c\boldsymbol{x}_c + \sum_{j\neq i}\overline{B}_j\boldsymbol{x}'\hat{\boldsymbol{v}}_j\hat{\boldsymbol{v}}_j'\boldsymbol{x}$. For this to be zero, it must either be $\boldsymbol{x} = 0$, or $\boldsymbol{x}$ orthogonal to all $\hat{\boldsymbol{v}}_j$ for $j \neq i$. Since the $\hat{\boldsymbol{v}}_j$ vectors include a basis, this means that $\hat{\boldsymbol{x}}$ is orthogonal to all $\hat{\boldsymbol{v}}_j$ for all $j \neq i$. If the $\hat{\boldsymbol{v}}_j$ for all $j \neq i$ contain a basis, then $\boldsymbol{x} = 0$. If not, by the above reasoning we know that they must at least span the $m-1$ dimensional space. This means that there is just one subspace orthogonal to all of them. So, $\boldsymbol{x}$ by construction must be parallel to the

$i-$th column of the inverse matrix $(I - \overline{F}')^{-1}$. Since $\overline{F}$ is an M-matrix, $\boldsymbol{x}$ must either be nonnegative or nonpositive. Now, if $\hat{\boldsymbol{v}}_i$ is linearly independent from some other vector, it must be that the network is nontrivial, so there is at least one firm, say h, that has an intermediate input. Without loss of generality, consider one such a firm that sells to the consumer. Suppose $\hat{\boldsymbol{v}}_h$ is linearly independent from $\hat{\boldsymbol{v}}_i$. Since $\boldsymbol{x}_c = 0$, it must be that the entry relative to good h is $x_{c,h} = 0$. So, $\boldsymbol{x}'\hat{\boldsymbol{v}}_h = -\sum_{g \in \mathcal{N}^{in}(i)} f_{h,g} x_{c,g} = 0$, but since all $x_{c,g}$ must have the same sign, this implies $x_{c,g} = 0$ for all $g \in \mathcal{N}^{in}(i)$. Iterating the reasoning, we find that all other $x_{c,g}$ must be zero for all goods g connected to the consumer, which implies $\boldsymbol{x} = 0$. We conclude that $M - \hat{B}_i$ is positive definite, which is what we wanted to show.

□

A.2 Proof of Lemma 3.2

Part 1 By Lemma 3.1, $M - \hat{B}_i$ is positive definite, so the inverse exists and is positive-definite. Then, Λ_i is positive definite because it is a principal submatrix of a positive-definite matrix.

Concerning the monotonicity, $\hat{B}_j = \overline{B}_j \hat{\boldsymbol{v}}_j \hat{\boldsymbol{v}}_j'$, so $\hat{B}_j$ is increasing in the coefficient $\overline{B}_j$. Moreover, $(M - \hat{B}_i)^{-1} = (\sum_{j \neq i} \hat{B}_j + \hat{B}_c)^{-1}$ is decreasing in each $\hat{B}_j$, and passing to a principal submatrix preserves the positive semidefinite ordering, so Λ_i is decreasing in each $\hat{B}_j$.

Part 2 The system (10) can equivalently be rewritten as:

$$\hat{\boldsymbol{q}}_i + (M - \hat{B}_i)\boldsymbol{p} = \overline{\boldsymbol{A}}, \tag{32}$$

$$\boldsymbol{q}_i = B_i \boldsymbol{p}_i - B_{i,f} \tag{33}$$

where $\overline{\boldsymbol{A}} := \hat{\boldsymbol{A}} + M_f$. By construction, we can rewrite the optimization of (9) by writing the market clearing and the linear schedule (14) as constraints:

$$\max_{\boldsymbol{q}_i, \boldsymbol{p}_i} \boldsymbol{q}_i' \boldsymbol{p}_i - \ell_i \tag{34}$$

subject to:

$$\boldsymbol{q}_i = \overline{B}_i(\boldsymbol{v}_i \boldsymbol{p}_i - f_{i,L}) \tag{35}$$

$$\hat{\boldsymbol{q}}_i + (M - \hat{B}_i)\boldsymbol{p} = \overline{\boldsymbol{A}} \tag{36}$$

$$\boldsymbol{q}_i = f_i(\ell_i)\boldsymbol{v}_i \tag{37}$$

because Equations (36) and (35) imply the pricing function in 3.1, and so the functional form in (9).

We can further simplify the problem inverting the market clearing constraint (36) to define the residual demand $\boldsymbol{p}_i^r$. From Lemma 3.1, $M - \hat{B}_i$ is positive definite, so we can invert it and write:

$$\boldsymbol{p}^r(\boldsymbol{q}_i) = (M - \hat{B}_i)^{-1}(\overline{\boldsymbol{A}} - \hat{\boldsymbol{q}}_i)$$

Reordering the equations so to have all the rows associated with inputs and ouputs of i first, we can write the matrix M in blocks as:

$$M - \hat{B}_i = \begin{pmatrix} M_{\mathcal{N}(i)} - B_i & M_{\mathcal{N}(i),\mathcal{N}(i)^c} \\ M'_{\mathcal{N}(i),\mathcal{N}(i)^c} & M_{\mathcal{N}(i)^c} \end{pmatrix}$$

Using the rule for block matrix inversion:

$$\boldsymbol{p}_i^r = [(M - \hat{B}_i)^{-1}]_{\mathcal{N}(i)} \left(\overline{\boldsymbol{A}}_{\mathcal{N}(i)} - \boldsymbol{q}_i - M_{\mathcal{N}(i),\mathcal{N}(i)^c}(M_{\mathcal{N}(i)^c})^{-1}\overline{\boldsymbol{A}}_{\mathcal{N}(i)^c}\right)$$

Defining $\Lambda_i := [(M - \hat{B}_i)^{-1}]_{\mathcal{N}(i)}$ and $\tilde{\boldsymbol{A}}_i := \overline{\boldsymbol{A}}_{\mathcal{N}(i)} - M_{\mathcal{N}(i),\mathcal{N}(i)^c}(M_{\mathcal{N}(i)^c})^{-1}\overline{\boldsymbol{A}}_{\mathcal{N}(i)^c}$, we obtain the equivalent formulation:

$$\max_{\boldsymbol{q}_i} \boldsymbol{q}_i'\boldsymbol{p}_i^r(\boldsymbol{q}_i) - \ell_i \tag{38}$$

$$\text{subject to:} \tag{39}$$

$$\boldsymbol{q}_i = \overline{B}_i(\boldsymbol{v}_i\boldsymbol{p}_i - f_{i,L}) \tag{40}$$

$$\boldsymbol{q}_i = f_i(\ell_i)\boldsymbol{v}_i \tag{41}$$

This differs from the problem (12) in the text of the Lemma only for the presence of the schedule constraint (40). To show that it is redundant, we solve the relaxed problem (12) and we show that the solution satisfies the constraint.

The optimization 12 is concave with a linear constraint, because the Hessian matrix is $-\begin{pmatrix} \Lambda_i & 0 \\ 0 & 1 \end{pmatrix}$, so the FOCs (19) are sufficient. To solve the system of FOCs (19) note that:

$$\frac{\partial f_i}{\partial \ell_i} = \frac{k_i}{\sqrt{2\ell_i k_i + f_{i,L}^2 k_i^2}} = \frac{k_i}{f_{i,L}k_i + f_i} = \left(f_{i,L} + \frac{1}{k_i}q_i^{out}\right)^{-1}$$

Then, from the FOC and the constraint $\boldsymbol{q}_i = q_i^{out}\boldsymbol{v}_i'$ we get:

$$\boldsymbol{v}_i'\Lambda_i\boldsymbol{v}_i q_i^{out} = (\boldsymbol{v}_i'\boldsymbol{p}_i - \boldsymbol{v}_i'\boldsymbol{\lambda}_i) \tag{42}$$

$$= \left(\boldsymbol{v}_i'\boldsymbol{p}_i - \left(\frac{\partial f_i}{\partial \ell_i}\right)^{-1}\right) \tag{43}$$

$$= \left(\boldsymbol{v}_i'\boldsymbol{p}_i - f_{i,L} - q_i^{out}k_i^{-1}\right) \tag{44}$$

which implies Equation (14) and (13). □

A.3 Proof of Theorem 1

Now, consider the game $\mathcal{G}' = (\mathcal{N}, (X_i, U_i)_{i\in\mathcal{N}})$ with payoffs $U_i = \ln \pi_i(e^{x_1}, \dots, e^{x_n})$, where $X_i = \mathbb{R}$. Since both $\ln(\cdot)$ and $exp(\cdot)$ are monotone, a profile $\overline{B}$ is a pure Nash equilibrium of $\mathcal{G}$ if and only if the profile $x = (\ln \overline{B}_1, \dots, \ln \overline{B}_n)$ is a pure Nash equilibrium of $\mathcal{G}'$. It follows that the Nash equilibrium of $\mathcal{G}'$ is unique if and only if the Nash equilibrium of $\mathcal{G}$ is unique. Moreover, since the log is order-preserving, the game $\mathcal{G}$ is also supermodular. In the following, we analyze the game $\mathcal{G}'$.

Existence The best reply equation (13) shows that $\overline{B}_i \in [0, k_i]$ and is continuous. So, by Brouwer's fixed-point theorem, there exists an equilibrium.

Potential To show that the modified game $\mathcal{G}'$ is a potential game, we show that the second cross-derivatives of the payoffs are equal. Since the supply is $\mathcal{S}_i = \overline{B}_i \boldsymbol{v}_i \boldsymbol{v}'_i \boldsymbol{p}_i$, the profit can be rewritten as:To compute the derivative of the payoffs, we must differentiate the matrix M^{-1}:

$$\frac{\partial}{\partial \overline{B}_i} M^{-1} = -M^{-1} \frac{\partial}{\partial \overline{B}_i} \left(\sum_j \overline{B}_j \hat{\boldsymbol{v}}_j \hat{\boldsymbol{v}}'_j + \hat{B}_c \right) M^{-1} = -M^{-1} \hat{\boldsymbol{v}}_i \hat{\boldsymbol{v}}'_i M^{-1}$$

Since $M_f = \sum_j \hat{B}_{j,f} f_{j,L}$ and $B_{j,f} = \overline{B}_j \boldsymbol{v}_j$, we have: $M_f = \sum_j \overline{B}_j \hat{\boldsymbol{v}}_j f_{j,L} = \sum_j \overline{B}_j \hat{\boldsymbol{v}}_j \hat{\boldsymbol{e}}_j \boldsymbol{f}_L$, where $\hat{\boldsymbol{e}}_j$ is the j-th canonical basis vector in $\mathbb{R}^n$. So:

$$\frac{\partial}{\partial \overline{B}_i} M_f = \frac{\partial}{\partial \overline{B}_i} \sum \overline{B}_j \hat{\boldsymbol{v}}_j \hat{\boldsymbol{e}}'_j \boldsymbol{f}_L = \hat{\boldsymbol{v}}_i \hat{\boldsymbol{e}}'_i \boldsymbol{f}_L$$

Moreover, $\hat{\boldsymbol{v}}'_i \boldsymbol{p} = \hat{\boldsymbol{v}}'_i M^{-1}(\boldsymbol{A} + M_f)$. So:

$$\begin{aligned} \frac{\partial}{\partial \overline{B}_i} \hat{\boldsymbol{v}}'_j \boldsymbol{p} &= -\hat{\boldsymbol{v}}'_j M^{-1} \hat{\boldsymbol{v}}_i \hat{\boldsymbol{v}}'_i M^{-1}(\boldsymbol{A} + M_f) + \hat{\boldsymbol{v}}'_j M^{-1} \hat{\boldsymbol{v}}_i \hat{\boldsymbol{e}}'_i \boldsymbol{f} \\ &= -\hat{\boldsymbol{v}}_j M^{-1} \hat{\boldsymbol{v}}_i (\hat{\boldsymbol{v}}'_i \boldsymbol{p} - f_{i,L}) \qquad (45) \end{aligned}$$

Using this, the derivative of the profit is:

$$\frac{\partial \pi_i}{\partial \overline{B}_i} = \overline{B}_i \left(1 - \frac{1}{2k_i} \overline{B}_i\right) \left(\hat{\boldsymbol{v}}'_i \boldsymbol{p} - f_{i,L}\right)^2 \left(\frac{1 - \overline{B}_i / k_i}{\overline{B}_i \left(1 - \frac{1}{2k_i} \overline{B}_i\right)} - 2\hat{\boldsymbol{v}}'_i M^{-1} \hat{\boldsymbol{v}}_i \right)$$

Notice that with our reparameterization $\overline{B}_i = e^{x_i}$ and $\frac{\partial U_i}{\partial x_i} = \frac{\partial \ln \pi_i}{\partial \ln \overline{B}_i}$. Since $\pi_i =$

$\overline{B}\left(1-\frac{1}{2k_i}\overline{B}_i\right)(\hat{\boldsymbol{v}}_i\boldsymbol{p}-f_{i,L})^2$, the derivative of U_i becomes:

$$\frac{\partial U_i}{\partial x_i}=1-\frac{1}{2k_i}\frac{e^{x_i}}{\left(1-\frac{1}{2k_i}e^{x_i}\right)}-2e^{x_i}\hat{\boldsymbol{v}}_i'M^{-1}\hat{\boldsymbol{v}}_i=1-\frac{1}{2k_i}\frac{\overline{B}_i}{\left(1-\frac{1}{2k_i}\overline{B}_i\right)}-2\overline{B}_i\hat{\boldsymbol{v}}_i'M^{-1}\hat{\boldsymbol{v}}_i$$

Using again 45:

$$\frac{\partial^2 U_i}{\partial x_j\partial x_i}=\begin{cases}2\overline{B}_i\overline{B}_j\hat{\boldsymbol{v}}_i'M^{-1}\hat{\boldsymbol{v}}_j\hat{\boldsymbol{v}}_j'M^{-1}\hat{\boldsymbol{v}}_i=2\overline{B}_i\overline{B}_j\left(\hat{\boldsymbol{v}}_i'M^{-1}\hat{\boldsymbol{v}}_j\right)^2 & i\neq j\\ =-\frac{1}{2k_i}\frac{\overline{B}_i}{\left(1-\frac{1}{2k_i}\overline{B}_i\right)^2}-2\overline{B}_i\hat{\boldsymbol{v}}_i'M^{-1}\hat{\boldsymbol{v}}_i+2\overline{B}_i^2\left(\hat{\boldsymbol{v}}_i'M^{-1}\hat{\boldsymbol{v}}_i\right)^2 & i=j\end{cases}$$

Since $\frac{\partial^2 U_i}{\partial x_j\partial x_i}=\frac{\partial^2 U_j}{\partial x_i\partial x_j}$, the game is a potential game. This means that there exists a twice differentiable function Ψ such that: $\frac{\partial\Psi}{\partial x_i}=\frac{\partial U_i}{\partial x_i}$ for each i and each profile x. In particular, this means that, even without knowing the expression of Ψ, we know its Hessian matrix H, we have $H_{ij}=\frac{\partial^2\Psi}{\partial x_i\partial x_j}=\frac{\partial^2 U_i}{\partial x_i\partial x_j}$.

Uniqueness Now, we prove that the potential is strictly concave. This proves that the game can have at most one Nash equilibrium. To prove it, we prove that the Hessian matrix H is negative definite, by proving that $-H$ is strictly diagonally dominant. Sum the off-diagonal entries:

$$\begin{aligned}\sum_{i\neq j}|H_{ij}|&=\sum_{i\neq j}H_{ij}\\&=\sum_{i\neq j}2\overline{B}_j\overline{B}_i\left(\hat{\boldsymbol{v}}_i'M^{-1}\hat{\boldsymbol{v}}_j\right)^2\\&=2\overline{B}_j\hat{\boldsymbol{v}}_j'M^{-1}\left(\sum_{i\neq j}\overline{B}_i\hat{\boldsymbol{v}}_i\hat{\boldsymbol{v}}_i'\right)M^{-1}\hat{\boldsymbol{v}}_j\\&=2\overline{B}_j\hat{\boldsymbol{v}}_j'M^{-1}\left(M-\hat{B}_c-\overline{B}_j\hat{\boldsymbol{v}}_j\hat{\boldsymbol{v}}_j'\right)M^{-1}\hat{\boldsymbol{v}}_j\\&=2\overline{B}_j\hat{\boldsymbol{v}}_j'M^{-1}MM^{-1}\hat{\boldsymbol{v}}_j-2\overline{B}_j\hat{\boldsymbol{v}}_j'M^{-1}\left(\hat{B}_c+\overline{B}_j\hat{\boldsymbol{v}}_j\hat{\boldsymbol{v}}_j'\right)M^{-1}\hat{\boldsymbol{v}}_j\\&<2\overline{B}_j\hat{\boldsymbol{v}}_j'M^{-1}\hat{\boldsymbol{v}}_j-2\overline{B}_j\hat{\boldsymbol{v}}_j'M^{-1}\left(\overline{B}_j\hat{\boldsymbol{v}}_j\hat{\boldsymbol{v}}_j'\right)M^{-1}\hat{\boldsymbol{v}}_j\\&=2\overline{B}_j\hat{\boldsymbol{v}}_j'M^{-1}\hat{\boldsymbol{v}}_j\left(1-\overline{B}_j\hat{\boldsymbol{v}}_j'M^{-1}\hat{\boldsymbol{v}}_j\right)\\&<-H_{jj}\end{aligned}$$

where the strict inequality is because $\hat{B}_c$ is positive semidefinite, and there must be at least a path from each firm j to the consumer, so that $\left[M^{-1}\hat{\boldsymbol{v}}_j\right]_c\neq 0$. Since the expression above is a sum of positive terms, it follows that $H_{jj} < 0$, and $-H_{jj} > \sum_{i \neq j} |H_{ij}|$, so $-H$ is strictly diagonally dominant and H is negative definite. Hence, Ψ is strictly concave and the Nash equilibrium is unique.

Moreover, since the diagonal of H is negative, it follows that the payoffs are concave, so the FOCs (13) are necessary and sufficient for the equilibrium. Since the payoffs are concave and the cross derivatives are positive, the game is also a supermodular game, and we can conclude that the unique Nash equilibrium is also the unique rationalizable action profile.

□

B Proofs of Section 4

B.1 Proof of Proposition 1

The proof follows from the following lemmas, proven in the Supplementary Appendix.

Lemma B.1. Define $\overline{BR}(\overline{\Lambda}, n, k)$ as the unique positive solution of:

$$X = \left(k^{-1} + (\overline{\Lambda}^{-1} + (n-1)X)^{-1}\right)^{-1}$$

Then, $\overline{BR}$ is increasing in n and k, and decreasing in $\overline{\Lambda}$.

Moreover, define $\overline{\Lambda}_i(\overline{B}_{-i}) := \overline{\Lambda}_i^{out} + \overline{\Lambda}_i^{in}$. The best reply equations (23) can be expressed as, for all i:

$$\overline{B}_i = \overline{BR}(\overline{\Lambda}_i(\overline{B}_{-i}), k_i, n_i)$$

Lemma B.2. In equilibrium, each $\overline{B}_i^*$ is increasing in each n_j and k_j.

Moreover, if $k_i = k$ for all i, then for all i, j $n_i \geq n_j$ implies $\overline{B}_i^* \geq \overline{B}_j^*$. If $n_i = n^*$ for all i, then for all i, j $k_i \geq k_j$ implies $\overline{B}_i^* \geq \overline{B}_j^*$

Lemma B.3. If $N = 1$ or $N \geq 2$ and $n_i \geq 2$, then the equilibrium slopes are nonzero.

To obtain the price impact expressions in (17), notice that for firms in layer 1, the slope of the residual demand is $B_{c,1} + (n_1 - 1)\overline{B}_1$. Firms in the upstream layer 2 face a demand $n_1\overline{B}_1(p_1 - p_2) + A_2 - B_{c,2}p_2 + (n_2 - 1)\overline{B}_2(p_2 - p_3)$, where it is now necessary to solve the first layer equations for p_1 as a function of p_2. Proceeding iteratively, we can rewrite the market-clearing conditions for goods i and $i+1$ as:

$$q_i = \underbrace{(\overline{\Lambda}_i^{out})^{-1}(A - p_i)}_{\substack{\text{Demand} \\ \text{from customers}}} - \underbrace{(n_i - 1)\overline{B}_i(p_i - p_{i+1} - f_{i,L})}_{\substack{\text{Supply of} \\ \text{competitors}}}$$

$$q_{i+1} = \underbrace{(\overline{\Lambda}_i^{in})^{-1} p_{i+1}}_{\substack{\text{Supply} \\ \text{from suppliers}}} - \underbrace{(n_i - 1)\overline{B}_i(p_i - p_{i+1} - f_{i,L})}_{\substack{\text{Demand of} \\ \text{competitors}}} \tag{46}$$

where $(\overline{\Lambda}_i^{out})$ and $(\overline{\Lambda}_i^{out})$ satisfy (18). The right-hand sides of Equations 46 constitute the (direct) residual demand and supply. The inverse Jacobian of this is schedule is exactly Λ_i in (17).

1. Suppose $n_i = n_j := n^*$. Denote Q the quantity consumed by the consumer in equilibrium. By Lemma B.2 in equilibrium $\overline{B}_i = \overline{B}_j := B^*$. By homogeneity and market clearing $q_i^{out} = q_j^{out} := Q/n^*$ for any i, j. As a consequence, by the expressions of the price impact in Equation (17), we have that the quantity $\overline{\Lambda} := \overline{\Lambda}_i^{in} + \overline{\Lambda}_i^{out} = B_c^{-1} + (n^*)^{-1} \sum_{j \neq i} (B^*)^{-1}$ is independent of i. Using the expression for markups computed in (24), we get:

$$\mu_i^{out} = \frac{\overline{\Lambda}_i^{out}}{1 + \overline{\Lambda}(n^* - 1)B^*} \frac{Q}{n^*}$$

$$\mu_i^{in} = \frac{\overline{\Lambda}_i^{in}}{1 + \overline{\Lambda}(n^* - 1)B^*} \frac{Q}{n^*}$$

   From Equation (18) we have that $\overline{\Lambda}_i^{out}$ is increasing in i (i.e., increasing upstream) while $\overline{\Lambda}_i^{in}$ is decreasing in i (i.e., increasing downstream). By the above expressions, the same is true of, respectively, the markup and markdown. The same is true for the relative markup $\overline{\mu}_i^{out} = \mu_i^{out}/p_i$, because since $Q = n^* B^*(p_i - p_{i+1}) > 0$, p_i is increasing in the layer i. Moreover, $p_{i+1} = \overline{\Lambda}_i^{in} Q + f_N$, so the relative markdown satisfies:

$$\overline{\mu}_i^{in} = \frac{\mu_i^{in}}{p_{i+1}} = cost \times \frac{\overline{\Lambda}_i^{in}}{\overline{\Lambda}^{in} Q + f_N}$$

   which again is increasing downstream.

2. If the quantity sold by firm i is q_i, by market clearing, it must be $Q = n_i q_i$. Moreover, $p_i - p_{i+1} = q_i/\overline{B}_i$, so the aggregate profit in layer i is:

$$\Pi_i = n_i q_i (p_i - p_{i+1}) - \frac{1}{2k} q_i^2 = \frac{Q^2}{n_i} \left(\frac{1}{\overline{B}_i} - \frac{1}{2k} \right),$$

   and the expression is valid also for firm 1. So, by Lemma B.2, $\Pi_i \leq \Pi_j$ if and only if $n_i \geq n_j$.

□

C Proofs of Section 5

C.1 Proof of Theorem 3

Part 1 The assumptions on Λ and Equation 13 show that $\overline{B}_i \in [0, k_i]$ and Λ_i is continuous, so the best reply map is also continuous. So, by Brouwer's fixed point theorem, there exists an equilibrium. Moreover, the best reply is increasing in the profile of slopes of other firms $\overline{B}_{-i}$. By Topkis' Theorem, the equilibrium set is a lattice, so it has a maximal and minimal element.

Part 2 Define BR^1, $BR^2 : \prod_i [0, k_i] \to \prod_i [0, k_i]$ the best reply maps for, respectively, model 1 and 2. We know that for any profile B we have, entrywise, $BR^1(B) > BR^2(B)$. Call $(B^*)^1$ the maximal equilibrium in model 1 and $(B^*)^2$ the maximal equilibrium in model 2. We have:

$$(B^*)^1 = BR^1((B^*)^1) > BR^2((B^*)^1)$$

Since the best reply is monotonic, we have that iterating the best reply of model 2 starting from $(B^*)^1$ we eventually reach the maximal equilibrium of model 2, so:

$$(B^*)^1 > BR^2((B^*)^1) > \cdots > (B^*)^2$$

which is what we wanted to show. The case of the minimal equilibrium works analogously.

□

C.2 Proof of Theorem 4

1. In the unilateral case, the best reply equation is:

$$\max_{\boldsymbol{q}_i, \ell_i} \boldsymbol{q}_i \boldsymbol{p}_i' - \ell_i$$

s.t. $\hat{\boldsymbol{q}}_i + (M - \hat{B}_i)\boldsymbol{p} = \overline{\boldsymbol{A}}$. From this, we get the residual demand. Now, instead, the prices of inputs are fixed. Only the output price is allowed to change. For consistency, only the output quantity can affect it. We can express the equations more conveniently by decomposing the matrix M as follows:

$$M = \begin{pmatrix} M_{\mathcal{N}_{in}(i)} & M_{\mathcal{N}_{in,out}(i)} \\ M'_{\mathcal{N}_{in,out}(i)} & M_{\mathcal{N}_{out}(i)} \end{pmatrix}$$

where, after reordering, $\mathcal{N}^{in}(i)$ contains the subset of all goods that are inputs of i, or all goods that are directly or indirectly connected to inputs of i. $\mathcal{N}^{out}(i)$ contains the remaining goods: the output of i, and possibly all downstream goods that are not connected to any inputs.

The equations involving $\boldsymbol{q}_i$ are:

$$\boldsymbol{q}_i + (M_{\mathcal{N}_{in}(i)} - \hat{B}_{\mathcal{N}_{in}(i)})\boldsymbol{p}_{\mathcal{N}^{in}(i)} + (M_{\mathcal{N}^{out}(i)} - \hat{B}_{\mathcal{N}^{out}(i)})\boldsymbol{p}_{\mathcal{N}^{out}(i)} = \overline{\boldsymbol{A}}_i$$

We can solve only for the downstream prices:

$$\boldsymbol{p}_{\mathcal{N}^{out}(i)} = (M_{\mathcal{N}^{out}(i)} - \hat{B}_{i,\mathcal{N}^{out}(i)})^{-1}\left(\overline{\boldsymbol{A}}_i - \boldsymbol{q}_i - (M_{\mathcal{N}_{in}(i)} - \hat{B}_{i,\mathcal{N}_{in}(i)})\boldsymbol{p}_{\mathcal{N}^{in}(i)}\right)$$

In particular, focusing only on the output good of firm i:

$$p_i^{out} = [(M_{\mathcal{N}^{out}(i)} - \hat{B}_{i,\mathcal{N}^{out}(i)})^{-1}]_{ii}(\overline{A}_i - q_i^{out}) + const$$

where $[\cdot]_{ii}$ denotes the diagonal element of the matrix inside the square brackets:

$$[(M_{\mathcal{N}^{out}(i)} - \hat{B}_{i,\mathcal{N}^{out}(i)})^{-1}]_{ii} = [(M_{ii} - B_{ii} - M'_{i,down}M^{-1}_{-i,down}M_{i,down})^{-1}]_{ii}.$$

So, we get:

$$\frac{\partial p_i^{out}}{\partial q_i^{out}} = [(M_{\mathcal{N}^{out}(i)} - \hat{B}_{\mathcal{N}^{out}(i)})^{-1}]_{ii}$$

Hence, the price impact is:

$$\frac{\partial \boldsymbol{p}_i}{\partial \boldsymbol{q}_i} = \Lambda_i^{unilateral} = \begin{pmatrix} [(M_{\mathcal{N}^{out}(i)} - \hat{B}_{\mathcal{N}^{out}(i)})^{-1}]_{ii} & 0 \\ 0 & 0 \end{pmatrix}$$

To compare with $\Lambda^{multilateral}$, note that we can also decompose $\Lambda^{multilateral}$ as:

$$\Lambda^{multilateral} = \left[\left(\begin{pmatrix} M_{i,down} & M_{i,down-up} \\ M'_{i,down-up} & M_{i,up} \end{pmatrix} - \hat{B}_i\right)^{-1}\right]_{\mathcal{N}_i}$$

For simplicity, from now on denote the blocks of the matrix $M - \hat{B}_i$ as:

$$M - \hat{B}_i = \begin{pmatrix} A_1 & A_2 \\ A'_2 & A_3 \end{pmatrix}$$

Using block inversion, $\Lambda_i^{unilateral}$ can also be written as:

$$\Lambda_i^{unilateral} = \lim_{T\to\infty} \left[\begin{pmatrix} A_1 & A_2 \\ A_2' & TA_3 \end{pmatrix}^{-1} \right]_{\mathcal{N}_i} \tag{47}$$

Now note that, for $T > 1$:

$$\begin{pmatrix} A_1 & A_2 \\ A_2' & A_3 \end{pmatrix} - \begin{pmatrix} A_1 & A_2 \\ A_2' & TA_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & (T-1)A_3 \end{pmatrix}$$

is positive semidefinite for $T > 1$. Since $\Lambda^{multilateral} = \left[\begin{pmatrix} A_1 & A_2 \\ A_2' & A_3 \end{pmatrix}^{-1} \right]_{\mathcal{N}_i}$, by inverting and passing to the limit we obtain $\Lambda_i^{unilateral} \leq \Lambda_i^{multilateral}$, which is what we wanted to show.

2. If $\hat{\boldsymbol{q}}_i + (M - \hat{B}_i)\boldsymbol{p} = \overline{\boldsymbol{A}}$, but the prices of the other markets are to be taken as given, then the equations involving $\boldsymbol{q}_i$ are:

$$\boldsymbol{q}_i + (M_{\mathcal{N}(i)} - B_i)\boldsymbol{p}_i + M_{\mathcal{N}(i),\mathcal{N}(i)^c}\boldsymbol{p}_{\mathcal{N}(i)^c} = \overline{\boldsymbol{A}}_i$$

Inverting, we obtain:

$$\boldsymbol{p}_i = (M_{\mathcal{N}(i)} - B_i)^{-1} \left(\overline{\boldsymbol{A}}_i - \boldsymbol{q}_i - M_{\mathcal{N}(i),\mathcal{N}(i)^c}\boldsymbol{p}_{\mathcal{N}(i)^c} \right)$$

Since $\boldsymbol{p}_{\mathcal{N}(i)^c}$ are to be considered constants, the price impact is:

$$\Lambda_i^{local} = (M_{\mathcal{N}(i)} - B_i)^{-1},$$

So, it is immediate to conclude:

$$\Lambda_i^{local} = (M_{\mathcal{N}(i)} - B_i)^{-1} < \left(M_{\mathcal{N}(i)} - B_i - M_{\mathcal{N}(i),\mathcal{N}(i)^c} M_{\mathcal{N}(i)^c}^{-1} \left(M_{\mathcal{N}(i),\mathcal{N}(i)^c} \right)' \right)^{-1} = \Lambda_i^{multilateral},$$

as we wanted to show.

C.3 Proof of Corollary 5.1

If there is a unique final good, say good 0, then the vector $\boldsymbol{A}$ has just one nonzero entry, corresponding to good 0. So, $\boldsymbol{A}'\boldsymbol{p} = A_0 p_0$, and as a consequence:

$$p_0 = \frac{1}{A_0} \boldsymbol{A}' M^{-1} \boldsymbol{A}$$

Since M is increasing in each B_i, p_0 is decreasing in each B_i. □

Now we know that M is increasing in each B_i, and so we obtain that p_0 is decreasing in each B_i. □

C.4 Proof of Proposition 2

1. Consider first unilateral market power on output. Plugging the price impact (29) in the best reply equation we get:

$$\overline{B}_i^{unilateral} = \left(k^{-1} + \frac{1}{(\overline{\Lambda}_i^{out})^{-1} + \overline{B}_i(n-1)}\right)^{-1} = \overline{BR}(\overline{\Lambda}_i^{out}, k, n) \quad (48)$$

   where $\overline{BR}$ is the same function defined in Lemma B.1. By Lemma B.1 and the fact that Λ_i^{out} is increasing upstream we conclude that, in equilibrium, $\overline{B}_i$ is decreasing upstream. The markup/markdown vector is:

$$\boldsymbol{\mu}_i = \frac{Q}{n}\Lambda_i^{unilateral,out} = \frac{Q}{n}\begin{pmatrix} \frac{1}{(\overline{\Lambda}_i^{out})^{-1} + \overline{B}_i(n-1)} \\ 0 \end{pmatrix}$$

   Using the best reply equation (48), we can rewrite the expression as:

$$\mu_i^{out} = \frac{Q}{n}\left(\frac{\overline{\Lambda}_i^{out}}{1 + \overline{\Lambda}_i^{out}\overline{B}_i(n-1)}\right) = \frac{Q}{n}\left(\overline{B}_i^{-1} - k^{-1}\right)$$

   So we conclude that both μ_i^{out} and the profit are increasing upstream.

2. Analogously to the previous case, we obtain that $\overline{B}_i$ is increasing *downstream*, the markup is zero and the markdown is:

$$\mu_i^{in} = \frac{Q}{n}\frac{\overline{\Lambda}_i^{in}}{1 + \overline{\Lambda}_i^{in}\overline{B}_i(n-1)} = \frac{Q}{n}\left(\overline{B}_i^{-1} - k^{-1}\right)$$

   As a consequence, both markdown and profits are increasing downstream. □

C.5 Proof of Proposition 3

The monopoly price in the after-merger setting is:

$$p^{post} = A\left(B_c + B_M\right)^{-1} = A\left(B_c + \frac{1}{k^{-1} + 1/B_c}\right)^{-1}$$

where B_M is the equilibrium coefficient of the supply of the only firm, and the best reply yields $B_M = (k^{-1} + \Lambda_M)^{-1} = \frac{1}{k^{-1}+1/B_c}$

In the pre-merger equilibrium instead the final price is:

$$p^{pre} = \frac{A}{B_c + \left(\frac{1}{n_1\overline{B}_1} + \frac{1}{\overline{B}_2}\right)} = A\left(B_c + \frac{1}{1/B_c + \frac{2}{n_1\overline{B}_1}}\right)^{-1}$$

where the last equality is obtained using the best reply equation for $\overline{B}_2$:

$$\overline{B}_2^{-1} = B_c^{-1} + (n_1\overline{B}_1)^{-1}$$

These expressions are valid under both unilateral and bilateral market power.

Hence we get that the price is higher after the merger if and only if $2k < n_1 B_1$. Now in both cases, $n_1\overline{B}_1(n_1)$ is increasing, so the equation $2k = n_1\overline{B}_1(n_1)$ has a unique solution. Moreover, if $n_1 = 2$ the RHS is lower, while for n_1 sufficiently large the RHS is higher. Define n^* as the solution in case of multilateral market power, and n_* as the solution with unilateral market power. By Theorem 3 for any n_1, $\overline{B}_1^{\text{unilateral}}(n_1) > \overline{B}_1^{\text{multilateral}}(n_1)$, so that $n_* < n^*$. Hence, it follows that for $n \in (n_*, n^*)$ the merger is welfare-increasing with multilateral market power, but welfare-decreasing with unilateral market power, which is what we wanted to show.

□

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Supplementary Appendix of "Multilateral market power in input-output networks"

D General model: non-linear schedules and uncertainty

In this section I show that the linear equilibrium analyzed in the main text remains an equilibrium even if firms are free to choose any nonlinear schedule. Moreover, it also survives introducing uncertainty in the intercept of the consumer demand $\boldsymbol{A}$ and the labor cost parameters $\boldsymbol{f}_L$. Specifically, we assume that the vector $\boldsymbol{\varepsilon} := (\boldsymbol{A}, \boldsymbol{f}_L)$ has a joint distribution $F_{\boldsymbol{\varepsilon}}$ with finite mean. We assume that $\boldsymbol{\varepsilon}$ has support $\mathcal{E}$, which we leave generic for now.[14]

D.1 The general game

The technology is the one defined by the Equations (5). The key difference is that the vectors $\boldsymbol{A}$ and $\boldsymbol{f}_L$ are stochastic. The schedules chosen by the firms $\mathcal{S}_i = (\mathcal{S}_i^{out}, -\mathcal{S}_i^{in})$ now are maps from $\mathbb{R}^{d_i} \times \mathbb{R}_+ \to \mathbb{R}^{d_i}$, mapping $\boldsymbol{p}_i, f_{i,L}$ to $\mathcal{S}_i(\boldsymbol{p}_i, f_{i,L})$, and the schedules are chosen *before* the realization of the vectors $\boldsymbol{A}$ and $\boldsymbol{f}_L$. Firms can condition their schedule on their value $f_{i,L}$: this can be equivalently interpreted by saying that firms are able to observe their own realization of the cost $f_{i,L}$. In this context, the market clearing conditions are analogous to (8), remembering that schedules are also functions of $\boldsymbol{A}$ and $f_{i,L}$ now:

$$\sum_i \hat{\mathcal{S}}_i(\boldsymbol{p}, f_{i,L}) = \hat{D}_c(\boldsymbol{p}, \boldsymbol{A}) \tag{49}$$

The set of feasible schedules for firm i is denoted $\mathcal{A}_i$, and $\mathcal{A} = \prod_{i \in \mathcal{N}} \mathcal{A}_i$. The set $\mathcal{A}_i$ is the set of schedules that:

1. satisfy Equations (5);

[14]For the uniqueness result in Lemma 52, we need $\mathcal{E}$ to be large: so, in this section, traded quantities may not be positive. Since trade has a direction, negative quantities can simply be interpreted as trade flowing in the opposite direction. This amounts to assuming that the production process is reversible, as in classic treatments of production theory. In the case of the consumer, this can be interpreted as the consumer holding some endowment of the relevant good. In this context, the primary input can be thought of as something else than labor (e.g., energy), for which the reversibility is more natural. This approach simplifies the analysis because it rules out corner solutions in which firms decide not to buy some inputs (or sell some outputs) at all. Notice that the constraints (5) are well-defined, because if $\ell_g = f_{g,L} q_g^{out} + \frac{1}{2k_g}(q_g^{out})^2$ then the square root in $f_g(\ell_g)$ always has a nonnegative argument even if $q_g^{out} < 0$.

2. are such that the market-clearing conditions (49) uniquely define a pricing function.[15]

Formally, we have the following definition.

Definition D.1 (Pricing function and payoffs)**.**
A function $\boldsymbol{p} : \mathcal{E} \times \mathcal{A} \to \mathbb{R}^m$ *that solves* (49) *is called a pricing function.*

The payoff of firm (player) i is the mapping from supply and demand schedules in $\mathcal{A}_i$ *to real numbers defined by the expected profits:*

$$\pi_i(\mathcal{S}_i, \mathcal{S}_{-i}) = \mathbb{E}_F\left(\boldsymbol{p}_i'(\boldsymbol{\varepsilon})\mathcal{S}_i(\boldsymbol{p}_i(\boldsymbol{\varepsilon}), f_i) - \mathcal{S}_{\ell,i}(\boldsymbol{p}_i(\boldsymbol{\varepsilon}), f_i) \mid f_i\right)$$

D.2 Equilibrium

Linear equilibrium We look for an equilibrium in which the schedules are linear, namely they have the expression of Equation (6):

$$\mathcal{S}_i(\boldsymbol{p}_i, f_{i,L}) = B_i \boldsymbol{p}_i - B_{i,f} f_{i,L} \tag{50}$$

with the difference that $f_{i,L}$ is interpreted as a stochastic variable unknown to other firms.

Market clearing and residual demand The best reply problem of firm i is:

$$\max_{\mathcal{S}_i \in \mathcal{A}_i} \pi_i(\mathcal{S}_i, \mathcal{S}_{-i}) \tag{51}$$

If the schedules of other players $\mathcal{S}_{-i}$ are linear as in Equation (50), then the market clearing equations and the residual schedule follow, respectively Equation (10), (11) of the main text, with the only difference that the intercept $\tilde{\boldsymbol{A}}_i$ is stochastic.

The key result is the following, that generalizes Lemma 3.2. It shows that the linear schedules represent the unique best reply among all the schedules satisfying the assumptions for which the game is well defined. The reason is the same as in Klemperer and Meyer (1989): with support large enough, uncertainty forces the firms to choose a specific locus of prices and quantities, and all the choices are relevant for some realization of the stochastic parameters.

Lemma D.1. The schedule $\mathcal{S}_i^*$ solves the best reply problem (51) if and only if for any

[15] Here we do not state explicitly sufficient conditions for this to be the case: in the parallel paper Bizzarri (2025), the reader can find a set of sufficient conditions.

$\boldsymbol{\varepsilon}$ the quantity vector $\boldsymbol{q}_i^* = \mathcal{S}_i^*(\boldsymbol{p}_i(\boldsymbol{\varepsilon}), f_i)$ solves:

$$\max_{\boldsymbol{q}_i, \ell_i} \boldsymbol{q}_i' \boldsymbol{p}_i^r(\boldsymbol{q}_i, \boldsymbol{\varepsilon}; B_{-i}) - \ell_i \tag{52}$$

subject to the technology constraint (5). Moreover, it is the unique such schedule if the support $\mathcal{E}$ is such that for all i, $[M^{-1}\overline{\boldsymbol{A}}]_i$ spans $\mathbb{R}^{d_i}$.

The condition on the support is satisfied if, for example $\mathcal{E} = \mathbb{R}^{n+m}$, corresponding to the case in which all goods are directly sold to the consumer. However, this is not necessary, for example for the Supply chain with layers the condition is easier to satisfy, it is sufficient that the consumer intercept A_0 and the cost parameter for the last layer f_N have full support.

Now the solution of (52) is immediate, as in the main text, and yields:

$$\mathcal{S}_i(\boldsymbol{p}_i, f_i) = \overline{B}_i(\boldsymbol{v}_i' \boldsymbol{p}_i - f_i)\boldsymbol{v}_i \tag{53}$$

where the coefficient $\overline{B}_i$ satisfies (13).

So, we conclude that the linear equilibrium of the generalized model of this section is unique and identical to the equilibrium of the main text, characterized by Equations (13) for all i. Moreover, since it solves (52), it is an ex-post equilibrium: firms would not want to revise their schedule even if they learned the realization of $\boldsymbol{\varepsilon}$.

D.3 Proof of Lemma D.1

The proof uses the following Lemma, stating in words that optimizing over a schedule ex-ante is equivalent to pointwise optimizing ex-post, provided the stochastic parameters span the whole domain of the schedule: this is the same property used in Klemperer and Meyer (1989) and Malamud and Rostek (2017).

Lemma D.2. If the distribution of the random variable ε has full support, optimizing over a function of ε is equivalent to optimizing ex-post after the realizations of the stochastic variables. Formally, if $\mathcal{F} \subseteq \{f : \mathcal{E} \to \mathbb{R}^n\}$:

$$f^* = \arg\max_{f \in \mathcal{F}} \mathbb{E}_\varepsilon g(f(\varepsilon), \varepsilon)$$

if and only if:

$$f^*(\varepsilon) = \arg\max_{y \in \mathcal{F}(\mathcal{E})} g(y, \varepsilon)$$

except possibly on a set of measure zero.

Proof. The second condition is equivalent to: $g(f^*(\varepsilon), \varepsilon) \geq g(y, \varepsilon)$ for all $y \in \mathcal{F}(\mathcal{E})$, ε.

Hence, it follows that for any function $f \in \mathcal{F}$:

$$\int g(f^*(\varepsilon), \varepsilon)\mathrm{d}F(\varepsilon) \geq \int g(f(\varepsilon), \varepsilon)\mathrm{d}F(\varepsilon)$$

which is the first condition.

For the vice-versa, suppose $f^* = \arg\max_{f \in \mathcal{F}} \mathbb{E}_\varepsilon g(f(\varepsilon), \varepsilon)$, but for some set A of positive measure, for all $\varepsilon' \in A$ there is a $y \in \mathcal{F}(\mathcal{E})$ such that $g(y, \varepsilon') > g(f^*(\varepsilon'), \varepsilon')$. Since all the y belong to $\mathcal{E}$, there is a function $y \in \mathcal{F}$ with this property. Then, define $y^*(\varepsilon)$ as identical to f^* except on A, where it is defined as y. By definition, also $y^* \in \mathcal{F}$. Hence, since the distribution has full support:

$$\begin{aligned}
\int g(y^*(\varepsilon), \varepsilon)\mathrm{d}F(\varepsilon) &= \int_A g(y^*(\varepsilon), \varepsilon)\mathrm{d}F(\varepsilon) + \int_{A^c} g(y^*(\varepsilon), \varepsilon)\mathrm{d}F(\varepsilon) && (54)\\
&= \int_A g(y(\varepsilon), \varepsilon)\mathrm{d}F(\varepsilon) + \int_{A^c} g(f^*(\varepsilon), \varepsilon)\mathrm{d}F(\varepsilon) && (55)\\
&> \int g(f^*(\varepsilon), \varepsilon)\mathrm{d}F(\varepsilon) && (56)
\end{aligned}$$

which is a contradiction. So, it must be $f^*(\varepsilon) = \arg\max_{y \in \mathcal{F}(\mathcal{E})} g(y, \varepsilon)$, which is what we wanted to show. □

First, we observe that by the proof in the main text, if $\mathcal{S}_i(\boldsymbol{p}_i, f_{i,L}) = \boldsymbol{q}_i$, then:

$$\boldsymbol{S}_i' \boldsymbol{p}_i(\mathcal{S}_i, \boldsymbol{\varepsilon}; B_{-i}) - \ell_i = \boldsymbol{q}_i' \boldsymbol{p}_i^r(\boldsymbol{q}_i, \boldsymbol{\varepsilon}; B_{-i}) - \ell_i.$$

So, the objective function in (51) can be substituted by the one above. Using the Lemma D.2, we obtain that the optimization (51) is equivalent to the pointwise ex-post optimization:

$$\max_{\boldsymbol{q}_i, \ell_i \in \mathcal{F}_i(\mathcal{E})} \boldsymbol{q}_i' \boldsymbol{p}_i^r(\boldsymbol{q}_i, \boldsymbol{\varepsilon}; B_{-i}) - \ell_i$$

under the relevant constraints, and denoting $\mathcal{F}_i(\mathcal{E})$ the codomain of the schedule of firm i:

$$\mathcal{F}_i(\mathcal{E}) = \{\boldsymbol{q}_i, \ell_i \mid \exists \boldsymbol{\varepsilon} \text{ s.t. } \mathcal{S}_i(\boldsymbol{p}_i(\boldsymbol{\varepsilon}), f_{i,L}) = \boldsymbol{q}_i, \ell_i = \mathcal{S}_{i,\ell}(\boldsymbol{p}_i(\boldsymbol{\varepsilon}), f_{i,L}), \text{and } (5)\}$$

Now, the above optimization has a unique solution, which is the schedule (53). However, to prove uniqueness, we need to ensure that the $\boldsymbol{\varepsilon}$ parameters span the whole domain of $\mathcal{S}_i$, or $\mathcal{F}_i(\mathcal{E}) = \mathbb{R}^{d_i}$: the optimal quantity is not constrained for those prices that are out of the support. The prices follow Equation (10), and so the schedule $\mathcal{S}_i$ has full support if $[M^{-1}\overline{\boldsymbol{A}}]_{\mathcal{N}(i)}$ spans $\mathbb{R}^{d_i}$, which is what we wanted to show. □

E Markups and markdowns

We show that the markups defined in Equation (21) agree with the standard approach of computing the gap between price and marginal cost, or marginal revenue products.

Definition E.1.
The total cost *of firm i is:* $C_i(q_i^{out}) = \sum_g p_g\left(\boldsymbol{q}_i\right) f_{ij} q_i^{out} + \frac{1}{2k_i}\left(q_i^{out}\right)^2$, *where everything is expressed as a function of q_i^{out} using the technology constraints, including the prices where we write explicitly the argument of the residual inverse schedule:* $\boldsymbol{p}_i^r\left(q_i^{out}\boldsymbol{v}_i\right)$. *Define the (absolute)* markup *as* $\mu_i := p_i - \frac{\partial C_i}{\partial q_i^{out}}$.

The revenue product *of input g is:* $R_{ig}(q_{ig}) = p_i^{out} q_i^{out} - \sum_{j \neq g} p_j\left(\boldsymbol{q}_i\right) f_{ij} q_i^{out} - \frac{1}{2k_i}\left(q_i^{out}\right)^2$ *where, again, everything must be expressed as a function of q_{ig} using the technology constraints. The* markdown on input g *is:* $\mu_{ig} := \frac{\partial R_{ig}}{\partial q_{ig}} - p_g$.

Lemma E.1. The vector $\boldsymbol{\mu}_i = (\mu_i, -\mu_{ig})$ satisfies:

$$\boldsymbol{\mu}_i = q_i^{out}\Lambda_i \boldsymbol{v}_i = \Lambda_i \boldsymbol{q}_i, \tag{57}$$

E.1 Proof of Lemma E.1

The total cost to produce q_i^{out} is:

$$C_i(q_i^{out}) := \sum_{g \in \mathcal{N}(i)^{in}} p_g^r(\boldsymbol{q}_i) q_{ig} + \ell_i \tag{58}$$

$$= \sum_{g \in \mathcal{N}(i)^{in}} p_g^r\left(q_i^{out}\boldsymbol{v}_i\right) q_i^{out} f_{ig} + f_{i,L} q_i^{out} + \frac{1}{2k_i}(q_i^{out})^2 \tag{59}$$

where everything is expressed as a function of q_i^{out} using the technology constraints, including the prices where we write explicitly the argument of the residual inverse schedule: $\boldsymbol{p}_i^r\left(q_i^{out}\boldsymbol{v}_i\right)$. In particular, $\partial \boldsymbol{p}_i^r\left(q_i^{out}\boldsymbol{v}_i\right)/\partial q_i^{out} = -\Lambda \times \boldsymbol{v}_i$. Analogously, the (net) revenue product generated by input g is:

$$\begin{aligned}
R_{ig}(q_{ig}) &:= \sum_{j \neq g} p_j^r q_{ij} - \ell_i = \boldsymbol{q}_i' \boldsymbol{p}_i^r + p_g^r q_{ig} - f_{i,L} q_i^{out} - \frac{1}{2k_i}\left(q_i^{out}\right)^2 \\
&= q_i^{out} \boldsymbol{v}_i' \boldsymbol{p}_i^r + p_g^r q_{ig} - f_{i,L} q_i^{out} - \frac{1}{2k_i}\left(q_i^{out}\right)^2 \\
&= \frac{q_{ig}}{f_{ig}} \boldsymbol{v}_i' \boldsymbol{p}_i^r(q_{i,g} f_{i,g}^{-1} \boldsymbol{v}_i) + p_g^r(q_{i,g} f_{i,g}^{-1} \boldsymbol{v}_i) q_{ig} - f_{i,L} \frac{q_{ig}}{f_{ig}} - \frac{1}{2k_i}\left(\frac{q_{ig}}{f_{ig}}\right)^2,
\end{aligned}$$

where again everything is expressed as a function of q_{ig} using the technology constraint $q_i^{out} = f_{ig}^{-1} q_{ig}$. Now, compute the marginal cost and the marginal revenue product:

$$
\begin{aligned}
\frac{\partial C_i}{\partial q_i^{out}} &= \sum_g p_g^r f_{ig} - \sum_g \left[\Lambda_i \boldsymbol{v}_i\right]_g f_{ig} q_i^{out} + f_{i,L} + \frac{1}{k_i} q_i^{out} \\
&= -\boldsymbol{v}_i' \boldsymbol{p}_i^r + p_i^{out} + \boldsymbol{v}_i' \Lambda_i \boldsymbol{v}_i q_i^{out} - \left[\Lambda_i \boldsymbol{v}_i\right]_i q_i^{out} + f_{i,L} + \frac{1}{k_i} q_i^{out} \\
&= -\boldsymbol{v}_i' \boldsymbol{p}_i^r + f_{i,L} + p_i^{out} + \left(\boldsymbol{v}_i' \Lambda_i \boldsymbol{v}_i + \frac{1}{k_i}\right) q_i^{out} - \left[\Lambda_i \boldsymbol{v}_i\right]_i q_i^{out} \\
&= -\left(\boldsymbol{v}_i' \boldsymbol{p}_i^r - f_{i,L} + \overline{B}_i^{-1} q_i^{out}\right) + p_i^{out} - \left[\Lambda_i \boldsymbol{v}_i\right]_i q_i^{out} = p_i^{out} - \left[\Lambda_i \boldsymbol{v}_i\right]_i q_i^{out} \\
\frac{\partial R_{ig}}{\partial q_{ig}} &= \frac{1}{f_{ig}} \boldsymbol{v}_i' \boldsymbol{p}_i - \frac{q_{ig}}{f_{ig}^2} \boldsymbol{v}_i' \Lambda_i \boldsymbol{v}_i - \frac{f_{i,L}}{f_{ig}} - \frac{q_{ig}}{k_i f_{ig}^2} - \frac{1}{f_{ig}} \left[\Lambda_i \boldsymbol{v}_i\right]_g q_{ig} + p_g^r \\
&= \frac{1}{f_{ig}} \left(\boldsymbol{v}_i' \boldsymbol{p}_i^r - f_{i,L} - \frac{q_{ig}}{f_{ig}} \left(\boldsymbol{v}_i' \Lambda_i \boldsymbol{v}_i + \frac{1}{k_i}\right)\right) - \frac{1}{f_{ig}} \left[\Lambda_i \boldsymbol{v}_i\right]_g q_{ig} + p_g^r \\
&= \frac{1}{f_{ig}} \left(\boldsymbol{v}_i' \boldsymbol{p}_i^r - f_{i,L} - \overline{B}_i^{-1} q_i^{out}\right) - \frac{1}{f_{ig}} \left[\Lambda_i \boldsymbol{v}_i\right]_g q_{ig} + p_g^r = -\frac{1}{f_{ig}} \left[\Lambda_i \boldsymbol{v}_i\right]_g q_{ig} + p_g^r,
\end{aligned}
$$

where, in both calculations, the terms with $\overline{B}_i$ disappear because of the best reply equation. So, the markup and markdowns are:

$$
\begin{aligned}
\mu_i^{out} &= p_i^{out} - \frac{\partial C_i}{\partial q_i^{out}} \\
&= p_i^{out} - \left(p_i^{out} - \left[\Lambda_i \boldsymbol{v}_i\right]_i q_i^{out}\right) \\
&= \left[\Lambda_i \boldsymbol{q}_i\right]_i \\
\mu_{i,g}^{in} &= \frac{\partial R_{ig}}{\partial q_{ig}} - p_g \\
&= \left(-\frac{1}{f_{ig}} \left[\Lambda_i \boldsymbol{v}_i\right]_g q_{ig} + p_g\right) - p_g \\
&= -\left[\Lambda_i \boldsymbol{v}_i\right]_g q_i^{out} = -\left[\Lambda_i \boldsymbol{q}_i\right]_g
\end{aligned}
$$

So, the markup-markdown vector is: $\Lambda_i \boldsymbol{q}_i$. □

F Substitute intermediate inputs

In this section, I show that the main results of the paper generalize to the case of intermediate inputs that are imperfect complements or substitutes. To keep tractability, we need the equilibrium to be in linear schedules. This means that we must preserve the linear-quadratic nature of the objective function in (12). Here, I introduce a parametric functional form for the technology that allows inputs to be imperfect complements or substitutes, while keeping tractability.

Denote $\boldsymbol{\omega}_i = (\omega_{ij})_{j\to i}$ a nonnegative parameter vector, representing the intensity of each input j in the technology of firm i, and denote $\boldsymbol{u}_i = (1, -\boldsymbol{\omega}_i)$. Assume that total labor hired $\tilde{\ell}_i$ can be allocated to different tasks: ℓ_i^A is allocated to dealing with the production inputs, while ℓ_i^B is used for other production tasks unrelated to intermediate inputs (e.g. management). The quantity ℓ_i^A represents the "handling costs" and can be interpreted as all the costs connected with storage, transportation, inventory, and tasks that have to be performed in order to use that input.

Assumption T: Technology constraints There is a positive definite matrix $\Sigma_i \in \mathbb{R}^{(d_i-1)\times(d_i-1)}$ and a vector $\boldsymbol{\omega}_i \in \mathbb{R}^{d_i-1}$ such that the handling costs for the input vector $\mathcal{S}_i^{in}$ are: $\ell_i^A = (\mathcal{S}_i^{in})'\Sigma_i(\mathcal{S}_i^{in})$.[16] Moreover, from the intermediate input vector $\mathcal{S}^{in}$ and labor ℓ_i^B firm i produces a quantity $\mathcal{S}_i^{out} = \sum_j \omega_{ij}\mathcal{S}_{ij}^{in} + \alpha_i\sqrt{2\ell_i^B}$. Relabel the labor allocated to generic tasks as: $\ell_i = \sqrt{2\ell_i^B}$; in this way: $\ell_i^B = \dfrac{1}{2}\ell_i^2$. In summary, the technology constraints are:

$$
\begin{aligned}
\boldsymbol{u}_i'\mathcal{S}_i &= \alpha_i\ell_i \\
\tilde{\ell}_i = \ell_i^A + \ell_i^B &= (\boldsymbol{q}_i^{in})'\Sigma_i\boldsymbol{q}_i^{in} + \frac{1}{2}\ell_i^2
\end{aligned}
\tag{60}
$$

where $\boldsymbol{u}_i$ is the vector $(1, -\boldsymbol{\omega}_i)$.

The matrix Σ_i codifies the (purely technological) patterns of substitutability and complementarity. For example, $\Sigma_{i,jk} > 0$ means that, for the same input prices, using both inputs j and k has a higher cost than only using one of the two: this captures substitutability; on the contrary, $\Sigma_{i,jk} < 0$ captures complementarity. When $\Sigma_i = I_i$ (the identity), inputs are nor substitutes nor complements.

This is the same assumption followed in Bimpikis et al. (2019), and can be seen as the extension to an input-output setting of the standard quadratic cost function commonly used: if a firm uses no intermediate inputs but only labor, the costs reduce to the standard form $\frac{1}{2}\ell_i^2$, and $q_i^{out} = \alpha_i\ell_i$.

Microfoundation of handling costs The structure for the labor costs above can be rationalized via a standard neoclassical production function. Define $\boldsymbol{\ell}_i = (\ell_{ij})_{j\in\mathcal{N}^{in}(i)}$, where $\ell_i^A = \sum_j \ell_{ij}$ a *labor allocation*. Suppose the production function of firm i is:

$$
\Phi(\boldsymbol{q}_i^{in}, \tilde{\ell}_i) = \max_{\boldsymbol{\ell}_i:\,\sum_j \ell_{ij}+\ell_i^A=\tilde{\ell}_i} \sum_j \omega_{ij}\min\{\phi_{ij}(\boldsymbol{\ell}_i), q_{ij}\} + \alpha_i\sqrt{2\ell_i^B}
\tag{61}
$$

[16] We could also include a linear term in the handling cost, that would result in a linear labor cost $f_{i,L}$ analogous to the main text. I don't do that here for simplicity.

where the functions $(\phi_{ij}(\boldsymbol{\ell}_i))_j$ are implicitly (and uniquely) defined by the equations:

$$\ell_{ij} = \omega_{ij}\phi_{ij} + \sum_h \Sigma_{i,jh}\phi_{ij}\phi_{ih} \quad \forall j \tag{62}$$

This generates the expressions of the previous paragraph, as proven by the next Proposition.

The interpretation is as follows. The firm hires a number of workers $\tilde{\ell}_i$. Such number of workers have to be allocated to different "tasks": $\ell_{i1}, \ell_{i2}, \ldots$. Suppose first that $\Sigma_i = \frac{1}{k_i} I_i$, so that inputs are neither complements nor substitutes. In this case the Equations (62) are independent, and can be explicitly solved as: $\phi_{ij} = \sqrt{2k_i\ell_{ij}}$. This means that tasks and inputs are in direct correspondence: each input, to be used in production, needs to be combined with a quantity of labor ℓ_{ij}. The efficiency of this quantity of labor depends on the parameter k_i[17], and is equal to $\phi_{ij} = \sqrt{2k_i\ell_{ij}}$. Since each intermediate input is perfectly complementary with the function of labor, at the optimum it must be $q_{ij} = \phi_{ij} = \sqrt{2k_i\ell_{ij}}$, and so $\ell_{ij} = \frac{1}{2k_i} q_{ij}^2$. Summing over j we recover the expression of the handling costs: $\ell_i^A = \sum_j \ell_{ij} = \frac{1}{2k_i}\sum_j q_{ij}^2$. If instead $\Sigma_i \neq I_i$, labor tasks and intermediate inputs are not in one-to-one association, but each input, to be used in production, needs a specific *combination* of labor tasks. The combination of labor tasks needed to complement input j is given by a function $\phi_{ij}(\boldsymbol{\ell}_i)$, that may depend on all the labor tasks. Specifically, if the quantity of labor allocated to task h increases both ϕ_{ij} and ϕ_{ik}, this creates complementarity between inputs j and k; otherwise, it creates substitutability. The proof is in F.5.

Proposition 5.
The production function in Equation (61) is equivalent to the relation in Equation (60).

F.1 The game

The game is the same defined in Section 2.2, with the only difference that the technology constraints are given by the Equations 5, so that the coefficient matrices are not restricted to have rank 1. We make the following assumption, instead.

Assumption A : for all i, B_i is symmetric and positive definite.

Call $B = (B_i)_{i\in\mathcal{N}}$ the profile of coefficients chosen by firms, and $\mathcal{B}_i$ the set of matrices satisfying the above assumptions, so that the set of feasible action profiles is $\mathcal{B} = \prod_i \mathcal{B}_i$.

[17] The parameter k_i can be thought as capital. Under this interpretation, the technology here defined has costant returns to scale. In this case, the analysis of the paper can be thought as analysing a short run in which capital is fixed.

The definition of Generalized SDFE is the same as Definition 5.1, where the price impact function Λ_i now has to be decreasing in each B_j in the positive semidefinite ordering.

F.2 Results

The following Lemma has the role of Lemma 3.1. The proof is in F.3.

Lemma F.1. The matrix $M - \hat{B}_i$ is positive definite, so invertible.

Given this, the proof of Lemma 3.2 is still valid, because it only uses the linearity of the schedules. In particular, this shows that the special case of the standard SDFE realizes for Λ_i with the same expression as in Lemma 3.2. So, the best reply problem is analogous to (12), with different technology constraints. In particular, using the technology constraint (60) to eliminate $\tilde{\ell}_i$, the best reply problem can be written as:

$$\max_{\boldsymbol{q}_i,\ell_i} (\boldsymbol{p}_i^r)'\boldsymbol{q}_i - (\boldsymbol{q}_i^{in})'\Sigma_i \boldsymbol{q}_i^{in} - \frac{1}{2}\ell_i^2 \tag{63}$$

subject to $\boldsymbol{u}_i'\boldsymbol{q}_i = \alpha_i \ell_i$. Since $\alpha_i > 0$, we can further eliminate the variable ℓ_i using the constraints and rewrite the problem as:

$$\max_{\boldsymbol{q}_i,\ell_i} (\boldsymbol{p}_i^r)'\boldsymbol{q}_i - (\boldsymbol{q}_i^{in})' C_i^{-1} \boldsymbol{q}_i^{in}, \tag{64}$$

where we define:

$$C_i := \begin{pmatrix} \frac{1}{\alpha_i^2} & \frac{1}{\alpha_i^2}\boldsymbol{\omega}_i' \\ \frac{1}{\alpha_i^2}\boldsymbol{\omega}_i & \Sigma_i + \frac{1}{\alpha_i^2}\boldsymbol{\omega}_i\boldsymbol{\omega}_i' \end{pmatrix}^{-1}$$

which thanks to the assumption on Σ_i is a positive definite matrix. The Hessian of the problem is $-\Lambda_i - C_i^{-1}$, so the problem is concave. The FOCs are:

$$\boldsymbol{p}_i^r - \Lambda_i \boldsymbol{q}_i - C_i^{-1}\boldsymbol{q}_i = 0$$

So, we get:

$$\mathcal{S}_i = (C_i^{-1} + \Lambda_i)^{-1}\boldsymbol{p}_i$$

So the coefficient matrix of firm i, in equilibrium, satisfies the expression:

$$B_i = (C_i^{-1} + \Lambda_i)^{-1} \tag{65}$$

Notice that C_i is the matrix of coefficients of the schedules that represent the competitive equilibrium of the economy, because price-taking corresponds to $\Lambda_i = 0$.

Existence of a solution of (65) needs a different proof. In particular, action spaces are not unidimensional, which means that the supermodular and potential structures are lost. However, the best replies are still increasing in the positive semidefinite ordering (by the assumption on Λ_i). The following theorem states conditions for existence and a characterization of the equilibrium, generalizing Part 1 of Theorem 3. The proof is in F.4.

Theorem 5. *1. The first-order condition for the best reply problem are:*

$$\Lambda_i \boldsymbol{q}_i = \boldsymbol{p}_i - \lambda_i \boldsymbol{u}_i \tag{66}$$

$$1 = \lambda_i \ell_i, \tag{67}$$

where λ_i is the multiplier on the technology constraint.

2. The best reply of firm i is (65). Moreover, it is increasing in B_{-i} in the positive semidefinite ordering.

3. There exist a minimal and a maximal Nash equilibrium.

In this context, the markup-markdown vector is $\lambda_i \boldsymbol{u}_i$, so we can see that the characterization of markups in terms of centrality in the goods network is still valid: $\boldsymbol{\mu}_i = \Lambda_i \boldsymbol{q}_i$, and also Theorem 2. What changes are possibly the signs of the entries of the matrix M, because intermediate inputs are possibly substitutes.

Theorems 3, part 2 and 4 are all valid under the more general model. This is because their proofs only rely on the fact that each coefficient matrix B_i, Λ_i, and $M - \hat{B}_i$ are symmetric and positive semi-definite, and Λ_i is decreasing as a function of each B_j.

F.3 Proof of Lemma F.1

If $\hat{B}_c$ is positive definite (which is the case when $\hat{B}_c = B_c$, that is all goods are consumed by the consumer), the thesis follows immediately. If $\hat{B}_c$ has some zero rows, consider a nonzero vector $\boldsymbol{x}$ such that $\boldsymbol{x}' M \boldsymbol{x} = 0$. For every subset $\boldsymbol{x}_i$, it must be $\boldsymbol{x}_i = 0$ or $\boldsymbol{x}_i = \lambda \boldsymbol{u}_i$, $\lambda \neq 0$ for some firm i with $\alpha_i = 0$. Moreover, it must be $\boldsymbol{x}_c = 0$ because B_c is positive definite.

Consider any good g. Since the network is connected it must exist a sequence $g_1, \ldots, g_s$ such that $g_1 = g$, $g_s \in \mathcal{C}$ and for every pair g_i, g_{i+1} in the sequence there is a firm that produces g_{i+1} and for which g_i is an input. Without loss of generality, choose this sequence so that the length s is minimal. Then, by induction on s, we can prove that $x_g = 0$.

If $s = 1$, $g \in \mathcal{C}$ and so $x_g = 0$ by the reasoning above. Now suppose the thesis holds for s, and consider a good for which the minimal sequence has length $s + 1$. By assumption, $x_{g_2} = 0$, so for each firm i producing g_2, it must be $\boldsymbol{x}_i = 0$. By assumption, there is a firm using g as input and producing g_2, and so it follows that $x_g = 0$. Then, by induction we proved that $x_g = 0$ for all goods g.

Now, suppose that each good is traded by at least three agents, and consider $M - \hat{B}_i$. This corresponds to the matrix of a network from which firm i has been removed. But if each good is traded by at least 3 firms, the network is still connected even when removing a firm. So, the reasoning above still holds, and we conclude that $M - \hat{B}_i$ is positive definite. □

□

F.4 Proof of Theorem 5

The payoffs are strictly concave, so Equation (65) is necessary and sufficient for optimization. We have to show that a profile of matrices satisfying it exists.

The best reply map BR_i defined by equation (65) is increasing and continuous, by the assumption on Λ_i. Moreover, the best reply belongs to the set $\{B_i \in \mathcal{B}_i \mid \|B_i\| \le \|C_i\|\}$, so it is bounded. Because of this, we have that the profile of best replies to $(C_i)_{i\in\mathcal{N}}$ satisfies $BR_i(C_{-i}) \le C_i$. Iterating the best reply, we find that it must converge to a profile B_i^*, that is the maximal equilibrium. Analogously, if $B_i^0 = 0$ for all i, we have $BR_i(B_{-i}^0) \ge B_i^0$, and iterating again we find that the sequence converges to the minimal equilibrium $B_{i,*}$. So, there are a maximal and a minimal equilibrium, possibly identical.

F.5 Proof of Proposition 5

First, we have to prove that the function $(\phi_{ij})_j : \mathbb{R}^{d_i-1} \to \mathbb{R}^{d_i-1}$, implicitly defined by:

$$\ell_{ij} = \frac{1}{2k_i} \sum_h \sigma_{i,jh}\phi_{ij}\phi_{ih} \quad \forall j$$

is well defined. We can rewrite this expression as: $F_{ij}(\boldsymbol{\ell}_i, \boldsymbol{\phi}_i) = 0$, for all j, where:

$$F_{ij}(\boldsymbol{\ell}_i, \boldsymbol{\phi}_i) = -\ell_{ij} + \frac{1}{2k_i} \sum_h \sigma_{i,jh}\phi_{ij}\phi_{ih}$$

We want to prove that this function of $\boldsymbol{\phi}_i$ has a zero for any $\boldsymbol{\ell}_i$. We use the intermediate value theorem, in the generalized form of the Poincaré-Miranda Theorem, Kulpa (1997). By this theorem, if for all j, when $\phi_{ij} = 0$ then $F_{ij}(\boldsymbol{\ell}_i, \boldsymbol{\phi}_i) < 0$, independently of the other ϕ_{ik}, $k \neq j$, and when $\phi_{ij} = \overline{F}$ then $F_{ij}(\boldsymbol{\ell}_i, \boldsymbol{\phi}_i) > 0$, independently of the other ϕ_{ik}, $k \neq j$, then there is a vector of zeros $\boldsymbol{\phi}_i^*$ such that $0 \le \phi_{ij}^* \le \overline{F}$ for all j.

Suppose $\ell_{ij} > 0$ for all j. Note that if $\phi_{ij} = 0$ then $F_{ij}(\boldsymbol{\ell}_i, \boldsymbol{\phi}_i) < 0$, independently of the other ϕ_{ik}, $k \neq j$. Now suppose there is an upper bound $\overline{F}$ on the ϕ_{ij}. We can write: $\sum_h \sigma_{i,jh}\phi_{ih} > \sigma_{i,jj}\phi_{ij} - \sum_{h\neq j} |\sigma_{i,jh}|\overline{F}$. Since Σ_i is diagonally dominant, the latter expression is positive if $\phi_{ij} = \overline{F}$: so, if $\phi_{ij} = \overline{F}$, it follows $\sum_h \sigma_{i,jh}\phi_{ih} > 0$. Now it is sufficient to choose $\overline{F}(\boldsymbol{\ell}_i)$ large enough, and we obtain that for $\phi_{ij} = \overline{F}$ then $F_{ij}(\boldsymbol{\ell}_i, \boldsymbol{\phi}_i) > 0$, independently of the other ϕ_{ik}, $k \neq j$. So, by the Poncaré-Miranda Theorem, there is a solution.

Now suppose that $\ell_{ij} = 0$ for some j: $\phi_{ij} = 0$ solves the corresponding equation, and we can simply apply the above reasoning to the remaining equations and variables. We conclude that the above equations implicitly define a map $\boldsymbol{\phi}_i : \mathbb{R}_+^{d_i} \to \mathbb{R}_+^{d_i}$.

For uniqueness, we want to use the Theorem 4 in Gale and Nikaido (1965) stating that if the Jacobian of a map is a P-matrix on a rectangular region, then the map is invertible. The domain of $\boldsymbol{\phi}_i$ is $\mathbb{R}_+^{d_i}$, a rectangular region.

The Jacobian of the function F_i defined above, with respect to $\boldsymbol{\phi}_i$ is the matrix with in position j, h the value $\frac{1}{2k_i}\sigma_{i,jh}\phi_{ij} + \frac{1}{2k_i}\delta_{jh}\sigma_{i,jj}\phi_{ij}$. This is:

$$J_i = \frac{1}{2k_i}\Sigma_i diag(\boldsymbol{\phi}_i) + \frac{1}{2k_i} diag(\Sigma_{i,k}) diag(\boldsymbol{\phi}_i)$$

The term $diag(\Sigma_i)diag(\boldsymbol{\phi}_i)$ is diagonal and positive. Moreover, we focus on the case in which the $\boldsymbol{\varepsilon}_i$ are small enough, so we neglect the first part.

Focus on the case in which $\phi_{ij} > 0$. The only non-diagonal part of this matrix is $\Sigma_i diag(\boldsymbol{\phi}_i)$: this is the product of the positive definite matrix Σ_i, and the positive diagonal matrix $diag(\boldsymbol{\phi}_i)$. A positive definite matrix is a P-matrix (see Gale and Nikaido (1965)). Moreover, an equivalent definition for A being a P-matrix is that for any vector $\boldsymbol{x}$ there exist a positive diagonal matrix D such that $\boldsymbol{x}'DA\boldsymbol{x} > 0$ (Horn et al. (1994), Ch 2). This means that if E_1, E_2 are positive diagonal matrices and A is a P-matrix, then also $E_1A + E_2$ is a P-matrix. Indeed, $\boldsymbol{x}'DE_1^{-1}(E_1A + E_2)\boldsymbol{x} = \boldsymbol{x}'DA\boldsymbol{x} + \boldsymbol{x}'DE_1^{-1}E_2\boldsymbol{x} > 0$, so the definition is satisfied.

So, J_i is a P-matrix and so, by Theorem 4 in Gale and Nikaido (1965), it follows that the map $F_{ij}(\cdot)$ is injective. Hence, the implicit equation has only one solution.

If $\phi_{ij} = 0$ for some j, this can only be a solution if $\ell_{ij} = 0$. We can esclude all the equations relative to zero labor, and apply the previous reasoning on the remaining variables: it follows that the solution is unique on the whole of $\mathbb{R}_+^{d_i}$.

G Sequential Cournot

The Sequential Oligopoly is a textbook model in Industrial organization (as in Salinger (1988), or Belleflamme and Peitz (2015)), both with Cournot or Bertrand or mixed approaches, used in various classic papers such as Spengler (1950) (monopoly), Salinger (1988) (Bertrand). Here I focus on Cournot, for simplicity.

Definition G.1 (Sequential Cournot).

Consider the setting of the layered supply chain of Example 3. Solve it as a standard Sequential Oligopoly as in Belleflamme and Peitz (2015): firms in layer 1 play a Cournot game on outputs, taking the input price p_2 as given. Then, compute the demand for good 2 from layer 1 implied by the Cournot equilibrium between downstream firms. Finally, firms in layer 2 play a Cournot game in outputs. If there are more layers, Firms take as given the input price p_3, and so on for all the layers.

Proposition 6.

The Sequential Cournot model is a Generalized SDFE where, for all $i < N$:

$$\Lambda_i^{sequential} = \begin{pmatrix} \overline{\Lambda}_i^{out} & 0 \\ 0 & 0 \end{pmatrix}$$

G.1 Proof of Proposition 6

Consider first the downstream layer, 1. The inverse demand is: $P_1(\sum_j q_{j,1}) = \frac{1}{B_c}(A - \sum_j q_{j,1})$ The FOC is:

$$\frac{\partial}{\partial q_i}\left((P_1 - P_2)q_{i,1} - \frac{1}{2k_i}q_{i,1}^2\right) = P_1 - P_2 - \frac{1}{B_c}q_{i,1} - \frac{1}{k_1}q_{i,1} = 0$$

So, the quantity and the price have the relation:

$$q_i = \left(\frac{1}{B_c} + \frac{1}{k_i}\right)^{-1}(P_1 - P_2)$$

This is a linear schedule with slope $\left(\frac{1}{B_c} + \frac{1}{k_i}\right)^{-1}$, which is exactly Equation (13), where the price impact is modified so that $\overline{\Lambda}^{in} = 0$ and the slope of competitors are set to 0, so that the price impact is:

$$\Lambda_i^{sequential} = \begin{pmatrix} \overline{\Lambda}_i^{out} & 0 \\ 0 & 0 \end{pmatrix}$$

Finally, the demand from layer 1 to the upstream layer can be obtained by solving the market-clearing conditions:

$$P_1\left(\sum_j q_{j,1}\right) = \frac{1}{B_c}\left(A - \sum_j q_{j,1}\right) \tag{68}$$

$$q_i = \left(\frac{1}{B_c} + \frac{1}{k_1}\right)^{-1}(P_1 - P_2) \tag{69}$$

which, since they are linear, have the same solution as the market clearing in the competition in schedules, and it is linear:

$$P_2 = A_2 - \overline{B}_2 \sum_j q_{j,2}$$

Since the demand has the same form as the first layer, just with different coefficients, the reasoning can be repeated for all layers.

□

H Additional Proofs

H.1 Proof of Lemma B.1

The equation $X = \left(k^{-1} + (\overline{\Lambda}^{-1} + (n-1)X)^{-1}\right)^{-1}$ is equivalent to the quadratic equation:

$$(n-1)X^2 + (\overline{\Lambda}^{-1} - (n-2)k)X - \overline{\Lambda}_i^{-1}k = 0 \tag{70}$$

$$\Delta = \left((n-2)k - \overline{\Lambda}^{-1}\right)^2 + 4(n-1)\overline{\Lambda}^{-1}k > 0$$

The quadratic formula gives:

$$X = \frac{\left((n-2)k - \overline{\Lambda}^{-1}\right) \pm \sqrt{\Delta}}{2(n-1)} \tag{71}$$

and since $n \geq 2$, we have that $\Delta > \left((n-2)k - \overline{\Lambda}^{-1}\right)^2$, so this has a positive and a negative root. So, the positive root is unique and the function $\overline{BR}$ is well-defined.

The monotonicity follows from the implicit function theorem applied to the function $F(\overline{BR}, n, k) := \overline{BR}^{-1} - \left(k^{-1} + \left(\overline{\Lambda}^{-1} + (n-1)\overline{BR}\right)^{-1}\right)$. In particular, since:

$$\begin{aligned}
\frac{\partial F}{\partial \overline{BR}} &= -\overline{BR}^{-2} + (n-1)\left(\overline{\Lambda}^{-1} + (n-1)\overline{BR}\right)^{-2} \\
&= \overline{BR}^{-2}\left(-1 + \frac{1}{n-1}\frac{(n-1)^2\overline{BR}^2}{\left(\overline{\Lambda}^{-1} + (n-1)\overline{BR}\right)^2}\right) < 0,
\end{aligned}$$

we have:

$$\frac{\partial \overline{BR}}{\partial k} = \frac{k^{-2}}{\overline{BR}^{-2} - (n-1)\left(\overline{\Lambda}^{-1} + (n-1)\overline{BR}\right)^{-2}} > 0$$

$$\frac{\partial \overline{BR}}{\partial n} = \frac{\left(\overline{\Lambda}^{-1} + (n-1)\overline{BR}\right)^{-2}\overline{BR}}{\overline{BR}^{-2} - (n-1)\left(\overline{\Lambda}^{-1} + (n-1)\overline{BR}\right)^{-2}} > 0$$

$$\frac{\partial \overline{BR}}{\partial \overline{\Lambda}} = -\frac{\left(\overline{\Lambda}^{-1} + (n-1)\overline{BR}\right)^{-2}\overline{\Lambda}^{-2}}{\overline{BR}^{-2} - (n-1)\left(\overline{\Lambda}^{-1} + (n-1)\overline{BR}\right)^{-2}} < 0$$

from which the monotonicities follow.

□

H.2 Proof of Lemma B.2

The proof needs the following Lemma.

Lemma H.1. Suppose $k_i = k$ for all layers i. Consider a profile B. If $n_i > n_j$ and $X \leq \overline{BR}(\overline{\Lambda}_j(X, B_{-i,j}), n_j, k)$, then $\overline{BR}(\overline{\Lambda}_i(X, B_{-i,j}), n_i, k) > \overline{BR}(\overline{\Lambda}_j(X, B_{-i,j}), n_j, k)$.

Proof. By definition we have:

$$\overline{\Lambda}_i(X, B_{-i,j}) = \mathcal{L} + \frac{1}{n_j X}, \quad \overline{\Lambda}_j(X, B_{-i,j}) = \mathcal{L} + \frac{1}{n_i X}$$

where $\mathcal{L} = \sum_{k \neq i,j} \frac{1}{n_k B_k}$. So, $\overline{\Lambda}_i(X, B_{-i,j}) \geq \overline{\Lambda}_j(X, B_{-i,j})$ if and only if $n_i \geq n_j$. We want to prove that $\overline{BR}(\overline{\Lambda}_i(X, B^*_{-i,j}), n_i, k) > \overline{BR}(\overline{\Lambda}_j(X, B^*_{-i,j}), n_j, k)$. Now define $\Delta n = n_i - n_j$, $\overline{\Lambda}(h) := \mathcal{L} + \frac{1}{(n_i - h\Delta n)X}$, and the function $F(h)$ as:

$$F(h) := \overline{BR}\left(\overline{\Lambda}(h), n_j + h\Delta n, k\right)$$

For $h = 0$, we have $F(0) = \overline{BR}_j(\overline{\Lambda}_j, n_j, k)$, while for $h = 1$ we have $F(1) = \overline{BR}_i(\overline{\Lambda}_i, n_i, k)$. Now we are going to prove that F is increasing in h when $n_i > n_j$ and $X \leq F(0)$, so proving our thesis. Using the calculations in Lemma B.1:

$$\begin{aligned}
\frac{\partial F}{\partial h} &= \frac{\partial \overline{BR}}{\partial n}\Delta n + \frac{\partial \overline{BR}}{\partial \overline{\Lambda}} \frac{\Delta n}{(n_i - h\Delta n)^2 X} \\
&= \frac{\Delta n \left(\overline{\Lambda}(h)^{-1} + (n_j + h\Delta n - 1)F(h)\right)^{-2}}{\left(\frac{(n_j + h\Delta n - 1)}{\left(\overline{\Lambda}^{-1}(h) + (n_j + h\Delta n - 1)F(h)\right)^2} - F(h)^{-2}\right)} \left(F(h) - \frac{\overline{\Lambda}(h)^{-2}}{(n_i - h\Delta n)^2 X}\right)
\end{aligned}$$

Because of Lemma B.1, the first fraction is positive when Δn is positive. So, this is positive if and only if

$$F(h) > \frac{\overline{\Lambda}(h)^{-2}}{(n_i - h\Delta n)^2 X} \tag{72}$$

$$= \left(\mathcal{L} + \frac{1}{(n_i - h\Delta n)X}\right)^{-2} \frac{1}{(n_i - h\Delta n)^2 X} \tag{73}$$

$$= \frac{X}{((n_i - h\Delta n)X\mathcal{L} + 1)^2} \tag{74}$$

Since $\mathcal{L} > 0$, this condition is always satisfied if $X \leq F(h)$. If this is true for a specific h^*, since F is increasing in h^* with continuous derivative, it must be increasing in an interval including h^*. Call h' the sup of the interval where F is increasing. So, it must be that for some $h' > h^*$ we have $F(h') > F(h^*) \geq X$. But then, the condition is still true, so we can repeat the reasoning, so h' cannot be the supremum. So, if $X \leq F(h^*)$ for some h^*, then it is increasing for all $h \in [h^*, 1]$. In particular, choosing $X \leq F(0)$ implies F is increasing for all $h \in [0, 1]$ and so we can conclude $F(0) < F(1)$, which is the thesis.

□

Consider the equilibrium profile B^*. The best reply function (23) is increasing in k_i and n_i by Lemma B.1. So, since the game is supermodular, it follows that in equilibrium each coefficient $\overline{B}_j$ is increasing in each k_i and n_i.

To compare the equilibrium values of the coefficients, we apply the theory of monotone comparative statics to modified best reply functions. The argument will follow Lemma 1 of Lazzati (2013), using the ranking of best replies provided in Lemma H.1 above.[18]

Consider the unique equilibrium profile B^*. All firms in the same layer are identical, and use an identical coefficient: so, we index the profile directly with the layers. Assume $n_i > n_j$. Suppose by contradiction that $B_i^* \leq B_j^*$. Now using Lemma H.1 and letting $X = B_i^*$, we have $\overline{BR}(\overline{\Lambda}_j(X, B_{-i,j}^*), n_j, k) = B_j^*$, and so the assumption $X \leq \overline{BR}(\overline{\Lambda}_j(X, B_{-i,j}^*), n_j, k)$ is satisfied by assumption. So, we can conclude that:

$$\overline{BR}(\overline{\Lambda}_i(B_i^*, B_{-i,j}^*), n_i, k) > \overline{BR}(\overline{\Lambda}_j(B_i^*, B_{-i,j}^*), n_j, k) \tag{75}$$

Moreover, since the best reply is increasing we get:

$$B_i^* = \overline{BR}(\overline{\Lambda}_i(B_j^*, B_{-i,j}^*), n_i, k) \geq \overline{BR}(\overline{\Lambda}_i(B_i^*, B_{-i,j}^*), n_i, k)$$

[18] The argument is a slight modification of Lazzati (2013), because in our context we cannot assume, as in that paper, that $\overline{BR}_i(X, B_{-i,j}) > \overline{BR}_j(X, B_{-i,j})$ for *all* X and $B_{i,j}$ (this is, in general, false). It turns out that the proof works with the weaker assumption that $\overline{BR}_i(X, B_{-i,j}^*) > \overline{BR}_j(X, B_{-i,j}^*)$ for all $X \leq B_i^*, B_j^*$, that is what Lemma H.1 provides.

Using again the assumption, we also conclude that:

$$B_j^* = \overline{BR}(\overline{\Lambda}_j(B_i^*, B_{-i,j}^*), n_j, k) \geq B_i^*$$

Combining the last two inequalities we find:

$$\overline{BR}(\overline{\Lambda}_j(B_i^*, B_{-i,j}^*), n_j, k) \geq \overline{BR}(\overline{\Lambda}_i(B_i^*, B_{-i,j}^*), n_i, k),$$

that contradicts Equation (75). Hence, it cannot be that $B_i^* \leq B_j^*$ and so we conclude that $B_i^* > B_j^*$. An analogous reasoning proves that the same is true when $n_i = n_j = n^*$ and we vary k_i. □

H.3 Proof of Lemma B.3

To see that the equilibrium slopes are nonzero, consider the profile in which for all i $\overline{B}_i = \varepsilon$. In this case, $\Lambda_i^{out} + \Lambda_i^{in} = \frac{1}{\varepsilon}\sum_{j\neq i}\frac{1}{n_j} + \frac{1}{B_c}$. So, the best reply is:

$$\begin{aligned}\overline{B}_i' &= \left(k_i^{-1} + \left(\left(\frac{1}{\varepsilon}\sum_{j\neq i}\frac{1}{n_j} + \frac{1}{B_c}\right)^{-1} + (n_i-1)\varepsilon\right)^{-1}\right)^{-1} \\ &= \varepsilon\left(\varepsilon k_i^{-1} + \left(\left(\sum_{j\neq i}\frac{1}{n_j} + \frac{\varepsilon}{B_c}\right)^{-1} + (n_i-1)\right)^{-1}\right)^{-1}\end{aligned}$$

If $N = 1$, then $\sum_{j\neq i}\frac{1}{n_j} = 0$, so $\overline{B}_i' \to k_i$.

If $N > 1$:

$$\lim_{\varepsilon\to 0}\frac{\overline{B}_i'}{\varepsilon} = \left(\sum_{j\neq i}\frac{1}{n_j}\right)^{-1} + (n_i - 1)$$

and if $n_i \geq 2$ then the above is larger than 1, meaning that $\overline{B}_i' > \varepsilon$. Iterating the best reply, we obtain an increasing sequence that must converge to the equilibrium $\overline{B}_i$. So, there exists an $\underline{\varepsilon} > 0$ such that for all i $\overline{B}_i > \underline{\varepsilon}$, and so the equilibrium is nonzero. □

H.4 Proof of Proposition 4

We need the computation in the following Lemma, proven below.

Lemma H.2. Assume $n_i \geq 2$ for all layers, so that the equilibrium is interior. Denote $\frac{\partial \overline{B}_i}{\partial \overline{\Lambda}_i}$ the derivative $\frac{\partial \overline{BR}(\overline{\Lambda}_i(\overline{B}_{-i}), n_i, k_i)}{\partial \overline{\Lambda}}$, where $\overline{BR}$ and $\overline{\Lambda}_i$ are the functions defined in Lemma B.1. Both in the multilateral and unilateral case, the impact of a productivity shock is:

$$\frac{\partial Q}{\partial k_i} = \frac{Q^2 B_c}{A} \frac{1}{1 - \frac{\partial \overline{B}_i}{\partial \overline{\Lambda}_i} \frac{\partial \overline{\Lambda}_i}{\partial \overline{B}_j} \frac{\partial \overline{B}_j}{\partial \overline{\Lambda}_j} \frac{\partial \overline{\Lambda}_j}{\partial \overline{B}_i}} \left(\frac{1}{n_i \overline{B}_i^2} + \frac{1}{n_j \overline{B}_j^2} \frac{\partial \overline{B}_j}{\partial \overline{\Lambda}_j} \frac{\partial \overline{\Lambda}_j}{\partial \overline{B}_i} \right) \left(\frac{\overline{B}_i}{k_i} \right)^2$$

The expression of $\frac{\partial \overline{\Lambda}_i}{\partial \overline{B}_j}$ depends on whether market power is uni or multilateral. With multilateral market power:

$$\frac{\partial \overline{\Lambda}_i}{\partial \overline{B}_j} = -\frac{1}{n_j \overline{B}_j^2},$$

so that:

$$\begin{aligned}
\frac{\partial Q}{\partial k_i} &= \frac{Q^2 B_c}{A} \frac{1}{1 - \frac{\partial \overline{B}_i}{\partial \overline{\Lambda}_i} \frac{\partial \overline{\Lambda}_i}{\partial \overline{B}_j} \frac{\partial \overline{B}_j}{\partial \overline{\Lambda}_j} \frac{\partial \overline{\Lambda}_j}{\partial \overline{B}_i}} \left(\frac{1}{n_i k_i^2} - \frac{1}{n_j \overline{B}_j^2} \frac{\partial \overline{BR}_j}{\partial \Lambda_j} \frac{1}{n_i \overline{B}_i^2} \left(\frac{\overline{B}_i}{k_i} \right)^2 \right) \\
&= \frac{Q^2 B_c}{A} \frac{1}{1 - \frac{\partial \overline{B}_i}{\partial \overline{\Lambda}_i} \frac{\partial \overline{\Lambda}_i}{\partial \overline{B}_j} \frac{\partial \overline{B}_j}{\partial \overline{\Lambda}_j} \frac{\partial \overline{\Lambda}_j}{\partial \overline{B}_i}} \left(\frac{1}{n_i k_i^2} - \frac{1}{n_j \overline{B}_j^2} \frac{\partial \overline{BR}_j}{\partial \Lambda_j} \frac{1}{n_i k_i^2} \right)
\end{aligned} \tag{76}$$

So, under homogeneity, we conclude $\frac{dQ}{dk_1} |_{k_1=k} = \frac{dQ}{dk_2} |_{k_2=k}$.

With unilateral market power, we have:

$$\frac{\partial \overline{\Lambda}_2}{\partial B_1} = -\frac{1}{n_1 B_1^2}, \quad \frac{\partial \overline{\Lambda}_1}{\partial B_2} = 0,$$

and so:

$$\begin{aligned}
\frac{\partial Q}{\partial k_1} &= \frac{Q^2 B_c}{A} \left(\frac{1}{n_1 k_1^2} - \frac{1}{n_2 B_2^2} \frac{\partial \overline{BR}_2}{\partial \overline{\Lambda}_2} \frac{1}{n_1 B_1^2} \left(\frac{B_1}{k_1} \right)^2 \right) \\
\frac{\partial Q}{\partial k_2} &= \frac{Q^2 B_c}{A} \frac{1}{n_2 k_2^2}
\end{aligned} \tag{77}$$

Since by Lemma B.1 $\frac{\partial \overline{BR}_2}{\partial \overline{\Lambda}_2} < 0$, with unilateral market power, the effect of a decrease in cost upstream is smaller: $\frac{dQ^{unilateral}}{dk_1} |_{k_1=k} > \frac{dQ^{unilateral}}{dk_2} |_{k_2=k}$.

□

H.5 Proof of Lemma H.2

The quantity consumed can be written as:

$$Q = A\frac{1}{B_c\Lambda_c + 1}, \quad \Lambda_c = \sum_i \frac{1}{n_i\overline{B}_i}$$

So, the response to a shock is:

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}k_i} &= -A\frac{B_c}{(B_c\Lambda_c+1)^2}\frac{d\Lambda_c}{dk_i}\\ &= A\frac{B_c}{(B_c\Lambda_c+1)^2}\left(\frac{1}{n_iB_i^2}\frac{dB_i}{dk_i} + \frac{1}{n_jB_j^2}\frac{dB_j}{dk_i}\right)\end{aligned}$$

Express the equilibrium equations using Lemma B.1. Then, totally differentiate:

$$\begin{aligned}\frac{dB_i}{dk_i} &= \frac{\partial\overline{BR}_i}{\partial k_i} + \frac{\partial\overline{BR}_i}{\partial\overline{\Lambda}_i}\frac{\partial\overline{\Lambda}_i}{\partial B_j}\frac{dB_j}{dk_i}\\ \frac{dB_j}{dk_i} &= \frac{\partial\overline{BR}_j}{\partial k_i} + \frac{\partial\overline{BR}_j}{\partial\overline{\Lambda}_j}\frac{\partial\overline{\Lambda}_j}{\partial B_i}\frac{dB_i}{dk_i}\end{aligned}$$

The direct impact of k_i is:

$$\frac{\partial\overline{BR}_i}{\partial k_i} = \left(\frac{B_i}{k_i}\right)^2, \quad \frac{\partial\overline{BR}_j}{\partial k_i} = 0$$

Solving the linear system, we get:

$$\begin{pmatrix}\frac{dB_i}{dk_i}\\ \frac{dB_j}{dk_i}\end{pmatrix} = \begin{pmatrix}1 & -\frac{\partial B_i}{\partial\overline{\Lambda}_i}\frac{\partial\overline{\Lambda}_i}{\partial B_j}\\ -\frac{\partial B_j}{\partial\overline{\Lambda}_j}\frac{\partial\overline{\Lambda}_j}{\partial B_i} & 1\end{pmatrix}^{-1}\begin{pmatrix}\left(\frac{B_i}{k_i}\right)^2\\ 0\end{pmatrix}$$

$$= \frac{1}{1-\frac{\partial B_i}{\partial\overline{\Lambda}_i}\frac{\partial\overline{\Lambda}_i}{\partial B_j}\frac{\partial B_j}{\partial\overline{\Lambda}_j}\frac{\partial\overline{\Lambda}_j}{\partial B_i}}\begin{pmatrix}1 & \frac{\partial B_i}{\partial\overline{\Lambda}_i}\frac{\partial\overline{\Lambda}_i}{\partial B_j}\\ \frac{\partial B_j}{\partial\overline{\Lambda}_j}\frac{\partial\overline{\Lambda}_j}{\partial B_i} & 1\end{pmatrix}\begin{pmatrix}\left(\frac{B_i}{k_i}\right)^2\\ 0\end{pmatrix}$$

Where the inverse matrix exists because, using the calculations in B.1:

$$\begin{aligned}
\frac{\partial B_i}{\partial \overline{\Lambda}_i}\frac{\partial \overline{\Lambda}_j}{\partial B_i} &= \frac{\left(\overline{\Lambda}_i^{-1} + (n_i - 1)B_i\right)^{-2}\overline{\Lambda}_i^{-2}}{B_i^{-2} - (n_i - 1)\left(\overline{\Lambda}_i^{-1} + (n_i - 1)B_i\right)^{-2}}\frac{1}{n_i B_i^2} \\
&= \frac{\overline{\Lambda}_i^{-2}}{\left(\overline{\Lambda}_i^{-1} + (n_i - 1)B_i\right)^{2} - (n_i - 1)B_i^2}\frac{1}{n_i} \\
&= \frac{\overline{\Lambda}_i^{-2}}{\overline{\Lambda}_i^{-2} + 2\overline{\Lambda}_i^{-1}(n_i - 1)B_i + \left[(n_i - 1)^2 - (n_i - 1)\right]B_i^2}\frac{1}{n_i} < \frac{1}{n_i} < 1
\end{aligned}$$

since $(n_i - 1)^2 - (n_i - 1) = n_i^2 - 3n + 2 = (n_i - 2)(n_i - 1) \geq 0$ if $n_i \geq 2$, and at least one between i, j must have $n \geq 2$ otherwise the interior equilibrium does not exist. □