Quality of students' explanations in describing multiplication algorithms: the case of Napier bones

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We describe a didactic cycle involving the Napier bones algorithm, included in a wider research whose aim is to investigate whether and how the use of different kinds of artifacts can affect students' explanations, supporting the understanding of some aspects of the multiplication. We present and discuss some data regarding group and individual tasks that allowed us to investigate how students' explanations were and what semiotic representations they chose when using Napier bones.

Keywords: Semiotic mediation, explanations, multiplication, artifacts.

Introduction

Algorithms for mathematical operations, which students experience at school, are not the only ones. In particular, for multiplication there are many alternative procedures associated with graphic, symbolic and/or material artifacts. Often, children, after knowing the traditional algorithm (multiplication in column), apply it in a mechanical way, without reflecting on the meaning of the foreseen steps, as a result of the fact that teaching multiplication is frequently based on memorising the multiplication table rather than on a meaningful understanding of the concept and related procedures (Vohra, 2007). Some studies (e.g., Haylock & Cockburn, 2008) suggest that children develop their mathematical understanding through language, symbols, concrete experience, images and manipulatives, which make mathematical concepts and operations more comprehensible to students (Swan & Marshall, 2010). In this sense, we believe that knowledge and use of different algorithms, instead a unique one, could help students to better understand some aspects of the multiplication. Knowledge about algorithms could change from being only instrumental to being also conceptual.

Students also find themselves having to use representations and symbols –typical of each algorithm– that, in addition to everyday language, are interpreted as carrying some meaning. At the same time, however, it should be kept in mind that semiotic mediation may not work. It depends on both the individuals and the characteristics of the tool involved in the mediation (Hasan, 2005). Explanations, produced by students during mathematical activities, could be a useful means for teachers both to see whether the mediation has worked and to observe whether understanding has been achieved.

The aim of our overall research is to investigate whether and how the use of different kinds of artifacts can affect students' explanations, and, therefore, the understanding of some aspects of the multiplication. We conducted a study in a primary school, involving 25 students who participated in mathematical activities focused on 5 different multiplication algorithms (multiplication in column, Chinese multiplication, Arabic multiplication, Napier bones multiplication, Genaille-Lucas rulers multiplication), each characterized by specific symbolic, graphic and/or material artifacts. In this paper we only focus on the Napier bones algorithm, answering to the following research questions: *what are the qualities in the students' explanations and what semiotic representations do the students' explanations draw upon when using Napier bones?*

Theoretical framework.

Our research is framed within the semiotic mediation paradigm, key aspect of the sociocultural theory of cognition, and developed in the work by Hasan (2005) and in that by Bartolini Bussi and Mariotti (2008). For Vygotsky (1978) cognitive development occurs through a *social activity* consisting of a progressive *interiorization* of strategies for using tools and mediation forms, allowing the child to attribute meaning to signs. The interiorization is carried out through *semiotic processes*: the construction of individual knowledge requires shared social experiences involving the production and the interpretation of different systems of signs (such as everyday language, images and symbols). Hasan (2005) refers to *mediation by semiosis*, i.e., mediation through the use of sign systems acting as an abstract tool in changing human activity, and emphasizes the semiotic function performed by language for the development of thought. Bartolini Bussi and Mariotti (2008) elaborated the Theory of Semiotic Mediation (TSM) in order to study links between artifacts, tasks, signs, and the mathematical knowledge to be mediated in the classroom. The teacher assumes the role of *cultural mediator* (Hasan, 2005; Bartolini Bussi & Mariotti, 2008): she mediates mathematical meanings to students through an artifact, exploiting its semiotic potential in relation to the tasks she has designed.

Admitting a link between language and thought, we can hypothesise that the quality of the explanations a student produce is related to the quality of her thinking (Albano et al., 2015; Ferrari, 2017). Within the field of mathematics, the explanation is closely related to 'mathematical understanding' and has to perform different functions: to clarify aspects of one's mathematical thinking; to describe the steps of a procedure used; to tell how one can arrive at a solution; to convince oneself or another person of some claim; to expand students' mathematics learning, as they communicate existing thoughts and can also generate new thought objects. In this way, explanations can be considered as constituent elements of the arguments (Levenson & Barkai, 2013) and, in our view, it is essential to analyse them in order to understand the development of knowledge that occurred during a semiotic activity involving artifacts.

Methodology.

The research is qualitative with an exploratory and descriptive aim: it is limited to collecting, analysing and interpreting data emerging from the specific context under investigation. In this study a single-group *pre-experimental research design* (Campbell & Stanley, 1963) was used, i.e., only the experimental group was considered, without a control group.

Context and overall research design.

The investigation involved 25 students from a fourth primary class of the Istituto Comprensivo 'Calcedonia' in Salerno (Italy), aged between 8 and 10 years.

The overall research design reflected the approach of the TSM (Bartolini Bussi & Mariotti, 2008). Researchers designed group and individual tasks, implemented in 5 didactic cycles, each one lasted about 1 hour, focused on a particular multiplication algorithm. In the first didactic cycle the work groups were formed and they remained the same for each cycle: the class teacher gave the role of 'captain' to five students with a good basic level and, then, four children were randomly assigned to each of them. Teacher was always present in the classroom, but her role was that of an external observer. The activities were conducted by one of the authors, a pre-service teacher who was doing her internship at that school. Each cycle was articulated in 3 phases: a *group activity* by means of tasks designed with the aim of exploring the artifact; a *collective discussion* on the structure and the way of using the artifact, based on the explanations produced by the students in the first phase; an *individual activity* by means of tasks designed to verify the understanding of the algorithm.

The didactic cycle n°4: Napier bones multiplication algorithm.

Napier bones multiplication algorithm is based on a material graphic artifact (Figure 1) consisting of 10 strips numbered from 0 to 9 and a further strip, called ruler, marked with the symbol ×. Each strip is divided into 9 squares, each—except the ruler—cut by a diagonal line running from top right to bottom left, in which are shown the multiples of the number marked at the top. In each of the 9 squares of the ruler there is a number from 0 to 9. To calculate the product between two multi-digit numbers, the bones of the digits of the first factor are placed side by side with the ruler on the right, creating a table with nine lines. Then, you have to consider the lines corresponding to the digits of the second factor, one after the other, starting from the units one. Each line will correspond to a partial product, obtained by summing along the diagonals. The desired product will be obtained as the sum of the partial products (Figure 1).

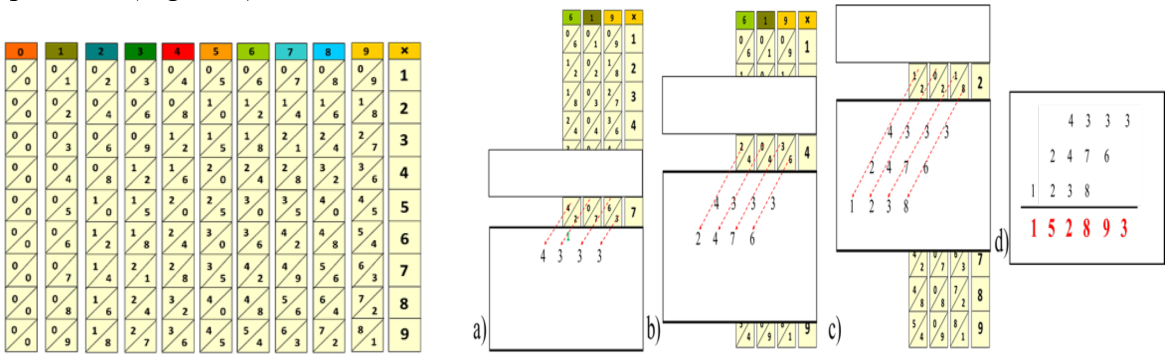

Figure 1: Napier bones (on the left) and four steps calculation of 619×247 (on the right)

The students were not familiar with this artifact, but it was introduced after they had already experienced multiplication in column, Chinese and Arabic algorithms, as part of the educational path. We expected not too many difficulties with the Napier bones, since some algorithm features are similar to those of Arabic multiplication. In the following we focus only on group and individual activities.

Group activity consisted of two tasks: Task 1 and Task 2. Each group was given a paper on which Napier bones were illustrated, without saying what they were. **Task 1** consisted of 3 questions (Bartolini Bussi & Mariotti, 2008): 1) What are they?; 2) What do they look like? You can use not only text, but also drawings to describe the objects; 3) What can they do? Justify your answer. The first two questions were strictly aimed at exploring the artifact and bringing out the students' prior knowledge, on which the new mathematical content would be anchored; the third question involved highlighting the utilization schemes of the artifact, which would then become a tool[1].

Then the students were told that the strips were named Napier bones and were asked to cut out the paper strips so that they could manipulate and use them for **Task 2**, which consisted of 3 questions:

1. Imagine you only have Napier bones available to carry out multiplication. Describe how you could execute the operation 123×6.
2. Describe how you could execute 436×72 multiplication using Napier bones.
3. What similarities and what differences do you notice between Napier bones multiplication and Arabic multiplication?

Starting from the assumption that Napier bones derive from Arabic multiplication, Question 1 aimed to investigate whether students, in exploring and using the artifact, were able to explain how to calculate a multiplication with a one-digit factor; it also aimed to observe what kind of semiotic resource the students used in their explanations. Question 2 had the same aim, but in the case of multiplication with a non-consecutive two-digit factor, which involves adding the two partial products, as in the traditional algorithm. The aim of Question 3 was to investigate whether the students were able to make a comparison between Napier bones algorithm and Arabic multiplication.

For *individual activity* was designed and implemented only **Task 3** consisted of 4 questions:

1. In a third-year class, the math teacher gives each child Napier bones without explaining what they are or what they are used for. Luca can't quite understand how to use them. Explain what they are and show him a few examples, describing step by step how to use them.
2. Luca still shows some difficulty. Execute the same multiplications as in your examples using multiplication in column and explain to him the differences between the two methods.
3. What are the advantages of using Napier bones? And what disadvantages?
4. In your opinion, does the use of Napier bones make multiplication easier than traditional calculus? Justify your answer.

In Question 1, by asking for examples, we wanted to find out if the students were able to explain how to do multiplication using the bones—highlighting algorithm features—and what representation they used to support their answers. The aim of Question 2 was to further investigate whether the students were able to explain the mathematical properties underlying both the traditional algorithm and the Napier bones. The Questions 3 and 4 were aimed at understanding how they perceived a possible use of bones in daily school practice.

Data collection and analysis.

The data were collected by means of *observation* during each phase of the didactic cycle, *structured group/individual tasks*; *field notes* regarding the collective discussions based on students' production.

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[1] According to the instrumental approach (Béguin & Rabardel, 2000) a subject, engaged in a goal-directed activity, can build schemes of instrumented action for an artifact. Thanks to the visible signs elaborated by the solver (e.g., text, words, representation), we can make inferences about the schemes she is developing for the tasks.

In this paper we consider only the explanations provided by the students in *structured group/individual tasks*. As we have just described, for each structured task, students were asked to explain their answers and to use different types of representation. In each group/ individual task, requests such as “Justify your answer”, “Explain how you arrived at the solution” or “What differences do you notice?”, “What are the advantages/disadvantages?” were included in order to stimulate explanations and continuous comparison. Once the data collection was completed, the name chosen by each group (Fulmine, Vento, Tempesta, Pioggia and Neve) was recorded for all group activity protocols. The protocols of the individual activities were numbered from 1 to 25, assigning the same number to each student in the different activities.

For the analysis of the explanations the attention was focused on the different types of representation used by the students: *graphic, verbal*. In addition, for analysing their qualities we were inspired by the criteria as used in the work by Cusi et al. (2017), namely *correctness*, which refers to the absence in the explanation of mathematical errors; *clarity*, which refers to the comprehensibility of the explanation by an interlocutor ; *completeness* refers to the description of all the steps of the algorithm, including the final step leading to the solution. As an example “12 and 17 are 29” can be considered *correct*, *no clear* and *no complete* because the description of the procedure is missing and a classmate might not understand that ‘and’ refers to an addition. Thus, if an explanation is correct, clear, and complete, it is able to perform the functions that Levenson and Barkai (2013) attributed to it.

Results.

In this paper we present only some results from data concerning Question 1 and Question 2 of Task 2 carried out in the group activity and Question 1 of Task 3 carried out in the individual activity.

Task 2 - Group activity.

Concerning **Task 2,** all groups provided an answer to **Question 1**. The analysis has revealed that all of them used *verbal representation* to explain their answer. Almost all of the explanations, however, was not qualitatively adequate, in relation to the chosen criteria.

In fact only one group (Fulmine) produced a *correct, clear and complete* explanation in that, even though it made use of a colloquial register, it explained with care the procedure to be used to do multiplication, showing that the group identified some similarity between the calculation of multiplication with Napier bones and Arabic algorithm.

Fulmine: We took the 1-2-3-X sticks (the one that contains the numbers up to 9) then we did 6+1=7, 2+1=3 and then 8+0=8, and came up with 738 using Arabic multiplication.

The other 4 groups, even though they did not describe the correct procedure and did not specify the various steps to be taken, still provided an explanation that, for 2 groups (Tempesta and Pioggia) was quite *clear*, while for the other 2 (Vento and Neve) was difficult to understand (*no correct, no clear, no complete*). We propose, as examples, the answers given by Pioggia and Neve.

Pioggia: To solve with Napier bones you have to put them in a row, counting only the numbers that you have to multiply. We put a 1-stick, a 2-stick, and another, 3-stick, then put sheets on the other numbers and add up the results of each number. The result is 18.

Neve:	To do the 123×6 multiplication we have to put the 6 in the 3, then the 6 in the 2 and finally the 6 in the 1 then we do the result and at the end the result is 7938.

The data regarding **Question 2** show that all groups provided the answer but the quality of the explanations worsened. None of the groups gave a *complete* answer and 3 out of 5 groups described the use of bones in a *no correct, no clear, no complete* way. As an example, the quality of the explanation provided by Fulmine seems to worsen with respect to the Question 1: the group described the procedure in a fairly *clear* way, but made an error in the sum of the partial products and, consequently, in the final result and they skipped some steps of the whole process.

Fulmine:	We took the 4-3-6-X sticks (the one that holds the numbers 1-9) then we did 436×7 first and then we did 436×2 and finally we added up the two results and it came out 30520.

More *clear and correct* (even if the final result is missing) is the explanation produced by Tempesta, as the group did not explain how to arrange the sticks, nor how to calculate the two partial products.

Tempesta:	First we looked at the numbers coming up in the 2's line and then the 7's line and finally we added up.

Task 3 - Individual activity.

Question 1 was completed by 21 out of 25 students and only 12 of these answered to the first part of it (e*xplain what they are*). As for the second part of the question (*show him a few examples, describing step by step how to use them*) only 16 out of 21 students provided explanations (Figure 2).

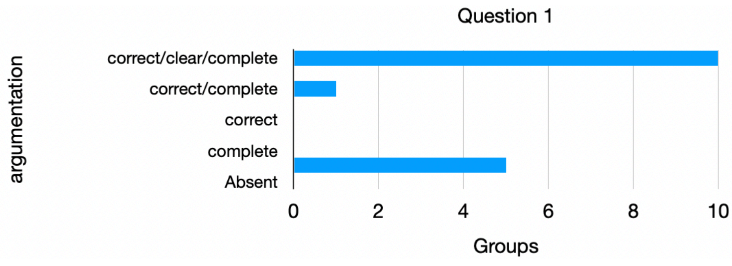
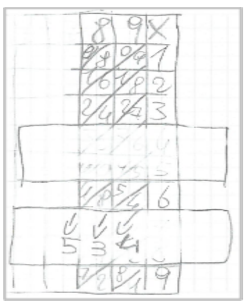


Figure 2: Task 3 - quality of the explanations in Question 1

Each one reported a single example and, 6 students used *verbal representation* to explain*,* describing in a more or less *correct and clear* way the various steps to be performed; 10 students, instead, used only the *graphic representation* by drawing the sticks they had to use to do the multiplication chosen as an example (Figure 2).

	Prot. n°12: I'll give you an example, 54×5; take the ruler and the 5-stick and 4-stick. Take a sheet of paper and divide it in half, you will come out with two equal sized sheets. Put the 5, 4 and ruler next to each other, take one half of the sheet and put it (ruler part) on 1-2-3-4 and the other on 6-7-8-9 and count the numbers in the diagonals that are created, like in Arabic multiplication.
Figure 3: A *correct/clear/complete*	

explanation (Prot. n°11) with graphic representation	

Regarding the quality, more than half of the sample (10 out of 16 students) provided *correct, clear, and complete* explanations, with some students using *verbal representation* (e.g. Prot. n° 12) and the majority using *graphic representation* (Prot. n°11 in Figure 3).

5 students out of 16 provided *no correct, no clear, no complete* explanations, using, often, a convoluted syntax. Only Protocol n°25 explained in a *correct and complete* way: using *verbal representation*, she explained all the steps to arrive at the result of the chosen multiplication, but, as for most of the class, she used a not very clear colloquial register. In addition, she was the only one to specify that it is important to move the second partial product one place to the left, because it is the product of the tens of the multiplier by the digits of the multiplicand.

Prot. n°25: Luca first you need to add up the numbers in the diagonals to get the result. If the multiplication is 172×47, to get the result you have to isolate first the second number i.e. 7 and then the first number i.e. 4. Finally you add it all up and the result will come out. But we must remember that the product of the tens must be under the other tens.

Discussion and Conclusions.

In this study we explored the qualities of students' explanations and what semiotic representations students draw upon in their explanations when using Napier's bones, in two different situations. Although this algorithm derives from Arabic multiplication, previously experienced, the analysis of the responses to Question 1 and Question 2 of Task 2 highlighted that the quality of students' explanations did not meet the chosen criteria. In addition they are all based on verbal representations. For the former only one group explained in a correct, clear, and complete way; for the latter only one group described the procedure of Napier bones algorithm in a correct and clear way. From the data analysis regarding Question 1 of Task 3 the quality of the explanations, with prevailing graphic representations, even if in colloquial register, seems to be improved according to the chosen criteria: 10 students provided correct, clear, and complete explanations and one student explained in a correct and complete way. This may also be a consequence of the collective discussion—not analysed in this paper—orchestrated by the preservice teacher (Vygotsky, 1978; Hasan, 2005; Bartolini & Mariotti, 2008) and based on the students' explanations collected in Task 2. In this way, students began to make sense of what they were doing and improved the quality of their explanations. Furthermore, the data seemed to suggest us that the use of graphic representations of Napier bones, in students' explanations in the individual activity, had helped them to perform multiplication correctly, to make connections with previous knowledge (Prot. n°. 12) and to understand some operation properties (Prot. n°. 25) (Swan & Marshall, 2010).

The study is at an early stage: data from other tasks, from other didactic cycles on different algorithms, and from collective discussions, still need to be analyzed. Nevertheless, we believe that it can be a starting point to reflect on artifacts mediation of a meaningful understanding of some aspects of the multiplication (Haylock & Cockburn, 2008).

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