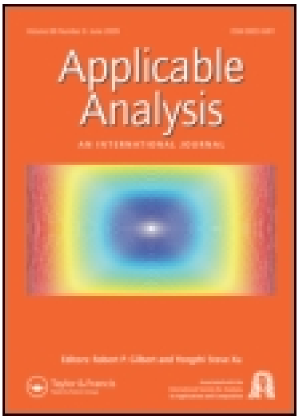

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A One-dimensional Variational Problem with Gradient Constraint

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In this note we give an explicit characterization of the minimum value of one-dimensional integral variational problem with gradient constraint by a positive measurable function.

Keywords: Variational problem; Gradient constraints; Limit problem; Explicit formula

AMS Subject Classification: 49J45

INTRODUCTION

The study of homogenization problems with gradient constraints is a widely developed topic studied by many authors (see for example [2–5,7]). The main purpose of homogenization is to understand the behavior of the so-called "limit problem", that can be done in many cases via a function characterized as the minimum value of an auxiliary variational problem on the unitary cell $Y = [0, 1]^n$. For example a typical setting in the quadratic case with gradient constraint in one dimension, we are lead to consider the function $f_\infty : \mathbb{R} \to [0, +\infty]$ given by

$$f_\infty(\xi) = \begin{cases} \min\limits_{u \in K_\xi} \displaystyle\int_0^1 a(x)(u'(x))^2\, dx & \text{if } K_\xi \notin \emptyset, \\ +\infty & \text{if } K_\xi \in \emptyset, \end{cases}$$

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where

$$a(x) \in L^\infty(Y) : 0 < \gamma_1 \le a(x) \le \gamma_2 \quad \text{a.e. } x \in [0,1], \\ \varphi : [0,1] \to [0, +\infty] \text{ is a measurable function} \tag{1}$$

and the set K_ξ of admissible functions is given by

$$K_\xi = \left\{v \in H^1([0,1]) \colon u(0) = 0,\ u(1) = \xi,\ |u'(x)| \le \varphi(x) \text{ a.e. in } [0,1]\right\}.$$

A more explicit characterization of the function f_∞ is in general not known if one has not further informations on $a(x)$ and $\varphi(x)$. The aim of this note is to give such a characterization in the one-dimensional case when $a(x)$ and $\varphi(x)$ satisfy (1).

In one dimension (about one-dimensional variational problems see [1]) another characterization of f_∞ is obtained in paper [6], even replacing $a(x)(u'(x))^2$ with a more general integrand $f(x, u'(x))$. This characterization involves duality arguments and it seems to us that can be effectively used to obtain our formula only when φ is constant.

Our arguments are effective only in the one-dimensional case and we are not aware of explicit formulas in dimensions of more than one.

The formula we obtain is the following

$$f_\infty(\xi) = \begin{cases} \int_0^1 a(x)\varphi^2_{c_\xi}(x)\,dx < +\infty & \text{if } 0 \le |\xi| < \int_0^1 \varphi(x)\,dx, \\ \int_0^1 a(x)\varphi^2(x)\,dx \le +\infty & \text{if } |\xi| = \int_0^1 \varphi(x)\,dx, \\ +\infty & \text{if } |\xi| > \int_0^1 \varphi(x)\,dx, \end{cases}$$

where

$$\varphi_c(x) = \min\left\{\frac{c}{a(x)}, \varphi(x)\right\} = \frac{c\varphi(x)}{\max\left\{a(x)\varphi(x), c\right\}}, \quad c \in [0, +\infty) \tag{2}$$

and c_ξ is such that

$$\int_0^1 \varphi_{c_\xi}(x)\,dx = |\xi| < \int_0^1 \varphi(x)\,dx. \tag{3}$$

PRELIMINARY RESULTS

Let us denote the Lebesgue measure of a subset $E \subseteq \mathbb{R}$ by $|E|$.

LEMMA 1 *Let $a(x)$ and $\varphi(x)$ verifying* (1). *Let us consider*, *fixed a measurable set $E \subseteq [0,1]$*, *the problem*

$$\lambda_\alpha = \min_{v \in C_\alpha} \int_E a(x)v^2(x)\,dx \tag{4}$$

where

$$C_\alpha = \left\{ v \in L^2(E)\colon |v(x)| \le \varphi(x) \text{ a.e. in } E, \int_E v(x)\,dx = \alpha \right\}.$$

Then

(i) *if* $C_\alpha \neq \emptyset$, *problem* (4) *has unique solution, i.e. there exists a unique function* $v_\alpha \in C_\alpha$ *such that* $\int_E a(x)v_\alpha^2(x)\,dx = \lambda_\alpha$;

(ii) $C_\alpha \neq \emptyset$ *if and only if*

$$|\alpha| < \int_E \varphi(x)\,dx \quad or \quad |\alpha| = \int_E \varphi(x)\,dx, \quad \varphi \in L^2(E);$$

(iii) *if* $C_\alpha \neq \emptyset$, *then* $\alpha \cdot v_\alpha(x) \ge 0$ *a.e. in* E;

(iv) *if* $\varphi(x) = +\infty$ *a.e. in* E, *then*

$$v_\alpha(x) = \frac{\alpha}{a(x)} \frac{1}{\int_E (1/a(x))\,dx};$$

(v) *if* $C_\alpha \neq \emptyset$ *and*

$$\left| \left\{ x\colon \frac{|\alpha|}{\int_E (1/a(x))\,dx} > a(x)\varphi(x) \right\} \right| > 0, \tag{5}$$

then

$$\left| \left\{ x\colon |v_\alpha(x)| = \varphi(x) \right\} \right| > 0.$$

Proof It is easy to see that C_α is a closed convex subset of $L^2(E)$. Moreover the bilinear form $A : L^2(E) \times L^2(E) \to [0, +\infty]$

$$A(u, v) = \int_E a(x)u(x)v(x)\,dx$$

is coercive and continuous; therefore (i) follows from Stampacchia theorem.

The proof of (ii) and (iii) is elementary.

Let us prove (iv). Let $\lambda = \alpha/(\int_E 1/a(x)\,dx)$ and consider $w(x) = v_\alpha(x) - (\lambda/a(x))$, and assume, by contradiction, $\int_E |w(x)|\,dx > 0$.

Let $w^+(x) = \max\{0, w(x)\}$ and $w^-(x) = -\min\{0, w(x)\}$. Since $\int_E w(x)\,dx = 0$, we have

$$\int_E w^+(x)\,dx = \int_E w^-(x)\,dx = \frac{1}{2}\int_E |w(x)|\,dx > 0. \tag{6}$$

Then there exist $\varepsilon > 0$ and two measurable subsets of positive measure F_1, F_2 of E such that $w^+(x) \ge \varepsilon$ in F_1 and $w^-(x) \ge \varepsilon$ in F_2.

We can assume, without loss of generality, that $|F_1| = |F_2| > 0$. We have

$$a(x)v_\alpha(x) \geq \lambda + \varepsilon a(x) \geq \lambda + \varepsilon\gamma_1 \quad \text{on } F_1,$$

$$a(x)v_\alpha(x) \leq \lambda - \varepsilon a(x) \leq \lambda - \varepsilon\gamma_1 \quad \text{on } F_2.$$

Then let $\mu > 0$ and $v_{\alpha,\mu}$ defined by

$$v_{\alpha,\mu}(x) = \begin{cases} v_\alpha(x) - \mu & \text{if } x \in F_1, \\ v_\alpha(x) + \mu & \text{if } x \in F_2, \\ v_\alpha(x) & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} \int_E a(x)\Big(v^2_{\alpha,\mu}(x) - v^2_\alpha(x)\Big)\,dx &= -2\mu \int_{F_1} a(x)v_\alpha(x)\,dx \\ &\quad + 2\mu \int_{F_2} a(x)v_\alpha(x)\,dx + \mu^2 \int_{F_1\cup F_2} a(x)v^2_\alpha(x)\,dx \\ &\leq -2\mu\varepsilon\gamma_1|F_1| + \mu^2 \int_{F_1\cup F_2} a(x)v^2_\alpha(x)\,dx \end{aligned}$$

that for a suitable choice of $\mu > 0$ is negative, that is a contradiction.

Let us prove (v). Let $\alpha \geq 0$ (the other case is analogous). By (iii) we have

$$v_\alpha(x) \geq 0 \quad \text{a.e. in } E. \tag{7}$$

Again by contradiction, let us assume

$$v_\alpha(x) < \varphi(x) \quad \text{a.e. in } E. \tag{8}$$

Let us set, as in the proof of (iv)

$$\lambda = \frac{\alpha}{\int_E (1/a(x))\,dx}$$

and

$$w(x) = v_\alpha(x) - \frac{\lambda}{a(x)}.$$

Then

$$\int_E w(x)\,dx = 0.$$

If $w(x) = 0$ a.e. on E, we will have $v_\alpha(x) = (\lambda/a(x))$ a.e. in E, and therefore

$$\frac{|\alpha|}{a(x)\int_E (1/a(x))\,dx} = |v_\alpha(x)| \leq \varphi(x) \quad \text{a.e. in } E,$$

that contradicts (5).

Arguing as in (6) and by (8) we have

$$\begin{aligned}\left|\left\{x \in E\colon w^+(x) > 0, \varphi(x) > 0\right\}\right| &= \left|\left\{x \in E\colon w^+(x) > 0\right\}\right| > 0\\ \left|\left\{x \in E\colon w^-(x) < 0, v_\alpha(x) < \varphi(x)\right\}\right| &= \left|\left\{x \in E\colon w^-(x) > 0\right\}\right| > 0.\end{aligned} \tag{9}$$

Then there exist $\varepsilon > 0$ and two measurable subsets of positive measure F_1, F_2 of E such that

$$\begin{aligned}\varphi(x) \geq \varepsilon \quad \text{and} \quad w^+(x) \geq \varepsilon \ \text{ if } x \in F_1,\\ w^-(x) \geq \varepsilon \quad \text{and} \quad v_\alpha(x) \leq \varphi(x) - \varepsilon \ \text{ if } x \in F_2\end{aligned}.$$

We can assume $|F_1| = |F_2| > 0$. Let $\varepsilon > \mu > 0$ and

$$v_{\alpha,\mu}(x) = \begin{cases} v_\alpha(x) - \mu & \text{if } x \in F_1,\\ v_\alpha(x) + \mu & \text{if } x \in F_2,\\ v_\alpha(x) & \text{if } x \notin F_1 \cup F_2.\end{cases}$$

Therefore by (9) and (7)

$$\begin{aligned} -\varphi(x) < -\mu \leq v_{\alpha,\mu}(x) \leq \varphi(x) \quad &\text{if} \quad x \in F_1,\\ 0 \leq v_{\alpha,\mu}(x) \leq \varphi(x) \quad &\text{if} \quad x \in F_2,\\ |v_{\alpha,\mu}(x)| = |v_\alpha(x)| \leq \varphi(x) \quad &\text{if} \quad x \notin F_1 \cup F_2\end{aligned}$$

and

$$\int_E v_{\alpha,\mu}(x)\,dx = \alpha - \mu|F_1| + \mu|F_2| = \alpha.$$

So $v_{\alpha,\mu} \in C_\alpha$. Then in the same way as in the proof of (iv), we can prove

$$\int_E a(x)v^2_{\alpha,\mu}(x)\,dx < \int_E a(x)v^2_\alpha(x)\,dx,$$

if μ is small enough. ∎

Corollary 1 *Let* $a(x)$, $\varphi(x)$ *satisfy* (1). *Let us consider the problem*

$$\min_{u \in K_\xi} \int_0^1 a(x)(u'(x))^2\,dx \tag{10}$$

where

$$K_\xi = \{u \in H^1([0,1]):\ u(0) = 0,\ u(1) = \xi,\ |u'(x)| \le \varphi(x)\ a.e.\ in\ [0,1]\}.$$

Then

(i) *if* $K_\xi \neq \emptyset$, *problem* (10) *has unique solution, i.e. there exists a unique function* $u_\xi(x)$ *such that* $f_\infty(\xi) = \int_0^1 a(x)(u'_\xi(x))^2\,dx$;
(ii) $K_\xi \neq \emptyset$ if and only if

$$|\xi| < \int_E \varphi(x)\,dx \quad or \quad |\xi| = \int_E \varphi(x)\,dx, \varphi \in L^2([0,1]);$$

(iii) *if* $K_\xi \neq \emptyset$, *then* $\xi \cdot u_\xi(x) \ge 0$ *a.e. in* E;
(iv) *if* $\varphi(x) = +\infty$, *then*

$$u'_\xi(x) = \frac{\xi}{a(x)} \frac{1}{\int_0^1 (1/a(x))\,dx};$$

(v) *if* $K_\xi \neq \emptyset$ *and*

$$\left|\left\{x:\ \frac{|\xi|}{\int_0^1 (1/a(x))\,dx} > a(x)\varphi(x)\right\}\right| > 0, \tag{11}$$

then

$$\left|\left\{x: |u'_\xi(x)| = \varphi(x)\right\}\right| > 0.$$

Proof Let $E = [0,1]$, $\alpha = \xi$ and let us consider minimum problem (4). Then it is easy to see that $K_\xi \neq \emptyset$ if and only if $C_\xi \neq \emptyset$, and u_ξ is solution of problem (10) if and only if u'_ξ is solution of problem (4). Then the thesis follows from Lemma 1. ■

LEMMA 2 *Let* $a(x)$ *and* $\varphi(x)$ *satisfy* (1), $c \in [0, +\infty]$, $\varphi_c(x)$ *defined in* (2) *and*

$$\psi(c) = \int_0^1 \varphi_c(x)\,dx. \tag{12}$$

Let c_0 *be defined by*

$$c_0 = \begin{cases} \|a \cdot \varphi\|_\infty & \text{if} \quad \varphi \in L^\infty([0,1]), \\ +\infty & \text{if} \quad \varphi \notin L^\infty([0,1]). \end{cases}$$

Then ψ *is strictly increasing and Lipschitz in* $[0, c_0)$, $\psi(0) = 0$ *and* $\lim_{c \to c_0^-} \psi(c) = \int_0^1 \varphi(x)\,dx$.

Proof The proof is elementary. ■

PROOF OF THE FORMULA

It is easy to verify the formula if

$$|\xi| > \int_0^1 \varphi(x)\,dx, \quad \text{(in this case } K_\xi = \emptyset)$$

or

$$|\xi| = \int_0^1 \varphi(x)\,dx,$$

since in this case

$$K_\xi = \begin{cases} \emptyset & \text{if } \varphi \notin L^2([0,1]), \\ \{u_\xi\} & \text{otherwise} \end{cases}$$

where $u_\xi(x) = \text{sign}(\xi)\int_0^x \varphi(t)\,dt$.

Let $0 \le |\xi| < \int_0^1 \varphi(x)\,dx$.

By Lemma 2 there exists a unique c_ξ such that $\psi(c_\xi) = |\xi|$.

We claim that the solution $u_\xi(x)$ of the problem (10) is

$$u_\xi(x) = \text{sign }(\xi)\int_0^x \varphi c_\xi(t)dt \tag{13}$$

and therefore

$$f_\infty(\xi) = \int_0^1 a(x)\varphi^2 c_\xi(x)\,dx. \tag{14}$$

It is trivial to verify that the function $\text{sign}(\xi)\int_0^x \varphi_{c_\xi}(t)\,dt \in K_\xi$ and so $K_\xi \neq \emptyset$.

Assume now $\xi \geq 0$ (the proof is analogous if $\xi < 0$). By (iii) of Lemma 1, $u'_\xi(x) \geq 0$ a.e. in $[0,1]$.

Let us observe that if u_ξ is the solution of problem (10), fixed a measurable set $E \subseteq [0,1]$, then u'_ξ is the solution of problem (4) in E with $\alpha = \int_E u'_\xi(x)\,dx$.

If not, let $v_\alpha(x)$ the solution of problem (4) and $\tilde{u}_\xi$ defined by

$$\tilde{u}'_\xi(x) = \begin{cases} u'_\xi(x) & \text{in } [0,1]\setminus E, \\ v_\alpha(x) & \text{in } E, \end{cases}$$

and

$$\tilde{u}_\xi(0) = 0.$$

Then it is easy to verify that $\tilde{u}_\xi \in K_\xi$. We have

$$\int_0^1 a(x)\Big(\tilde{u}'_\xi(x)\Big)^2 dx = \int_{[0,1]\setminus E} a(x)\Big(\tilde{u}'_\xi(x)\Big)^2 dx + \int_E a(x)v_\alpha^2(x)\,dx < \int_0^1 a(x)\Big(u'_\xi(x)\Big)^2 dx$$

that is a contradiction.

Let us set $E = \{x\colon u'_\xi(x) < \varphi(x)\}$, $\alpha = \int_E u'_\xi(x)\,dx$ and $\lambda = (\alpha/\int_E (1/a(x))\,dx)$.

Let us observe that $|E| > 0$. Otherwise

$$\left|\int_0^1 u'_\xi(x)\,dx\right| = \int_0^1 \varphi(x)\,dx > |\xi|,$$

that is a contradiction.

We know that $u'_\xi(x) = v_\alpha(x) < \varphi(x)$ a.e. in E. Then by (v) of Lemma 1

$$\frac{\alpha}{\int_E (1/a(x))\,dx} \le a(x)\varphi(x) \quad \text{a.e. in } E$$

therefore

$$\frac{\lambda}{a(x)} \le \varphi(x) \quad \text{a.e. in } E. \tag{15}$$

Since $\int_E (\lambda/a(x))\,dx = \alpha$ by (iv) of Lemma 1 and (15) we have $v_\alpha(x) = (\lambda/a(x))$ and $u'_\xi(x) = (\lambda/a(x))$ a.e. in E. Then, by definition of E, we have

$$\frac{\lambda}{a(x)} < \varphi(x) \quad \text{a.e. in } E. \tag{16}$$

Let us now show that $\lambda = c_\xi$.

By (16) it follows

$$|u'_\xi(x)| \ge \varphi_\lambda(x) \quad \text{a.e. in } [0,1].$$

If $\lambda > c_\xi$ then

$$\xi = \int_0^1 u'_\xi(x)\,dx \ge \int_0^1 \varphi_\lambda(x)\,dx = \psi(\lambda) > \psi(c_\xi) = \xi$$

that is a contradiction.

If $\lambda < c_\xi$, let $G = \left\{x\colon (\lambda/a(x)) < \varphi(x)\right\}$. By (16) we have $|E \setminus G| = 0$.

Moreover $|G \setminus E| > 0$; if not, we have

$$\xi = \int_0^1 u'_\xi(x)\,dx = \int_E \frac{\lambda}{a(x)}\,dx + \int_{[0,1]\setminus E} \varphi(x)\,dx = \int_G \frac{\lambda}{a(x)}\,dx + \int_{[0,1]\setminus G} \varphi(x)\,dx = \psi(\lambda) < \xi.$$

Therefore

$$|E| = \left|\left\{x\colon u'_\xi(x) = \frac{\lambda}{a(x)} < \varphi(x)\right\}\right| > 0 \text{ and } |G \setminus E| = \left|\left\{x\colon u'_\xi(x) = \varphi(x) > \frac{\lambda}{a(x)}\right\}\right| > 0.$$

Then there exist $\varepsilon > 0$ and two measurable sets of positive measure F_1 and F_2 such that

$$u'_\xi(x) = \frac{\lambda}{a(x)} \leq \varphi(x) - \varepsilon \quad \text{a.e. in } F_1,$$
$$u'_\xi(x) = \varphi(x) > \frac{\lambda}{a(x)} + \varepsilon \quad \text{a.e. in } F_2.$$

Without loss of generality we can assume $|F_1| = |F_2| > 0$.

Let $\mu < \varepsilon$ and $\tilde{u}_\xi$ defined by $\tilde{u}_\xi(0) = 0$ and

$$\tilde{u}'_\xi(x) = \begin{cases} u'_\xi(x) + \mu & \text{if } x \in F_1, \\ u'_\xi(x) - \mu & \text{if } x \in F_2, \\ u'_\xi(x) & \text{if } x \notin F_1 \cup F_2. \end{cases}$$

We observe that $\tilde{u}_\xi$ is an admissible function. Moreover

$$\begin{aligned} \int_0^1 a(x)\Big(\tilde{u}'_\xi(x)\Big)^2 dx - \int_0^1 a(x)\Big(u'_\xi(x)\Big)^2 dx &= \int_{F_1} a(x)\left[\left(\frac{\lambda}{a(x)} + \mu\right)^2 - \left(\frac{\lambda}{a(x)}\right)^2\right] dx \\ &\quad + \int_{F_2} a(x)\big[(\varphi(x) - \mu)^2 - \varphi^2(x)\big]\, dx \\ &\leq 2\mu\lambda|F_1| - 2\mu \int_{F_2} a(x)\varphi(x)\, dx \\ &\quad + \mu^2 \int_{F_1 \cup F_2} a(x)\, dx \\ &\leq -2\mu\lambda\varepsilon\gamma_1 + \mu^2 \int_{F_1 \cup F_2} a(x)\, dx \end{aligned}$$

that for a suitable choice of $\mu > 0$ is negative, that is a contradiction.

Then

$$a(x)\varphi(x) > \lambda + \varepsilon a(x) > \lambda + \varepsilon\gamma_1.$$

Once proved $\lambda = c_\xi$, denoted by $G = \big\{x\colon (c_\xi/a(x)) < \varphi(x)\big\}$, as above it is easy to see that

$$|G \setminus E| + |E \setminus G| = 0; \tag{17}$$

by (17) we get

$$u'_\xi(x) = \varphi_{c_\xi}(x) \quad \text{a.e in } [0, 1],$$

and so the thesis. ∎

References

[1] G. Buttazzo, M. Giaquinta and S. Hildebrandt (1998). *One-dimensional Variational Problems, An Introduction*. Oxford Lecture Series in Mathematics and its Applications, 15. The Clarendon Press, Oxford University Press, New York.

[2] L. Carbone, D. Cioranescu, R. de Arcangelis and A. Gaudiello (2002). Homogenization of unbounded functionals and nonlinear elastomers. The general case. *Asymptot. Anal.*, **29**(3–4), 221–272.

[3] L. Carbone and R. de Arcangelis (2002). Unbounded functionals in the Calculus of variations. Representation, relaxation and homogenization. *Monographs and Surveys in Pure and Applied Mathematics*, Vol. 125. Chapman & Hall/CRC, Boca Raton.

[4] L. Carbone and S. Salerno (1984/85). Further results on a problem of homogenization with constraints on the gradient. *J. Analyse Math.*, **44**, 1–20.

[5] G. Cardone, A. Corbo Esposito, G.A. Yosifian and V.V. Zhikov. Homogenization of some problems with gradient constraints. *Asymptot. Anal.* (to appear).

[6] N. Fusco (1979). One-dimensional Γ-convergence. *Boll. Un. Mat. Ital. B (5)*, **16**(1), 74–86.

[7] V.V. Jikov (Zhikov), S.M. Kozlov and O.A. Oleinik (1994). *Homogenization of Differential Operators and Integral Functionals*. Springer-Verlag, Berlin.