Nonlinear Anal. TMA 272 (2026) 114196

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Research Paper

Asymptotic behaviour of a nonlinear coupled system for uptake processes with metabolic competition phenomena

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ARTICLE INFO

Editor: Prof Matteo Novaga

2000 MSC:
35B27
35J25
35B40
35K57
35Q92

Keywords:
Homogenization
Unfolding operators
Oscillating boundary
Nonlinear flux conditions
Reaction-diffusion equation

ABSTRACT

In this paper, we analyze the steady state uptake and coupled diffusion-reaction of two metabolites such as, for instance, glucose and galactose, within a single enterocyte. The considered mathematical model consists of a non-linear coupled system posed in a domain with a highly oscillating boundary, whose boundary oscillation has a ε-periodicity in its first variable but fixed amplitude. At the microscopic scale, the boundary conditions encode a competitive interaction between the two metabolites for the same "carrier" influenced by the oscillation amplitude through a parameter $k \geq 1$. Using the periodic unfolding operator in a comb-shaped domain and some monotonicity techniques, we study the asymptotic behaviour of the inhibition phenomena that occur in the uptake process, when ε tends to zero. Depending on the values of the parameter k in the surface reaction term, two different limit regimes arise: for $k > 1$, the system asymptotically decouples into separate systems of equations for each metabolite, without any trace of their competition. The most relevant case is $k = 1$, where the limit problem remains coupled through a lower-order term describing the effect of competition/inhibition between the two metabolites.

1. Introduction

1.1. Motivation

Metabolic processes continuously take place in all living organisms, where biochemical species can diffuse, react and are transported across intracellular and extracellular spaces. Many of these reactions involve proteins that bind selectively to specific substrates. When a protein has a single binding site, the reaction is described by Michaelis-Menten kinetics, typically yielding a hyperbolic response. However, many proteins have multiple binding sites. In this case, cooperativity or inhibition may arise, since the binding of one substrate molecule can affect positively or negatively the binding at other sites. Such behaviour may still be modeled by Micaelis Menten kinetics or, more often by a sigmoidal Hill curve (see [1] for more details). Here we focus on the uptake and the eventual diffusion-reaction of two metabolites, for instance glucose and galactose, within each enterocyte, neglecting the transport through the basolateral membrane.

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https://doi.org/10.1016/j.na.2026.114196
Received 15 December 2025; Accepted 1 June 2026

1.2. The mathematical model

We consider for simplicity the two dimensional case, but the following arguments will hold also in a three dimensional setting. Given a small and positive parameter ε, the vertical section of each cell of the intestine epithelium, can be modeled by a periodic highly oscillating domain $\Omega_\varepsilon \subseteq \mathbb{R}^2$ which is made up of a fixed part Ω^- with lower boundary Γ_b and lateral boundary Γ_s (the section of the basolateral membrane) and a part Ω_ε^+ whose upper boundary γ_ε (the section of the apical membrane)) is represented by the graph of a highly oscillating function, ε-periodic in the first variable and having fixed oscillations' amplitude (see Section 2.1 for details). This model domain corresponds to a $2:1:1$ thick junction in the sense considered by Mel'nyk and Nazarov [2] which means that the body is two dimensional while the junction (interface with the body) is one dimensional with several periodic one dimensional protrusions.

In the domain described above, fixed f_1 and f_2 in $L^2(\Omega)$, we consider the following nonlinear coupled system of equations

$$\begin{cases} -\operatorname{div}(D_1^\varepsilon \nabla u_1^\varepsilon) + h_1(u_1^\varepsilon) = f_1 & \text{in } \Omega_\varepsilon, \\ -\operatorname{div}(D_2^\varepsilon \nabla u_2^\varepsilon) + h_2(u_2^\varepsilon) = f_2 & \text{in } \Omega_\varepsilon, \\ D_1^\varepsilon \nabla u_1^\varepsilon \cdot \nu_\varepsilon = -\varepsilon^k \left(g_1(x_2, u_1^\varepsilon) - g(x_2, u_2^\varepsilon)\right) & \text{on } \gamma_\varepsilon, \\ D_2^\varepsilon \nabla u_2^\varepsilon \cdot \nu_\varepsilon = -\varepsilon^k \left(g_2(x_2, u_2^\varepsilon) - g(x_2, u_1^\varepsilon)\right) & \text{on } \gamma_\varepsilon, \\ u_1^\varepsilon = u_2^\varepsilon = 0 & \text{on } \Gamma_b, \\ u_i^\varepsilon \text{ is } \Gamma_s - \text{periodic} \end{cases} \tag{1}$$

where ν_ε is the unit outward normal to γ_ε and $k \in \mathbb{R}$. The boundary condition on the apical membrane γ_ε shows the competition between glucose and galactose for the same "carrier". We assume that the diffusion matrices D_1^ε and D_2^ε are uniformly elliptic and bounded and have ε-periodicity in Ω_ε^+, whereas in Ω^-, they are independent of ε. The nonlinear functions h_1, h_2, g, g_1 and g_2 satisfy some natural regularity and growth conditions (cf. Example 2.1). For more details about the setting of the problem, we refer the reader to Section 2.1.

Coupled systems of equations, with nonlinear surface and reaction terms can be found in [3–12] and references therein, to name a few. Although the nonlinear boundary conditions in (1) are a particular case of the one assumed in [11], in our paper the particular shape of the domain makes the analysis of the problem an hard task. For boundary-value problems in a domain with rough boundary or rough interface we may cite [13–31] and references therein. Furthermore, homogenization problems involving source terms in nonstandard spaces such as models with L^1 or $L \log L$ data are discussed in [32,33] and the references therein.

The goal of this paper is to study the asymptotic behaviour of the problem (1) when the period of the oscillations tends to zero. The homogenisation process is developed using the periodic unfolding operator in a comb domain (see [34,35] and [28]) and some monotonicity arguments. The values of the parameter $k \in \mathbb{R}$ in the surface reaction term play a crucial role and give rise to two different limit problems. In fact, for $k > 1$, we obtain a system of equations for each metabolite without any memory of the existing competition between them, as they acted separately. The most relevant case is $k = 1$, when the limit problem is coupled with a lower order term describing the effect of competition between the two metabolites (see (14) in Theorem 2.2).

1.3. Outline

The paper is organized as follows. In Section 2, we describe in details our domain and its boundary. Eventually, after introducing suitable assumptions on the data, we state the main result (see Theorem 2.1). Moreover, we mention some practical examples of nonlinear functions involved in some biological processes that our problem aims to describe and, as a consequence, we justify our assumptions which could seem merely technical at first sight. In Section 4, following the same notation as in [28], we recall the definition and properties of the unfolding operator for a domain with oscillating boundary of comb type and related results. Moreover, we state an existence result for the solution of problem (1) (see Proposition (4.1)) together with a uniform a priori estimate. Hence, in Proposition (4.2) and Proposition (4.3), we prove the uniqueness of the solution of Problem (1) when $k > 1$ and $k = 1$, respectively. At last, since we are dealing with concentrations of metabolites, in Proposition 4.4 and Proposition 4.5, we prove the non-negativity and state the boundedness of the solution of Problem (1). In Section 5, we state some crucial compactness result as a consequence of the a priori estimate and the properties of the unfolding operators recalled in Section 4. Finally, in Section 6 and Section 7, we demonstrate the asymptotic analysis for problem (1) and prove Theorem 2.1 and Theorem 2.2 separately for the two cases $k > 1$ and $k = 1$. Both proofs contain several steps and take advantage of a monotonicity argument.

2. Setting of the problem and main result

2.1. Description of the domain

Let $L > 0$ and let $\varepsilon = \dfrac{L}{n}$, $n \in \mathbb{N}$, be a small positive parameter vanishing to zero when n tends to $+\infty$. We consider a domain $\Omega_\varepsilon \subset \mathbb{R}^2$ made up of two parts, Ω^- which is fixed and Ω_ε^+ whose upper boundary γ_ε, oscillates rapidly with ε (see Fig. 1).

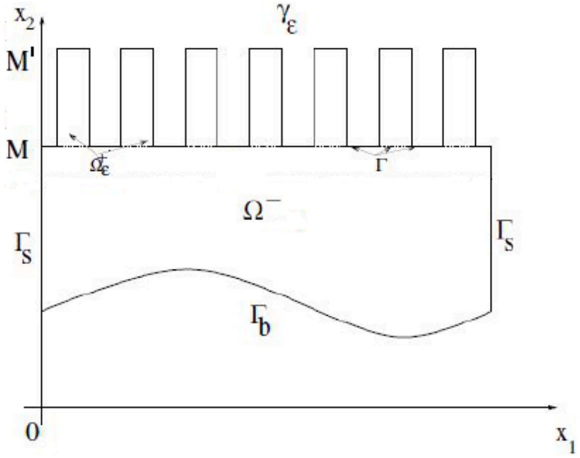

Fig. 1. Ω_ε.

More precisely, let $w : \mathbb{R} \to \mathbb{R}$ be a smooth L-periodic function and m its maximum value in $[0, L]$. Further, for positive constants M, M', a, b such that $0 < a < b < L$ and $M' > M > m$, let η be the L-periodic function defined as

$$\eta(x_1) = \begin{cases} M' & \text{if } x_1 \in (a, b), \\ M & \text{if } x_1 \in [0, L) \setminus (a, b). \end{cases}$$

Hence the domain Ω_ε can be written as

$$\Omega_\varepsilon = \left\{ (x_1, x_2) : 0 < x_1 < L,\, w(x_1) < x_2 < \eta\left(\frac{x_1}{\varepsilon}\right) \right\}$$

and its bottom boundary Γ_b is defined as

$$\Gamma_b = \left\{ \left(x_1, x_2\right) : 0 \le x_1 \le L,\, x_2 = w(x_1) \right\}.$$

We observe that the upper part of the domain Ω_ε, denoted by Ω_ε^+, is the union of thin slabs of fixed height, namely

$$\Omega_\varepsilon^+ = \bigcup_{j=0}^{n-1} (j\varepsilon L + \varepsilon a, j\varepsilon L + \varepsilon b) \times \left(M, M'\right).$$

Further the fixed bottom region of the domain Ω_ε, denoted by Ω^-, can be described by

$$\Omega^- = \left\{ (x_1, x_2) : 0 < x_1 < L,\, w(x_1) < x_2 < M \right\}$$

and its vertical and top boundaries, denoted by Γ_s and Γ respectively, are defined as

$$\Gamma_s = \left\{ \left(0, x_2\right) : w(0) \le x_2 \le M \right\} \cup \left\{ \left(L, x_2\right) : w(L) \le x_2 \le M \right\},$$

and

$$\Gamma = \left\{ \left(x_1, M\right) : 0 \le x_1 \le L \right\}.$$

The oscillating part of the boundary of Ω_ε can be written as

$$\gamma_\varepsilon = \partial\Omega_\varepsilon \setminus \left\{ \Gamma_s \cup \Gamma_b \right\}.$$

We will prove that our limit problem will be posed in the following fixed domain

$$\Omega = \left\{ \left(x_1, x_2\right) : 0 < x_1 < L,\, w(x_1) < x_2 < M' \right\}$$

made up of two parts, Ω^- previously described and Ω^+ defined as

$$\Omega^+ = \left\{ \left(x_1, x_2\right) : 0 < x_1 < L,\, M < x_2 < M' \right\},$$

see Fig. 2.

The above mentioned Γ is the interface between Ω^- and Ω^+ whereas Γ_b is also the bottom boundary of Ω. Moreover, the vertical and the top boundary of Ω, denoted by $\Gamma_{s'}$ and Γ_u respectively, are defined as

$$\Gamma_{s'} = \left\{ \left(0, x_2\right) : w(0) \le x_2 \le M' \right\} \cup \left\{ \left(0, x_2\right) : w(L) \le x_2 \le M' \right\}$$

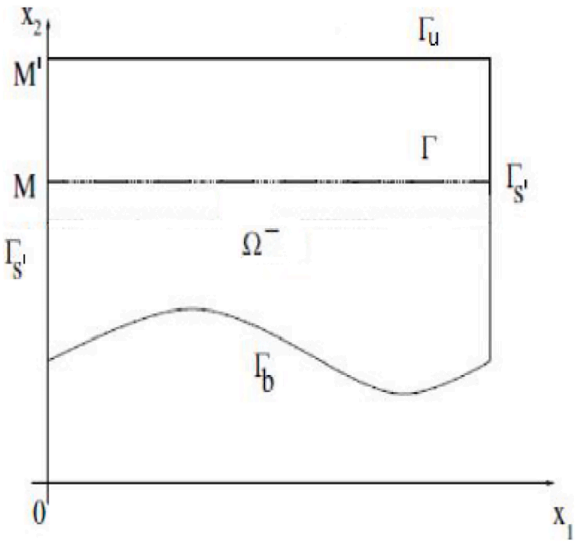

Fig. 2. Limit domain.

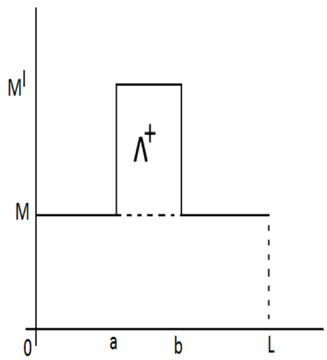

Fig. 3. Reference cell.

and

$$\Gamma_u = \left\{ \left(x_1, M' \right) : 0 \leq x_1 \leq L \right\}.$$

Formally we can write $\Omega_\varepsilon = \text{Interior}\left\{ \overline{\Omega^-} \cup \overline{\Omega^+_\varepsilon} \right\}$ and $\Omega = \text{Interior}\left\{ \overline{\Omega^-} \cup \overline{\Omega^+} \right\}$, where Interior$\{S\}$ and $\overline{S}$ denote the interior and the closure of the set S in $\mathbb{R}^2$ with respect to the standard topology, respectively.

The reference cell domain is defined as follows

$$\Lambda^+ = \{(y_1, x_2) : a < y_1 < b,\ M < x_2 < M'\}$$

and its lateral boundary is $\gamma_{sl} \cup \gamma_{sr}$, where

$$\gamma_{sl} = \quad \left\{ \left(a, x_2 \right) : M \leq x_2 \leq M' \right\}$$

$$\gamma_{sr} = \quad \left\{ \left(b, x_2 \right) : M \leq x_2 \leq M' \right\},$$

see Fig. 3.

2.2. Assumptions on the data

We make the following assumptions on the data:

$(\mathbf{H}_1)$ For any fixed $i \in \{1, 2\}$, the diffusion matrix is given by

$$D_i^\varepsilon(x) = D_i^+\left(\frac{x_1}{\varepsilon}, x_2 \right) \chi_{\Omega_\varepsilon^+}(x) + D_i^-\left(x_1, x_2 \right) \chi_{\Omega^-}(x), \quad \text{a. e. in } \Omega_\varepsilon$$

where

1. $D_i^+ = (d_{m,n}^{i+}) \in (L^\infty(\Lambda^+))^{2\times 2}$ is a L-periodic matrix field in y_1 such that for any $\xi \in \mathbb{R}^2$, $\alpha_i^+|\xi|^2 \le (D_i^+(y_1,x_2)\xi,\xi) \le \beta_i^+|\xi|^2$, almost everywhere in Λ^+, for some positive real constants α_i^+ and β_i^+;
2. $D_i^- = (d_{m,n}^{i-}) \in (L^\infty(\Omega^-))^{2\times 2}$ is and elliptic fixed matrix field such that for any $\xi \in \mathbb{R}^2$, $\alpha_i^-|\xi|^2 \le (D_i^-(x_1,x_2)\xi,\xi) \le \beta_i^-|\xi|^2$, almost everywhere in Ω^-, for some positive real constants α_i^- and β_i^-.

($\mathbf{H}_2$) The reaction rates h_1, $h_2 : \mathbb{R} \to \mathbb{R}$ are increasing, globally Lipschitz continuous and such that $h_1(0) = h_2(0) = 0$.
($\mathbf{H}_3$) The function $g : [M, M'] \times \mathbb{R} \to \mathbb{R}$ in the surface reaction terms is such that

1. $g(x_2,\cdot)$ is an increasing function such that $g(x_2,0) = 0$ for every $x_2 \in [M, M']$.
2. $g(x_2,\cdot)$ is Lipschitz continuous with constant independent of x_2.

($\mathbf{H}_4$) Functions g_1, $g_2 : [M, M'] \times \mathbb{R} \to \mathbb{R}$ appearing in the surface reaction terms are differentiable with respect to their arguments and there exists a constant C_0 such that for both arguments

$$\left|\frac{\partial g_i}{\partial x_2}(x_2,s)\right| \le C_0, \quad i = 1,2. \tag{2}$$

Moreover, for $i = 1,2$

1. $(g_i - g)(x_2,\cdot)$ is an increasing function such that $(g_i - g)(x_2,0) = 0$ for every $x_2 \in [M, M']$.
2. $(g_i - g)(x_2,\cdot)$ is Lipschitz continuous with constant independent of x_2.

($\mathbf{H}_5$) The right hand data f_1 and f_2 are in $L^2(\Omega)$ and supposed to be nonnegative.

Remark 2.1. As a consequence of assumption ($\mathbf{H}_1$), for any fixed $i = 1,2$, if we denote $\alpha_i = \min\{\alpha_i^+, \alpha_i^-\}$ and $\beta_i = \max\{\beta_i^+, \beta_i^-\}$, we get that the diffusion matrix D_i^ε is an elliptic matrix field such that for any $\xi \in \mathbb{R}^2$

$$\alpha_i|\xi|^2 \le (D_i^\varepsilon(x_1,x_2)\xi,\xi) \le \beta_i|\xi|^2, \quad \text{a. e. in } \Omega_\varepsilon.$$

Remark 2.2. Assumptions ($\mathbf{H}_2$) and $(\mathbf{H}_4)_1$ imply

$$sh_i(s) \ge 0 \quad \forall s \in \mathbb{R}$$

and

$$\big(g_i(x_2,s) - g(x_2,s)\big)s \ge 0 \quad \forall (x_2,s) \in [M, M'] \times \mathbb{R}$$

for any $i = 1,2$.

Remark 2.3. As a consequence of assumptions $(\mathbf{H}_3)_1$ and $(\mathbf{H}_4)_1$, we get that, for any $i = 1,2$, the function $g_i(x_2,\cdot)$ is increasing and $g_i(x_2,0) = 0$, for every $x_2 \in [M, M']$. Hence

$$g_i(x_2,s)s \ge 0 \quad \forall (x_2,s) \in [M, M'] \times \mathbb{R}.$$

Remark 2.4. Assumptions ($\mathbf{H}_3$) and ($\mathbf{H}_4$) in particular implies the following growth conditions

$$|g(x_2,s)| \le l|s| \quad \forall (x_2,s) \in [M, M'] \times \mathbb{R} \tag{3}$$

and for $i = 1,2$

$$|(g_i - g)(x_2,s)| \le l'|s| \quad \forall (x_2,s) \in [M, M'] \times \mathbb{R} \tag{4}$$

$$|g_i(x_2,s)| \le (l' + l)|s| \quad \forall (x_2,s) \in [M, M'] \times \mathbb{R} \tag{5}$$

where l and l' are positive constants independent of x_2.

Let us observe that (5) easily implies

$$\left|\frac{\partial g_i}{\partial s}(x_2,s)\right| \le C_0', \ \forall s \in \mathbb{R} \quad i = 1,2 \tag{6}$$

with C_0' independent of x_2.

Remark 2.5. Our assumptions ensure well-posedness and uniform a priori estimates for the ε-problem. They are also physically meaningful: monotonicity and Lipschitz continuity are standard for chemical reactions, while the condition that the nonlinearities vanish at zero expresses the absence of reaction when concentrations vanish. The nonnegativity of f_1 and f_2 is used to guarantee nonnegative solutions, as expected in biological models (see, for example, [11]).

Example 2.1. For the reaction rates h_1, h_2 and the transport mechanisms g_1 and g_2 (see [36]), we were inspired by the so-called Michaelis-Menten kinetics. Hence we can take

$$h_i(s) = \frac{a_i s}{s + b_i}, \qquad g_i(s) = \frac{c_i s}{d_i + s}, \quad a_i, b_i, c_i, d_i > 0, \ i = 1,2.$$

Concerning the surface term g we can think to the following functions

$$g(x_2,s) = \bar{g}(x_2)s^2 \quad (k > 1), \qquad g(x_2,s) = \bar{g}(x_2)s \quad (k = 1),$$

where $\bar{g}(\cdot)$, under suitable assumptions, measures the inhibition depending on the position of the metabolite on the apical membrane. Since solutions are nonnegative and bounded (see Proposition 4.4 and Proposition 4.5 in Section 4), possible singularities for negative values can be handled by using the classical arguments as in [11] and one can get nonlinearities satisfying ($\mathbf{H}_4$) under suitable assumptions on $\bar{g}$ and for some suitable values of the constants involved.

2.3. Functional framework and main result

Let $H^1_{per}(\Omega_\varepsilon)$ be the space of the $H^1(\Omega_\varepsilon)$-functions which are periodic in the x_1- direction with period L. Analogously $C^\infty_{per}(\overline{\Omega})$ and $L^2_{per}(\Omega)$ denote the $C^\infty(\overline{\Omega})$ and $L^2(\Omega)$ functions which are periodic in the x_1- direction with period L. Moreover, throughout the paper, for $v \in L^2(\Omega)$ (or $v \in L^2(\Omega_\varepsilon)$) we shall denote by v^+ the restriction of v to Ω (or Ω^+_ε respectively) and by v^- the restriction of v to Ω^-.

We consider the closed subspace of $H^1_{per}(\Omega_\varepsilon)$

$$V^\varepsilon := \{v^\varepsilon \in H^1_{per}(\Omega_\varepsilon) : v^\varepsilon_{|\Gamma_b} = 0\}.$$

Due to the Poincaré uniform inequality

$$\|v\|_{L^2(\Omega_\varepsilon)} \le C_P \|\nabla v\|_{L^2(\Omega_\varepsilon)}, \tag{7}$$

V^ε is a Hilbert space endowed with the equivalent norm

$$\|v\|_{V^\varepsilon} = \|\nabla v\|_{L^2(\Omega_\varepsilon)}. \tag{8}$$

Definition 2.1. We say that the couple $(u^\varepsilon_1, u^\varepsilon_2) \in V^\varepsilon \times V^\varepsilon$ is a weak solution of problem (1) if it holds

$$\begin{aligned}
&\int_{\Omega_\varepsilon} D^\varepsilon_1 \nabla u^\varepsilon_1 \nabla \varphi_1 \, dx + \int_{\Omega_\varepsilon} D^\varepsilon_2 \nabla u^\varepsilon_2 \nabla \varphi_2 \, dx + \int_{\Omega_\varepsilon} h_1(u^\varepsilon_1)\varphi_1 \, dx + \int_{\Omega_\varepsilon} h_2(u^\varepsilon_2)\varphi_2 \, dx + \\
&+\varepsilon^k \int_{\gamma_\varepsilon} \big(g_1(x_2, u^\varepsilon_1) - g(x_2, u^\varepsilon_2)\big)\varphi_1 \, ds + \varepsilon^k \int_{\gamma_\varepsilon} \big(g_2(x_2, u^\varepsilon_2) - g(x_2, u^\varepsilon_1)\big)\varphi_2 \, ds = \\
&= \int_{\Omega_\varepsilon} f_1 \varphi_1 \, dx + \int_{\Omega_\varepsilon} f_2 \varphi_2 \, dx, \quad \forall (\varphi_1, \varphi_2) \in V^\varepsilon \times V^\varepsilon.
\end{aligned} \tag{9}$$

Under our assumptions, problem (1) admits a unique weak solution in the sense of Definition 2.1 (see Section 4 for details).

Let us introduce the Hilbert space

$$V(\Omega) := \left\{ v \in L^2_{per}(\Omega) : \frac{\partial v^+}{\partial x_2} \in L^2(\Omega^+), v^- \in H^1(\Omega^-), \, v^+ = v^- \text{ on } \Gamma, \, v^-_{|\Gamma_b} = 0 \right\} \tag{10}$$

endowed with the following norm

$$\|v\|_{V(\Omega)} = \|v^-\|_{H^1(\Omega^-)} + \|v^+\|_{L^2(\Omega^+)} + \left\| \frac{\partial v^+}{\partial x_2} \right\|_{L^2(\Omega^+)}.$$

In what follows with $\tilde{\cdot}$ we denote the classical extension to zero to the whole Ω^+ of a function defined only on Ω^+_ε.

Now we are able to state the main results of our paper, stressing out that the asymptotic behaviour of the problem (1) depends on the value of the parameter k

Theorem 2.1. *Let $k > 1$ and $(u^\varepsilon_1, u^\varepsilon_2) \in V^\varepsilon \times V^\varepsilon$ be the sequence of weak solutions to (1). Under the assumptions $(\mathbf{H}_1) - (\mathbf{H}_5)$, for any $i = 1, 2$, there exists a function $u_i \in V(\Omega)$ such that it holds*

$$\begin{aligned}
&\widetilde{u_i^{\varepsilon+}} \rightharpoonup \left(\frac{b-a}{L}\right) u^+_i \quad \textit{weakly in } L^2(0, L; H^1(M, M')) \\
&\widetilde{u_i^{\varepsilon-}} \rightharpoonup \left(\frac{b-a}{L}\right) u^-_i \quad \textit{weakly in } H^1(\Omega^-),
\end{aligned} \tag{11}$$

and u_i is the unique solution of the following system of equations

$$\begin{cases}
-\dfrac{\partial}{\partial x_2}\left(\mathcal{M}\left(\dfrac{\mathrm{Det}(D^+_i)}{d^{i+}_{11}}\right)\dfrac{\partial u^+_i}{\partial x_2}\right) + h_i(u^+_i) = \dfrac{(b-a)}{L} f_i & \textit{in } \Omega^+, \\
-\mathrm{div}(D^-_i \nabla u^-_i) + h_i(u^-_i) = f_i & \textit{in } \Omega^- \\
\mathcal{M}\left(\dfrac{\mathrm{Det}(D^+_i)}{d^{i+}_{11}}\right)\dfrac{\partial u^+_i}{\partial x_2} = 0, & \textit{on } \Gamma_u, \\
u^+_i = u^-_i & \textit{on } \Gamma \\
\left(d^{i-}_{21}\dfrac{\partial u^-_i}{\partial x_1} + d^{i-}_{22}\dfrac{\partial u^-_i}{\partial x_2}\right) = \mathcal{M}\left(\dfrac{\mathrm{Det}(D^+_i)}{d^{i+}_{11}}\right)\dfrac{\partial u^+_i}{\partial x_2} & \textit{on } \Gamma, \\
u^-_i = 0 & \textit{on } \Gamma_b \\
u_i \, \Gamma_{s'} - \textit{periodic},
\end{cases} \tag{12}$$

being $\mathcal{M}(\cdot)$ the mean value on the interval (a,b).

Theorem 2.2. *Let $k=1$ and $(u_1^\varepsilon,u_2^\varepsilon)\in V^\varepsilon\times V^\varepsilon$ be the sequence of weak solutions to (1). Let $g(x_2,\cdot)=\bar g(x_2)\cdot$ where $\bar g:[M,M']\to[0,+\infty)$ is continuous. Under the assumptions $(\mathbf{H}_1)-(\mathbf{H}_5)$, for any $i=1,2$, there exists a function $u_i\in V(\Omega)$ such that it holds*

$$\begin{aligned} &\widetilde{u_i^{\varepsilon+}} \rightharpoonup \left(\frac{b-a}{L}\right)u_i^+ \quad \text{weakly in } L^2(0,L;H^1(M,M'))\\ &\widetilde{u_i^{\varepsilon-}} \rightharpoonup \left(\frac{b-a}{L}\right)u_i^- \quad \text{weakly in } H^1(\Omega^-), \end{aligned} \tag{13}$$

and the pair (u_1,u_2) is the unique solution of the following coupled system of equations

$$\begin{cases} -\dfrac{\partial}{\partial x_2}\left(\mathcal{M}\left(\dfrac{\mathrm{Det}(D_1^+)}{d_{11}^{1+}}\right)\dfrac{\partial u_1^+}{\partial x_2}\right)+h_1(u_1^+)+\dfrac{2}{L}\left(g_1(\cdot,u_1^+)-\bar g(\cdot)u_2^+\right)= & \\ =\dfrac{(b-a)}{L}f_1 & \text{in } \Omega^+,\\ -\dfrac{\partial}{\partial x_2}\left(\mathcal{M}\left(\dfrac{\mathrm{Det}(D_2^+)}{d_{11}^{2+}}\right)\dfrac{\partial u_2^+}{\partial x_2}\right)+h_2(u_2^+)+\dfrac{2}{L}\left(g_2(\cdot,u_2^+)-\bar g(\cdot)u_1^+\right)= & \\ =\dfrac{(b-a)}{L}f_2 & \text{in } \Omega^+\\ -\mathrm{div}(D_i^-\nabla u_i^-)+h_i(u_i^-)=f_i,\quad i=1,2 & \text{in } \Omega^-\\ \mathcal{M}\left(\dfrac{\mathrm{Det}(D_i^+)}{d_{11}^{i+}}\right)\dfrac{\partial u_i^+}{\partial x_2}=0,\quad i=1,2 & \text{on } \Gamma_u,\\ u_i^+=u_i^-,\quad i=1,2 & \text{on } \Gamma\\ \left(d_{21}^{i-}\dfrac{\partial u_i^-}{\partial x_1}+d_{22}^{i-}\dfrac{\partial u_i^-}{\partial x_2}\right)=\mathcal{M}\left(\dfrac{\mathrm{Det}(D_i^+)}{d_{11}^{i+}}\right)\dfrac{\partial u_i^+}{\partial x_2},\quad i=1,2 & \text{on } \Gamma,\\ u_i^-=0,\quad i=1,2 & \text{on } \Gamma_b\\ u_i\,\Gamma_{s'}-\text{periodic},\quad i=1,2 & \end{cases} \tag{14}$$

being $\mathcal{M}(\cdot)$ the mean value on the interval (a,b).

3. Unfolding operator in a comb domain

In this section, following the same notation as in [28], we recall the definition, the properties and related results of the unfolding operator for a domain with oscillating boundary of a comb type . The basic idea is to write any $x\in\mathbb{R}$ as

$$x=[x]_L+\{x\}_L.$$

where $[x]_L=mL$ and $\{x\}_L$ are the integral and fractional parts, respectively, of x with respect to L (m is the largest integer such that $mL\le x$).

Definition 3.1. Let $v:\overline{\Omega_\varepsilon^+}\to\mathbb{R}$. We define the operator $T^\varepsilon v:\overline{\Omega^+}\times[a,b]\to\mathbb{R}$ as follows

$$T^\varepsilon v(x_1,x_2,y_1)=v\left(\varepsilon\left[\frac{x_1}{\varepsilon}\right]_L+\varepsilon y_1,x_2\right). \tag{15}$$

Proposition 3.1. *The operator T^ε has the following properties:*

(a) T^ε is linear. Further for any measurable functions $v_1,v_2:\overline{\Omega_\varepsilon^+}\to\mathbb{R}$ it holds

$$T^\varepsilon(v_1v_2)=T^\varepsilon(v_1)T^\varepsilon(v_2).$$

(b) If $v\in L^1(\Omega_\varepsilon^+)$, then we have:

$$\int_{\Omega^+\times(a,b)}T^\varepsilon v\,dx_1dx_2dy_1=L\int_{\Omega_\varepsilon^+}v\,dx_1dx_2. \tag{16}$$

(c) If $v\in H^1(\Omega_\varepsilon^+)$, then $T^\varepsilon v\in L^2((0,L);H^1((M,M')\times(a,b)))$ and

$$\frac{\partial}{\partial x_2}T^\varepsilon v=T^\varepsilon\frac{\partial v}{\partial x_2},\quad \frac{\partial}{\partial y_1}T^\varepsilon v=\varepsilon T^\varepsilon\frac{\partial v}{\partial x_1},$$

(d) If $v \in L^1(\gamma_\varepsilon)$, then:

$$
\begin{aligned}
&i) \quad L\int_{\gamma_\varepsilon\cap\Gamma} v = \int_0^L \int_{(0,a)\cup(b,L)} T^\varepsilon v(x_1,M,y_1)\,dx_1\,dy_1,\\
&ii) \quad L\int_{\gamma_\varepsilon\cap\Gamma_u} v = \int_0^L \int_{(a,b)} T^\varepsilon v(x_1,M',y_1)\,dx_1\,dy_1,\\
&iii) \quad L\int_{\gamma_\varepsilon\setminus\{\Gamma\cup\Gamma_u\}} v = \frac{1}{\varepsilon}\int_0^L \int_M^{M'} \left(T^\varepsilon v(x_1,x_2,a)+T^\varepsilon v(x_1,x_2,b)\right)dx_1\,dx_2.
\end{aligned}
\tag{17}
$$

As a consequence of (a) and (b) in Proposition 3.1, we get that if $v \in L^2(\Omega_\varepsilon^+)$, then $T^\varepsilon v \in L^2(\Omega^+\times(a,b))$

$$\|T^\varepsilon v\|_{L^2(\Omega^+\times(a,b))} = \sqrt{L}\|v\|_{L^2(\Omega_\varepsilon^+)}$$

and by (c) in Proposition 3.1, if $v \in H^1(\Omega_\varepsilon^+)$, then we have

$$\|T^\varepsilon v\|_{L^2((0,L);H^1((M,M')\times(a,b)))} \le C\|v\|_{H^1(\Omega_\varepsilon^+)}, \tag{18}$$

where $C > 0$ is a constant independent of ε.

Proposition 3.2. *The operator T^ε satisfies the following convergences:*

(a) For every $v \in L^2(\Omega^+)$,

$$T^\varepsilon(v|_{\Omega_\varepsilon^+}) \xrightarrow{\varepsilon\to 0} v \text{ strongly in } L^2(\Omega^+\times(a,b)).$$

(b) If $v_\varepsilon \in L^2(\Omega_\varepsilon^+)$ is such that $T^\varepsilon v_\varepsilon \rightharpoonup v$ weakly in $L^2(\Omega^+\times(a,b))$ as $\varepsilon \to 0$, then

$$\widetilde{v_\varepsilon} \rightharpoonup \frac{1}{L}\int_a^b v\,dy_1 \text{ weakly in } L^2(\Omega^+).$$

(c) If $T^\varepsilon v_\varepsilon \rightharpoonup v$ weakly in $L^2((0,L);H^1((M,M')\times(a,b)))$ for a sequence $v_\varepsilon \in H^1(\Omega_\varepsilon^+)$ then

$$\widetilde{v_\varepsilon} \rightharpoonup \frac{1}{L}\int_a^b v\,dy \text{ weakly in } L^2((0,L);H^1((M,M'))).$$

We recall some boundary estimates proved in [28] and useful in the sequel.

Lemma 3.1. *For any $v \in H^1(\Omega_\varepsilon)$ it holds*

$$\|v\|_{L^2(\gamma_\varepsilon\cap\Gamma)} \le C\|v\|_{H^1(\Omega_\varepsilon)},$$

$$\|v\|_{L^2(\gamma_\varepsilon\cap\Gamma_u)} \le C\|v\|_{H^1(\Omega_\varepsilon)},$$

$$\|v\|_{L^2(\gamma_\varepsilon\setminus(\Gamma\cup\Gamma_u))} \le \frac{C}{\sqrt{\varepsilon}}\|v\|_{H^1(\Omega_\varepsilon)},$$

with C a positive constant independent of ε.

4. Preliminary results

Let us observe that (9) can also be written as

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} D_1^\varepsilon \nabla u_1^\varepsilon \nabla\varphi_1\,dx + \int_{\Omega_\varepsilon} D_2^\varepsilon \nabla u_2^\varepsilon \nabla\varphi_2\,dx + \int_{\Omega_\varepsilon} h_1(u_1^\varepsilon)\varphi_1\,dx + \int_{\Omega_\varepsilon} h_2(u_2^\varepsilon)\varphi_2\,dx+\\
&+\varepsilon^k\int_{\gamma_\varepsilon}\left(g_1(x_2,u_1^\varepsilon)-g(x_2,u_1^\varepsilon)\right)\varphi_1\,ds + \varepsilon^k\int_{\gamma_\varepsilon}\left(g_2(x_2,u_2^\varepsilon)-g(x_2,u_2^\varepsilon)\right)\varphi_2\,ds+\\
&+\varepsilon^k\int_{\gamma_\varepsilon}\left(g(x_2,u_1^\varepsilon)-g(x_2,u_2^\varepsilon)\right)\left(\varphi_1-\varphi_2\right)ds = \int_{\Omega_\varepsilon} f_1\varphi_1\,dx + \int_{\Omega_\varepsilon} f_2\varphi_2\,dx\\
&\forall(\varphi_1,\varphi_2)\in V^\varepsilon\times V^\varepsilon.
\end{aligned}
\tag{19}
$$

The Browder fixed point Theorem and classical arguments, as an easy consequence of our assumptions, lead to the following result

Proposition 4.1. *Under assumptions* $(\mathbf{H}_1)-(\mathbf{H}_5)$*, problem (9) admits at least one solution* $(u_1^\varepsilon, u_2^\varepsilon) \in V^\varepsilon \times V^\varepsilon$ *satisfying the following a priori estimate*

$$\|(u_1^\varepsilon, u_2^\varepsilon)\|_{V^\varepsilon \times V^\varepsilon} \le C. \tag{20}$$

with C *a positive constant independent of* ε.

In the following results, we prove, a uniqueness result for the weak solution to problem (9) for the cases $k > 1$ and $k = 1$ separately. In the second one, we assume that the function g is linear with respect to the second argument.

Proposition 4.2. *Let* $k > 1$*. Under the assumption* $(\mathbf{H}_1)-(\mathbf{H}_5)$*, there exists* ε_0 *such that for any* $\varepsilon \le \varepsilon_0$ *problem (9) admits a unique weak solution.*

Proof. We shall prove the uniqueness by contradiction. Let $(u_1^\varepsilon, u_2^\varepsilon)$ and $(\bar u_1^\varepsilon, \bar u_2^\varepsilon)$ be two weak solutions to problem (9). Taking $\varphi_1 = U_1 = u_1^\varepsilon - \bar u_1^\varepsilon$ and $\varphi_2 = U_2 = u_2^\varepsilon - \bar u_2^\varepsilon$ as test functions in (9) written for the solutions $(u_1^\varepsilon, u_2^\varepsilon)$ and $(\bar u_1^\varepsilon, \bar u_2^\varepsilon)$ respectively and subtracting, we get

$$\begin{aligned}
&\int_{\Omega_\varepsilon} D_1^\varepsilon \nabla U_1^\varepsilon \nabla U_1^\varepsilon \, dx + \int_{\Omega_\varepsilon} D_2^\varepsilon \nabla U_2^\varepsilon \nabla U_2^\varepsilon \, dx + \\
&+ \int_{\Omega_\varepsilon} \big(h_1(u_1^\varepsilon) - h_1(\bar u_1^\varepsilon)\big) U_1^\varepsilon \, dx + \int_{\Omega_\varepsilon} \big(h_2(u_2^\varepsilon) - h_2(\bar u_2^\varepsilon)\big) U_2^\varepsilon \, dx + \\
&+ \varepsilon^k \int_{\gamma_\varepsilon} \big(g_1(x_2, u_1^\varepsilon) - g_1(x_2, \bar u_1^\varepsilon)\big) U_1^\varepsilon \, ds + \varepsilon^k \int_{\gamma_\varepsilon} \big(g_2(x_2, u_2^\varepsilon) - g_2(x_2, \bar u_2^\varepsilon)\big) U_2^\varepsilon + \\
&= \varepsilon^k \int_{\gamma_\varepsilon} \big(g(x_2, u_2^\varepsilon) - g(x_2, \bar u_2^\varepsilon)\big) U_1^\varepsilon \, ds + \varepsilon^k \int_{\gamma_\varepsilon} \big(g(x_2, u_1^\varepsilon) - g(x_2, \bar u_1^\varepsilon)\big) U_2^\varepsilon \, ds.
\end{aligned} \tag{21}$$

The second and third lines in (21) are clearly non negative by assumption $(\mathbf{H}_2)$, and Remark 2.3. Let us consider the right hand member. Due to assumption $(\mathbf{H}_3)_2$ we get

$$\begin{aligned}
&\left| \int_{\gamma_\varepsilon} \big(g(x_2, u_2^\varepsilon) - g(x_2, \bar u_2^\varepsilon)\big) U_1^\varepsilon \, ds - \int_{\gamma_\varepsilon} \big(g(x_2, u_1^\varepsilon) - g(x_2, \bar u_1^\varepsilon)\big) U_2^\varepsilon \, ds \right| \le \\
&\le C \int_{\gamma_\varepsilon} |U_1^\varepsilon| |U_2^\varepsilon| \, ds,
\end{aligned} \tag{22}$$

with C a positive constant independent of ε.

Hence, by Remark 2.1 and (22), (21) becomes

$$\alpha_1 \|\nabla U_1\|^2_{L^2(\Omega_\varepsilon)} + \alpha_2 \|\nabla U_2\|^2_{L^2(\Omega_\varepsilon)} \le \varepsilon^k C \Big(\|U_1\|^2_{L^2(\gamma_\varepsilon)} + \|U_2\|^2_{L^2(\gamma_\varepsilon)} \Big). \tag{23}$$

By Lemma 3.1, (7) and (8) and since $k > 1$, for ε sufficiently small, we get the result. □

Proposition 4.3. *Let* $k = 1$ *and* $g(x_2, s) = \bar g(x_2) s$*, where* $\bar g : [M, M'] \to [0, +\infty)$ *is a continuous function. Under the assumption* $(\mathbf{H}_1)-(\mathbf{H}_5)$*, problem (9) admits a unique weak solution.*

Proof. We argue as in the proof of the previous proposition starting from (19). Since in this case g is linear, due to the properties of $\bar g$, Remark 2.1, assumptions $(\mathbf{H}_2)$, and $(\mathbf{H}_4)_1$, we easily get

$$\alpha_1 \|\nabla U_1\|^2_{L^2(\Omega_\varepsilon)} + \alpha_2 \|\nabla U_2\|^2_{L^2(\Omega_\varepsilon)} \le 0,$$

which by (8) easily implies the result. □

Now we want to prove that, for any fixed ε, each component of the solution to problem (9) is nonnegative.

Proposition 4.4. *Under the assumptions* $(\mathbf{H}_1)-(\mathbf{H}_5)$*, if* $(u_1^\varepsilon, u_2^\varepsilon) \in V^\varepsilon \times V^\varepsilon$ *is a solution of the problem (9), then* $u_1^\varepsilon \ge 0, u_2^\varepsilon \ge 0$ *a.e. in* Ω_ε.

Proof. For any function v in V^ε, let us consider

$$v_+ = \max\{v, 0\}, \; v_- = -\min\{v, 0\} \quad \text{a.e. in } \Omega_\varepsilon$$

which are both nonnegative and still belong to V^ε. Clearly, $v = v_+ - v_-$ and for $(v_1, v_2) \in V^\varepsilon \times V^\varepsilon$ it is easy to prove that

$$(v_1 - v_2)((v_1)_- - (v_2)_-) \le 0, \tag{24}$$

as well as for their traces (see [10] for details). Now let us choose $\varphi_1 = -(u_1^\varepsilon)_-, \varphi_2 = -(u_2^\varepsilon)_-$ as test functions in (19) which is equivalent to (9). Since f_1 and f_2 are nonnegative, it holds

$$-\int_{\Omega_\varepsilon} D_1^\varepsilon \nabla u_1^\varepsilon \nabla (u_1^\varepsilon)_- \, dx - \int_{\Omega_\varepsilon} D_2^\varepsilon \nabla u_2^\varepsilon \nabla (u_2^\varepsilon)_- \, dx - \int_{\Omega_\varepsilon} h_1(u_1^\varepsilon)(u_1^\varepsilon)_- \, dx$$

$$
\begin{aligned}
&-\int_{\Omega_\varepsilon} h_2(u_2^\varepsilon)(u_2^\varepsilon)_-\,dx-\varepsilon^k\int_{\gamma_\varepsilon}(g(x_2,u_1^\varepsilon)-g(x_2,u_2^\varepsilon))((u_1^\varepsilon)_- -(u_2^\varepsilon)_-)\,ds\\
&-\varepsilon^k\int_{\gamma_\varepsilon}(g_2(x_2,u_2^\varepsilon)-g(x_2,u_2^\varepsilon))(u_2^\varepsilon)_-\,ds-\varepsilon^k\int_{\gamma_\varepsilon}(g_1(x_2,u_1^\varepsilon)-g(x_2,u_1^\varepsilon))(u_1^\varepsilon)_-\,ds\le 0.
\end{aligned}
$$

By assumptions $(\mathbf{H}_2)$, $(\mathbf{H}_3)_1$ and $(\mathbf{H}_4)_1$, Remark 2.1, and since $\nabla u_i^\varepsilon\nabla(u_i^\varepsilon)_-=-|\nabla(u_i^\varepsilon)_-|^2$ for $i=1,2$, we get

$$
\alpha_1\|\nabla(u_1^\varepsilon)_-\|_{L^2(\Omega_\varepsilon)}+\alpha_2\|\nabla(u_2^\varepsilon)_-\|_{L^2(\Omega_\varepsilon)}\le 0,
$$

which, due to (8), means

$$
(u_1^\varepsilon)_-=(u_2^\varepsilon)_-=0
$$

and we get the claimed result. $\square$

Finally, by adapting the arguments of [37] (see also [10]) to our case and by a classical result of Stampacchia (see Lemma 4.1 in [38]), it is easy to get that, for any fixed ε, each component of the solution to problem (9) is bounded.

Proposition 4.5. *Under the assumptions $(\mathbf{H}_1)-(\mathbf{H}_5)$, for any fixed ε, any solution $(u_1^\varepsilon,u_2^\varepsilon)\in V^\varepsilon\times V^\varepsilon$ to problem (1) is bounded in $(L^\infty(\Omega_\varepsilon))^2$.*

5. Compactness results

The properties of the unfolding operator, recalled in Section 3, together with the a priori estimate (20) imply the following compactness result which is crucial for studying the asymptotic behaviour of problem (9)

Proposition 5.1. *Let $(u_1^\varepsilon,u_2^\varepsilon)\in V^\varepsilon\times V^\varepsilon$ be the unique weak solution to problem (1). Under assumptions $(\mathbf{H}_1)-(\mathbf{H}_5)$, there exist $(u_1,u_2)\in V(\Omega)\times V(\Omega)$, $\left(P_1,P_2\right)\in(L^2(\Omega^+\times(a,b)))^2$ and $(\xi_1,\xi_2)\in(L^2(\Omega^+\times(a,b)))^2$ such that for any fixed $i=1,2$, up to a subsequence, the following convergences hold*

$$
u_i^{\varepsilon-}\rightharpoonup u_i^-\quad\text{weakly in } H^1(\Omega^-),\tag{25}
$$

$$
\begin{cases}
i) & T^\varepsilon u_i^{\varepsilon+}\rightharpoonup u_i^+ \qquad \text{weakly in } L^2(0,L;H^1((M,M')\times(a,b))),\\[2mm]
ii) & T^\varepsilon\dfrac{\partial u_i^{\varepsilon+}}{\partial x_2}\rightharpoonup\dfrac{\partial u_i^+}{\partial x_2} \qquad \text{weakly in } L^2(\Omega^+\times(a,b)),\\[2mm]
iii) & T^\varepsilon\dfrac{\partial u_i^{\varepsilon+}}{\partial x_1}\rightharpoonup P_i \qquad \text{weakly in } L^2(\Omega^+\times(a,b)),
\end{cases}\tag{26}
$$

where

$$
d_{11}^{i+}P_i+d_{12}^{i+}\frac{\partial u_i^+}{\partial x_2}=0\quad\text{a.e. in }\Omega^+\times(a,b).\tag{27}
$$

Moreover

$$
h_i(u_i^{\varepsilon-})\to h_i(u_i^-)\quad\text{strongly in } L^2(\Omega^-)\tag{28}
$$

and

$$
T^\varepsilon h_i(u_i^{\varepsilon+})\rightharpoonup\xi_i\ \text{weakly in } L^2(\Omega^+\times(a,b)).\tag{29}
$$

Proof. Let us fix $i\in\{1,2\}$. The convergences (25) and (26) are consequences of (20) and the argument in [28]. Moreover by acting as in [28], ones can get that u_1^+ and u_2^+ are independent of y_1 variable and $u_i^+=u_i^-$ on Γ. Hence $u_i\in V(\Omega)$. Concerning the nonlinear reaction terms, by assumption $(\mathbf{H}_2)$ and the properties of the unfolding operator, we get (28) due to (25) and the uniform boundness of $T^\varepsilon h_i(u_i^{\varepsilon+})$ in $L^2(\Omega^+\times(a,b))$. Hence, there exists $\xi_i\in L^2(\Omega^+\times(a,b))$ such that, up to a subsequence, (29) holds.

Now we want to prove (27). To this aim, let us consider the following function

$$
\varphi^\varepsilon=\varepsilon\phi(x_1,x_2)\psi\left(\left\{\frac{x_1}{\varepsilon}\right\}_L\right),\qquad\phi\in\mathcal{D}(\Omega^+),\psi\in C_c^\infty([0,L]).\tag{30}
$$

By the definition of the unfolding operator and as a consequence of Proposition 3.2 we obtain

$$
\begin{cases}
i) & T^\varepsilon\varphi^\varepsilon\to 0 \qquad \text{in } L^2\big(\Omega^+\times(a,b)\big),\\[2mm]
ii) & T^\varepsilon\dfrac{\partial\varphi^\varepsilon}{\partial x_1}\to\phi\dfrac{\partial\psi}{\partial y_1} \qquad \text{in } L^2\big(\Omega^+\times(a,b)\big),\\[2mm]
iii) & T^\varepsilon\dfrac{\partial\varphi^\varepsilon}{\partial x_2}\to 0 \qquad \text{in } L^2\big(\Omega^+\times(a,b)\big).
\end{cases}\tag{31}
$$

By taking $(\varphi^\varepsilon, 0)$ as test function in (9), we get

$$\int_{\Omega_\varepsilon^+} D_1^\varepsilon \nabla u_1^\varepsilon \nabla \varphi^\varepsilon\, dx + \int_{\Omega_\varepsilon^+} h_1(u_1^\varepsilon)\varphi^\varepsilon\, dx + \varepsilon^k \int_{\gamma_\varepsilon \setminus (\Gamma \cup \Gamma_u)} \big(g_1(x_2, u_1^\varepsilon) - g(x_2, u_2^\varepsilon)\big)\varphi^\varepsilon\, ds =$$

$$= \int_{\Omega_\varepsilon^+} f_1 \varphi^\varepsilon\, dx.$$

Passing to the unfolding, by convergences (26), (31) and assumption $(\mathbf{H}_2)$ we have

$$\int_{\Omega_\varepsilon^+} D_1^\varepsilon \nabla u_1^\varepsilon \nabla \varphi^\varepsilon\, dx \to \frac{1}{L}\int_{\Omega^+ \times (a,b)} \left(d_{11}^{1+} P_1 + d_{12}^{1+} \frac{\partial u_1^+}{\partial x_2} \right) \phi(x_1, x_2) \frac{\partial \psi}{\partial y_1}(y_1)\, dx\, dy_1, \tag{32}$$

$$\int_{\Omega_\varepsilon^+} h_1(u_1^\varepsilon)\varphi^\varepsilon\, dx \to 0 \tag{33}$$

and

$$\int_{\Omega_\varepsilon^+} f_1 \varphi^\varepsilon\, dx \to 0. \tag{34}$$

Now we need to estimate the boundary term. To this aim let us observe that by $(d)_{iii)}$ in Proposition 3.1 we can write

$$\varepsilon^k \int_{\gamma_\varepsilon \setminus (\Gamma \cup \Gamma_u)} \big(g_1(x_2, u_1^\varepsilon) - g(x_2, u_2^\varepsilon)\big)\varphi^\varepsilon\, ds =$$

$$= \varepsilon^{k-1} \int_0^L \int_M^{M'} \Big(g_1(x_2, (T^\varepsilon u_1^\varepsilon)_{|\gamma_{sl}}) - g(x_2, (T^\varepsilon u_2^\varepsilon)_{|\gamma_{sl}} \Big) (T^\varepsilon \varphi^\varepsilon)_{|\gamma_{sl}}\, dx_1\, dx_2 +$$

$$+ \varepsilon^{k-1} \int_0^L \int_M^{M'} \Big(g_1(x_2, (T^\varepsilon u_1^\varepsilon)_{|\gamma_{sr}}) - g(x_2, (T^\varepsilon u_2^\varepsilon)_{|\gamma_{sr}} \Big) (T^\varepsilon \varphi^\varepsilon)_{|\gamma_{sr}}\, dx_1\, dx_2.$$

Hence, by assumption $(\mathbf{H}_3)_2$, (5) in Remark 2.4, (18), (20), (30) and the trace property, since $k \geq 1$ we get

$$\left| \varepsilon^k \int_{\gamma_\varepsilon \setminus (\Gamma \cup \Gamma_u)} \big(g_1(x_2, u_1^\varepsilon) - g(x_2, u_2^\varepsilon)\big)\varphi^\varepsilon\, ds \right| \to 0. \tag{35}$$

Finally, by collecting together (32), (33), (34) and (35) we obtain

$$\int_{\Omega^+ \times (a,b)} \left(d_{11}^{1+} P_1 + d_{12}^{1+} \frac{\partial u_1^+}{\partial x_2} \right) \phi(x_1, x_2) \frac{\partial \psi}{\partial y_1}(y_1)\, dx\, dy_1 = 0$$

for every $\phi \in \mathcal{D}(\Omega^+)$ and $\psi \in C_c^\infty([0, L])$, which, by following the same argument as in [28], implies

$$d_{11}^{1+} P_1 + d_{12}^{1+} \frac{\partial u_1^+}{\partial x_2} = 0 \quad \text{a.e. in } \Omega^+ \times (a, b).$$

Analogously, it holds

$$d_{11}^{2+} P_1 + d_{12}^{2+} \frac{\partial u_2^+}{\partial x_2} = 0 \quad \text{a.e. in } \Omega^+ \times (a, b).$$

Hence we get (27) which completes the proof. □

Due to the compactness result proved in Proposition (5.1) we are able to prove Theorem 2.1 and Theorem 2.2 passing to the limit in the variational problem (9). The main issues are finding the limits of the surface integrals and identifying the functions ξ_1 and ξ_2 found in Proposition 5.1 as limits of the nonlinear reaction terms in the upper part Ω^+ of the fixed domain Ω. We are led to two different limit problems according to the value of the parameter k.

6. Homogenization process for $k > 1$

In this section, where no ambiguity arises we omit + when denoting the elements of the matrix D_i^+ for $i = 1, 2$. The proof is performed into three steps.

6.1. Step 1: the limit problem

Let us start by considering φ_1 and φ_2 in $C^\infty_{per}(\overline{\Omega})$ vanishing on Γ_b and take their restrictions to Ω_ε as test functions in (9). Hence we can write

$$\begin{aligned}
&\sum_{i=1}^{2}\left[\int_{\Omega_\varepsilon^+} D_i^\varepsilon \nabla u_i^\varepsilon \nabla\varphi_i\,dx + \int_{\Omega_\varepsilon^+} h_i(u_i^\varepsilon)\varphi_i\,dx + \int_{\Omega^-} D_i^\varepsilon \nabla u_i^\varepsilon \nabla\varphi_i\,dx + \int_{\Omega^-} h_i(u_i^\varepsilon)\varphi_i\,dx\right]+\\
&+\varepsilon^k\int_{\gamma_\varepsilon}\big(g_1(x_2,u_1^\varepsilon)-g(x_2,u_2^\varepsilon)\big)\varphi_1\,ds+\varepsilon^k\int_{\gamma_\varepsilon}\big(g_2(x_2,u_2^\varepsilon)-g(x_2,u_1^\varepsilon)\big)\varphi_2\,ds=\\
&=\sum_{i=1}^{2}\left[\int_{\Omega_\varepsilon^+} f_i\varphi_i\,dx+\int_{\Omega^-} f_i\varphi_i\,dx\right].
\end{aligned}\tag{36}$$

Let us fix $i\in\{1,2\}$. By assumption $(\mathbf{H}_1)$, (25), (28) and since $f_i\in L^2(\Omega)$, we easily get

$$\begin{cases}
\displaystyle\int_{\Omega^-} D_i^\varepsilon\nabla u_i^\varepsilon\nabla\varphi_i\,dx \longrightarrow \int_{\Omega^-} D_i^-\nabla u_i^-\nabla\varphi_i\,dx,\\
\displaystyle\int_{\Omega^-} h_i(u_i^\varepsilon)\varphi_i\,dx \longrightarrow \int_{\Omega^-} h_i(u_i^-)\varphi_i\,dx,\\
\displaystyle\int_{\Omega^-} f_i\varphi_i\,dx \longrightarrow \int_{\Omega^-} f_i\varphi_i\,dx.
\end{cases}\tag{37}$$

Concerning the oscillating part Ω_ε^+ of the domain, passing to the unfolding, by (26), (27) and (29), we get

$$\int_{\Omega_\varepsilon^+} D_i^\varepsilon\nabla u_i^\varepsilon\nabla\varphi_i\,dx \longrightarrow \frac{1}{L}\int_{\Omega^+\times(a,b)}\left(d_{21}^i P_i + d_{22}^i\frac{\partial u_i^+}{\partial x_2}\right)\frac{\partial\varphi_i}{\partial x_2}\,dx\,dy_1,\tag{38}$$

$$\int_{\Omega_\varepsilon^+} h_i(u_i^\varepsilon)\varphi_i\,dx \to \frac{1}{L}\int_{\Omega^+\times(a,b)}\xi_i\varphi_i\,dx\,dy_1,\tag{39}$$

$$\int_{\Omega_\varepsilon^+} f_i\varphi_i\,dx \to \frac{1}{L}\int_{\Omega^+\times(a,b)} f_i\varphi_i\,dx\,dy_1=\frac{(b-a)}{L}\int_{\Omega^+} f_i\varphi_i\,dx\,dy_1.\tag{40}$$

Now it remains to analyze the boundary terms. Let us observe that by the Lipschitz continuity of g_1 and g, Lemma 3.1 and(20) we immediately get

$$\varepsilon^k\int_{\gamma_\varepsilon\cap(\Gamma\cup\Gamma_u)}\big(g_1(x_2,u_1^\varepsilon)-g(x_2,u_2^\varepsilon)\big)\varphi_1\,ds\to 0.\tag{41}$$

Moreover by arguing as in the proof of (35) and, since $k>1$, we obtain as well

$$\varepsilon^k\int_{\gamma_\varepsilon\setminus(\Gamma\cup\Gamma_u)}\big(g_1(x_2,u_1^\varepsilon)-g(x_2,u_2^\varepsilon)\big)\varphi_1\,ds\to 0.\tag{42}$$

By (41) and (42) it holds

$$\varepsilon^k\int_{\gamma_\varepsilon}\big(g_1(x_2,u_1^\varepsilon)-g(x_2,u_2^\varepsilon)\big)\varphi_1\,ds\to 0.\tag{43}$$

Analogously we prove

$$\varepsilon^k\int_{\gamma_\varepsilon}\big(g_2(x_2,u_2^\varepsilon)-g(x_2,u_1^\varepsilon)\big)\varphi_2\,ds\to 0.\tag{44}$$

By convergences (37), (38), (39), (40), (43), (44) we are able to pass to the limit in (36) and get by density the following limit problem

$$\begin{aligned}
&\frac{1}{L}\sum_{i=1}^{2}\left[\int_{\Omega^+\times(a,b)}\left(d_{21}^i P_i+d_{22}^i\frac{\partial u_i^+}{\partial x_2}\right)\frac{\partial\varphi_i}{\partial x_2}\,dx\,dy_1+\int_{\Omega^+\times(a,b)}\xi_i\varphi_i\,dx\,dy_1\right]+\\
&+\sum_{i=1}^{2}\left[\int_{\Omega^-} D_i^-\nabla u_i^-\nabla\varphi_i\,dx+\int_{\Omega^-} h_i(u_i^-)\varphi_i\,dx\right]=\\
&=\sum_{i=1}^{2}\left[\frac{(b-a)}{L}\int_{\Omega^+} f_i\varphi_i\,dx+\int_{\Omega^-} f_i\varphi_i\,dx\right],\quad \forall(\varphi_1,\varphi_2)\in V(\Omega)\times V(\Omega).
\end{aligned}\tag{45}$$

To identify the functions ξ_1 and ξ_2 we need a monotone relation which will be proved in the next step.

6.2. Step 2: a monotone relation

We want to prove that for all v_1, v_2 in $L^2(\Omega)$ and $\Phi_1 = (\Phi_{11}, \Phi_{12})$, $\Phi_2 = (\Phi_{21}, \Phi_{22})$ in $(L^2(\Omega \times (a,b)))^2$ the following variational inequality holds

$$\begin{aligned}&\frac{1}{L}\sum_{i=1}^{2}\int_{\Omega^+\times(a,b)}\left[\left(d_{21}^i P_i + d_{22}^i\frac{\partial u_i^+}{\partial x_2}\right)\left(\frac{\partial u_i^+}{\partial x_2} - \Phi_{i2}\right) + D_i\Phi_i\left(\Phi_i - \left(P_i, \frac{\partial u_i^+}{\partial x_2}\right)\right)\right]dx\,dy_1 +\\ &+\sum_{i=1}^{2}\int_{\Omega^-}\left[D_i\left(\nabla u_i^- - \Phi_i\right)\left(\nabla u_i^- - \Phi_i\right) + \left(h_i(u_i^-) - h_i(v_i)\right)\left(u_i^- - v_i\right)\right]dx +\\ &+\frac{1}{L}\sum_{i=1}^{2}\int_{\Omega^+\times(a,b)}\left(\xi_i - h_i(v_i)\right)\left(u_i^+ - v_i\right)dx\,dy_1 \geq 0.\end{aligned} \tag{46}$$

To this aim, let us observe that assumptions $(\mathbf{H}_3)_1$ and $(\mathbf{H}_4)_1$ imply

$$\begin{aligned}&\int_{\gamma_\varepsilon}\left(g_1(x_2,u_1^\varepsilon) - g(x_2,u_2^\varepsilon)\right)u_1^\varepsilon\,ds + \int_{\gamma_\varepsilon}\left(g_2(x_2,u_2^\varepsilon) - g(x_2,u_1^\varepsilon)\right)u_2^\varepsilon\,ds =\\ &= \int_{\gamma_\varepsilon}\left(g_1(x_2,u_1^\varepsilon) - g(x_2,u_1^\varepsilon)\right)u_1^\varepsilon\,ds + \int_{\gamma_\varepsilon}\left(g_2(x_2,u_2^\varepsilon) - g(x_2,u_2^\varepsilon)\right)u_2^\varepsilon\,ds\\ &+ \int_{\gamma_\varepsilon}\left(g(x_2,u_1^\varepsilon) - g(x_2,u_2^\varepsilon)\right)\left(u_1^\varepsilon - u_2^\varepsilon)\right)ds \geq 0.\end{aligned}$$

Hence, due to assumptions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ we can use monotonicity and write

$$\begin{aligned}&\sum_{i=1}^{2}\left[\int_{\Omega_\varepsilon} D_i^\varepsilon\left(\nabla u_i^\varepsilon - \Phi_i\right)\left(\nabla u_i^\varepsilon - \Phi_i\right)dx + \int_{\Omega_\varepsilon}\left(h_i(u_i^\varepsilon) - h_i(v_i)\right)\left(u_i^\varepsilon - v_i\right)\right)dx\right] +\\ &+\varepsilon^k\int_{\gamma_\varepsilon}\left(g_1(x_2,u_1^\varepsilon) - g(x_2,u_2^\varepsilon)\right)u_1^\varepsilon\,ds + \varepsilon^k\int_{\gamma_\varepsilon}\left(g_2(x_2,u_2^\varepsilon) - g(x_2,u_1^\varepsilon)\right)u_2^\varepsilon\,ds \geq 0,\end{aligned} \tag{47}$$

for v_1, v_2 in $L^2(\Omega)$ and Φ_1, Φ_2 in $(L^2(\Omega \times (a,b)))^2$.

Let us expand the left hand member of the previous inequality and divide it into three blocks

$$\begin{aligned}B1 = &\sum_{i=1}^{2}\left[\int_{\Omega_\varepsilon} D_i^\varepsilon\nabla u_i^\varepsilon\nabla u_i^\varepsilon\,dx + \int_{\Omega_\varepsilon} h_i(u_i^\varepsilon)u_i^\varepsilon\,dx\right] +\\ &+\varepsilon^k\int_{\gamma_\varepsilon}\left(g_1(x_2,u_1^\varepsilon) - g(x_2,u_2^\varepsilon)\right)u_1^\varepsilon\,ds + \varepsilon^k\int_{\gamma_\varepsilon}\left(g_2(x_2,u_2^\varepsilon) - g(x_2,u_1^\varepsilon)\right)u_2^\varepsilon\,ds,\end{aligned}$$

$$\begin{aligned}B2 = &\sum_{i=1}^{2}\left[\int_{\Omega^-} D_i^-\Phi_i\Phi_i\,dx - \int_{\Omega^-} D_i^-\nabla u_i^\varepsilon\Phi_i\,dx - \int_{\Omega^-} D_i^-\Phi_i\nabla u_i^\varepsilon\,dx\right] +\\ &+\sum_{i=1}^{2}\left[\int_{\Omega^-} h_i(v_i)v_i\,dx - \int_{\Omega^-} h_i(u_i^\varepsilon)v_i\,dx - \int_{\Omega^-} h_i(v_i)u_i^\varepsilon\,dx\right],\end{aligned}$$

$$\begin{aligned}B3 = &\sum_{i=1}^{2}\left[\int_{\Omega_\varepsilon^+} D_i^\varepsilon\Phi_i\Phi_i\,dx - \int_{\Omega_\varepsilon^+} D_i^\varepsilon\nabla u_i^\varepsilon\Phi_i\,dx - \int_{\Omega_\varepsilon^+} D_i^\varepsilon\Phi_i\nabla u_i^\varepsilon\,dx\right] +\\ &+\sum_{i=1}^{2}\left[\int_{\Omega_\varepsilon^+} h_i(v_i)v_i\,dx - \int_{\Omega_\varepsilon^+} h_i(u_i^\varepsilon)v_i\,dx - \int_{\Omega_\varepsilon^+} h_i(v_i)u_i^\varepsilon\right].\end{aligned}$$

Since we want to pass to the limit in (47), let us analyze the asymptotic behaviour of each of the previous blocks.

Let us start by block B1.

By taking $\varphi_1 = u_1^\varepsilon$ and $\varphi_2 = u_2^\varepsilon$ as test functions in (19) and passing to the limit, by (25) and $(26)_{i)}$ we get

$$\begin{aligned}\lim_{\varepsilon\to 0} B1 = &\lim_{\varepsilon\to 0}\sum_{i=1}^{2}\left[\int_{\Omega_\varepsilon^+} f_i u_i^\varepsilon\,dx + \int_{\Omega^-} f_i u_i^\varepsilon\,dx\right] =\\ &= \sum_{i=1}^{2}\left[\frac{b-a}{L}\int_{\Omega^+} f_i u_i^+\,dx + \int_{\Omega^-} f_i u_i^-\,dx\right],\end{aligned}$$

since u_1^+ and u_2^+ don't depend on the local variable y_1.

By considering $\varphi_1 = u_1$ and $\varphi_2 = u_2$ in the limit problem (45), we can write

$$\begin{aligned}\lim_{\varepsilon\to 0} B1 &= \sum_{i=1}^{2}\left[\frac{1}{L}\int_{\Omega^+\times(a,b)}\left(d^i_{21}P_i + d^i_{22}\frac{\partial u_i^+}{\partial x_2}\right)\frac{\partial u_i^+}{\partial x_2}dx\,dy_1 + \frac{1}{L}\int_{\Omega^+\times(a,b)}\xi_i u_i^+ dx\,dy_1\right]+\\ &+\sum_{i=1}^{2}\left[\int_{\Omega^-} D_i^-\nabla u_i^-\nabla u_i^- dx + \int_{\Omega^-} h_i(u_i^-)u_i^- dx\right].\end{aligned} \tag{48}$$

Concerning the block B2, by (25) and (28) we get

$$\begin{aligned}\lim_{\varepsilon\to 0} B2 &= \sum_{i=1}^{2}\left[\int_{\Omega^-} D_i^-\Phi_i\Phi_i dx - \int_{\Omega^-} D_i^-\nabla u_i^-\Phi_i dx - \int_{\Omega^-} D_i^-\Phi_i\nabla u_i^- dx\right]+\\ &+\sum_{i=1}^{2}\left[\int_{\Omega^-} h_i(v_i)v_i dx - \int_{\Omega^-} h_i(u_i^-)v_i dx - \int_{\Omega^-} h_i(v_i)u_i^- dx\right].\end{aligned} \tag{49}$$

Finally, for the block B3, passing to the unfolding, by (26), (27) and (29) we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0} B_3 =\\ &\frac{1}{L}\sum_{i=1}^{2}\left[\int_{\Omega^+\times(a,b)} D_i\Phi_i\Phi_i dx\,dy_1 - \int_{\Omega^+\times(a,b)}\left(d^i_{21}P_i + d^i_{22}\frac{\partial u_i^+}{\partial x_2}\right)\Phi_{i2}dx\,dy_1\right.\\ &\left.-\int_{\Omega^+\times(a,b)} D_i\Phi_i\left(P_i,\frac{\partial u_i^+}{\partial x_2}\right)dx\,dy_1\right]+\\ &+\frac{1}{L}\sum_{i=1}^{2}\left[\int_{\Omega^+\times(a,b)} h_i(v_i)v_i dx\,dy_1 - \int_{\Omega^+\times(a,b)}\xi_i v_i dx\,dy_1 - \int_{\Omega^+\times(a,b)} h_i(v_i)u_i^+ dx\,dy_1\right].\end{aligned} \tag{50}$$

Hence, by (48), (49) and (50), we can pass to the limit in (47) and get (46).

6.3. Step 3: identification of ξ_1 and ξ_2

By taking in (46)

$$\Phi_i = \left(P_i,\frac{\partial u_i^+}{\partial x_2}\right)\chi_{\Omega^+} + \nabla u_i^-\chi_{\Omega^-} \quad i = 1,2$$

and for any $\lambda > 0$

$$v_1 = u_1 - \lambda\psi, \quad v_2 = u_2$$

with $\psi \in C_c^1(\Omega^+)$, we obtain

$$\frac{\lambda}{L}\int_{\Omega^+\times(a,b)}\left(\xi_1 - h_1(u_1^+ - \lambda\psi)\right)\psi\,dx\,dy_1 \geq 0.$$

By the continuity of h_1 and since u_1 doesn't depend on y_1, the previous inequality implies

$$\int_a^b \xi_1 dy_1 = Lh_1(u_1^+) \quad \text{a. e. in } \Omega^+. \tag{51}$$

Analogously, by taking in (46) the same Φ_i as before, $v_1 = u_1$, $v_2 = u_2 - \lambda\psi$ with $\psi \in C_c^1(\Omega^+)$ and since u_2 doesn't depend on y_1, we get

$$\int_a^b \xi_2 dy_1 = Lh_2(u_2^+) \quad \text{a. e. in } \Omega^+. \tag{52}$$

Finally the limit problem (45) can be written as

$$\begin{aligned}
&\sum_{i=1}^{2}\left[\frac{1}{L}\int_{\Omega^+\times(a,b)}\left(d^i_{21}P_i+d^i_{22}\frac{\partial u_i^+}{\partial x_2}\right)\frac{\partial\varphi_i}{\partial x_2}dx\,dy_1\right]+\\
&+\sum_{i=1}^{2}\left[\int_{\Omega^-}D_i^-\nabla u_i^-\nabla\varphi_i dx+\int_{\Omega^-}h_i(u_i^-)\varphi_i dx+\int_{\Omega^+}h_i(u_i^+)\varphi_i dx\right]=\\
&=\sum_{i=1}^{2}\left[\frac{(b-a)}{L}\int_{\Omega^+}f_i\varphi_i dx+\int_{\Omega^-}f_i\varphi_i dx\right].
\end{aligned}\tag{53}$$

Let us fix $i\in\{1,2\}$. Due to assumption $(\mathbf{H}_1)_1$, we get that the determinant $\mathrm{Det}(D_i)>0$ and $d^i_{11}>0$ in $\Omega^+\times(a,b)$. Hence, by (27) and since u_i^+ doesn't depend on y_1, we can write

$$\begin{aligned}
&\frac{1}{L}\int_{\Omega^+\times(a,b)}\left(d^i_{21}P_i+d^i_{22}\frac{\partial u_i^+}{\partial x_2}\right)\frac{\partial\varphi_i}{\partial x_2}dx\,dy_1=\frac{1}{L}\int_{\Omega^+\times(a,b)}\frac{Det(D_i)}{d^i_{11}}\frac{\partial u_i^+}{\partial x_2}\frac{\partial\varphi_i}{\partial x_2}dx\,dy_1\\
&=\int_{\Omega^+}\mathcal{M}\left(\frac{Det(D_i)}{d^i_{11}}\right)\frac{\partial u_i^+}{\partial x_2}\frac{\partial\varphi_i}{\partial x_2}dx,
\end{aligned}\tag{54}$$

being $\mathcal{M}(\cdot)$ the mean value on the interval (a,b). Therefore (53) is the weak formulation of problem (12). Due to our assumptions $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ and $(\mathbf{H}_5)$, by classical arguments one can prove that the limit problem (12) admits a unique solution. Hence convergences stated in Proposition 5.1 hold for the entire sequence and this ends the proof.

7. Homogenization process for $k=1$

Let us recall that in this case we take g linear with respect to the second variable. We observe that if $\bar g:[M,M']\to[0,+\infty)$ is a continuous function, $g(x_2,\cdot)=\bar g(x_2)\cdot$ satisfies assumptions $(\mathbf{H}_3)$. The proof is performed into four steps and it follows almost the same argument as in the previous section. Hence we can write (36) and get convergences (37), (38), (39), (40). Moreover we can prove again (41) whilst the convergence (42) is rather different. Like the previous section, where no ambiguity arises we omit + when denoting the elements of the matrix D_i^+ for $i=1,2$.

7.1. Step 1: asymptotic analysis of a nonlinear surface term

Let us fix $i\in\{1,2\}$ and denote

$$\mathcal{G}^i_\varepsilon(x_1,x_2)=g_i(x_2,u_i^{\varepsilon+}(x_1,x_2))\text{ a. e. in }\in\Omega^+_\varepsilon.$$

By unfolding we get

$$T^\varepsilon\mathcal{G}^i_\varepsilon(x_1,x_2,y_1)=g_i\big(x_2,T^\varepsilon u_i^{\varepsilon+}(x_1,x_2,y_1)\big).$$

Assumption (2), the Lipschitz property (5) in Remark 2.4, (18) and (20) imply

$$\|T^\varepsilon\mathcal{G}^i_\varepsilon\|_{L^2((0,L);H^1((M,M')\times(a,b)))}\le C$$

with C a positive constant independent of ε.

Hence there exist ζ_i in $L^2\big((0,L);H^1((M,M')\times(a,b))\big)$ such that

$$T^\varepsilon\mathcal{G}^i_\varepsilon=g_i\big(\cdot,T^\varepsilon(u_i^{\varepsilon+})\big)\rightharpoonup\zeta_i\quad\text{weakly in }L^2((0,L);H^1((M,M')\times(a,b)))\tag{55}$$

up to a subsequence. Let us prove that ζ_i doesn't depend on the variable y_1. To this aim we observe that by the differenziability of g_i we can write

$$\frac{\partial\mathcal{G}^i_\varepsilon}{\partial x_1}(x_1,x_2)=\frac{\partial g_i}{\partial s}(x_2,u_i^{\varepsilon+}(x_1,x_2))\frac{\partial u_i^{\varepsilon+}}{\partial x_1}(x_1,x_2).$$

By (6), the properties of the unfolding and (20) we get

$$\left\|T^\varepsilon\left(\frac{\partial\mathcal{G}^i_\varepsilon}{\partial x_1}\right)\right\|_{L^2(\Omega^+\times(a,b))}=\left\|\frac{\partial g_i}{\partial s}(\cdot,T^\varepsilon(u_i^{\varepsilon+}))T^\varepsilon\left(\frac{\partial u_i^{\varepsilon+}}{\partial x_1}\right)\right\|_{L^2(\Omega^+\times(a,b))}\le C.$$

Hence by $c)$ in Proposition 3.1, when ε tends to zero, we can write

$$\frac{\partial}{\partial y_1}(T^\varepsilon\mathcal{G}^i_\varepsilon)=\varepsilon T^\varepsilon\left(\frac{\partial\mathcal{G}^i_\varepsilon}{\partial x_1}\right)\to0\quad\text{strongly in }L^2(\Omega^+\times(a,b)),$$

which, by (55), implies that ζ_i doesn't depend on y_1. As a consequence, by continuity of the trace operator, we get in particular

$$\begin{aligned} & T^\varepsilon \mathcal{G}_i^\varepsilon(\cdot,a) \rightharpoonup \zeta_i \quad \text{in } L^2(\Omega^+) \\ & T^\varepsilon \mathcal{G}_i^\varepsilon(\cdot,b) \rightharpoonup \zeta_i \quad \text{in } L^2(\Omega^+). \end{aligned} \tag{56}$$

By $(d)_{iii}$ in Proposition 3.1 and since $\bar{g}$ depends only on x_2, we get

$$\begin{aligned} \varepsilon \int_{\gamma_\varepsilon \setminus (\Gamma\cup\Gamma_u)} \big(g_1(x_2,u_1^\varepsilon) - \bar{g}(x_2)u_2^\varepsilon\big)\varphi_1\, ds = \frac{1}{L}\Bigg[\int_0^L \int_M^{M'} \Big(T^\varepsilon \mathcal{G}^\varepsilon_{1_{|\gamma_{sl}}} - (T^\varepsilon \bar{g}(x_2))_{|\gamma_{sl}} (T^\varepsilon u^\varepsilon_{2+})_{|\gamma_{sl}}\Big) T^\varepsilon \varphi_{1_{|\gamma_{sl}}}\, dx + \\ + \int_0^L \int_M^{M'} \Big(T^\varepsilon \mathcal{G}^\varepsilon_{1_{|\gamma_{sr}}} - (T^\varepsilon \bar{g}(x_2))_{|\gamma_{sr}} (T^\varepsilon u^\varepsilon_{2+})_{|\gamma_{sr}}\Big) T^\varepsilon \varphi_{1_{|\gamma_{sr}}}\Bigg] dx. \end{aligned} \tag{57}$$

Hence by (56) we get the following convergence

$$\varepsilon \int_{\gamma_\varepsilon \setminus (\Gamma\cup\Gamma_u)} \big(g_1(x_2,u_1^\varepsilon) - \bar{g}(x_2)u_2^\varepsilon\big)\varphi_1\, ds \to \frac{2}{L}\int_{\Omega^+} (\zeta_1 - \bar{g}(x_2)u_2^+)\varphi_1,$$

which, by (41) holding even in this case, implies

$$\varepsilon \int_{\gamma_\varepsilon} \big(g_1(x_2,u_1^\varepsilon) - \bar{g}(x_2)u_2^\varepsilon\big)\varphi_1\, ds \to \frac{2}{L}\int_{\Omega^+} (\zeta_1 - \bar{g}(x_2)u_2^+)\varphi_1. \tag{58}$$

Analogously it holds

$$\varepsilon \int_{\gamma_\varepsilon} \big(g_2(x_2,u_2^\varepsilon) - \bar{g}(x_2)u_1^\varepsilon\big)\varphi_1\, ds \to \frac{2}{L}\int_{\Omega^+} (\zeta_2 - \bar{g}(x_2)u_1^+)\varphi_2. \tag{59}$$

Now we can pass to the limit in (36). By convergences (37), (38), (39), (40), (41), (58) and (59), by density, the limit problem reads as the following one

$$\begin{aligned} & \frac{1}{L}\sum_{i=1}^2 \left[\int_{\Omega^+\times(a,b)} \left(d^i_{21} P_i + d^1_{22}\frac{\partial u_i^+}{\partial x_2}\right)\frac{\partial \varphi_i}{\partial x_2}\, dx\, dy_1 + \int_{\Omega^+\times(a,b)} \xi_i \varphi_i\, dx\, dy_1\right] + \\ & + \sum_{i=1}^2 \left[\int_{\Omega^-} D_i^- \nabla u_i^- \nabla \varphi_i\, dx + \int_{\Omega^-} h_i(u_i^-)\varphi_i\, dx\right] + \\ & + \frac{2}{L}\int_{\Omega^+} (\zeta_1 - \bar{g}(x_2)u_2^+)\varphi_1\, dx + \frac{2}{L}\int_{\Omega^+} (\zeta_2 - \bar{g}(x_2)u_1^+)\varphi_2\, dx, \\ & = \sum_{i=1}^2 \left[\frac{(b-a)}{L}\int_{\Omega^+} f_i\varphi_i\, dx + \int_{\Omega^-} f_i \varphi_i\, dx\right], \quad \forall(\varphi_1,\varphi_2) \in V(\Omega)\times V(\Omega). \end{aligned} \tag{60}$$

Let us observe that

$$\begin{aligned} & \int_{\Omega^+} (\zeta_1 - \bar{g}(x_2)u_2^+)\varphi_1 + \int_{\Omega^+} (\zeta_2 - \bar{g}(x_2)u_1^+)\varphi_2 = \\ & = \int_{\Omega^+} (\zeta_1 - \bar{g}(x_2)u_1^+)\varphi_1 + \int_{\Omega^+} (\zeta_2 - \bar{g}(x_2)u_2^+)\varphi_2 + \int_{\Omega^+} \bar{g}(x_2)\big(u_1^+ - u_2^+\big)\big(\varphi_1 - \varphi_2\big). \end{aligned}$$

Hence (60) becomes

$$\begin{aligned} & \frac{1}{L}\sum_{i=1}^2 \left[\int_{\Omega^+\times(a,b)} \left(d^i_{21} P_i + d^1_{22}\frac{\partial u_i^+}{\partial x_2}\right)\frac{\partial \varphi_i}{\partial x_2}\, dx\, dy_1 + \int_{\Omega^+\times(a,b)} \xi_i \varphi_i\, dx\, dy_1\right] + \\ & + \sum_{i=1}^2 \left[\int_{\Omega^-} D_i^- \nabla u_i^- \nabla \varphi_i\, dx + \int_{\Omega^-} h_i(u_i^-)\varphi_1\, dx\right] + \\ & + \frac{2}{L}\sum_{i=1}^2 \int_{\Omega^+} (\zeta_i - \bar{g}(x_2)u_i^+)\varphi_i\, dx + \frac{2}{L}\int_{\Omega^+} \bar{g}(x_2)\big(u_1^+ - u_2^+\big)\big(\varphi_1 - \varphi_2\big) = \\ & = \sum_{i=1}^2 \left[\frac{(b-a)}{L}\int_{\Omega^+} f_i\varphi_i\, dx + \int_{\Omega^-} f_i \varphi_i\, dx\right], \quad \forall(\varphi_1,\varphi_2) \in V(\Omega)\times V(\Omega). \end{aligned} \tag{61}$$

It remains to identify ξ_1, ξ_2 ζ_1 and ζ_2.

7.2. Step 2: a monotone relation

We want to prove for all v_1, v_2 in $L^2(\Omega)$, $\Phi_1 = (\Phi_{11}, \Phi_{12})$, $\Phi_2 = (\Phi_{21}, \Phi_{22})$ in $(L^2(\Omega \times (a,b)))^2$ and τ_1, τ_2 in $V(\Omega)$ the following inequality

$$
\begin{aligned}
&\frac{1}{L}\sum_{i=1}^{2}\int_{\Omega^+\times(a,b)}\left[\left(d_{21}^i P_i + d_{22}^i \frac{\partial u_i^+}{\partial x_2}\right)\left(\frac{\partial u_i^+}{\partial x_2} - \Phi_{i2}\right) + D_i\Phi_i\left(\Phi_i - \left(P_i, \frac{\partial u_i^+}{\partial x_2}\right)\right)\right] dx\, dy_1 + \\
&+\sum_{i=1}^{2}\int_{\Omega^-}\left[D_i\left(\nabla u_i^- - \Phi_i\right)\left(\nabla u_i^- - \Phi_i\right) + \left(h_i(u_i^-) - h_i(v_i)\right)\left(u_i^- - v_i\right)\right] dx + \\
&+\frac{1}{L}\sum_{i=1}^{2}\int_{\Omega^+\times(a,b)}\left(\xi_i - h_i(v_i)\right)\left(u_i^+ - v_i\right) dx\, dy_1 + \\
&+\frac{2}{L}\sum_{i=1}^{2}\int_{\Omega^+}\left[\left(\zeta_i - \bar g(x_2)u_i^+\right) - \left(g_i(x_2,\tau_i) - \bar g(x_2)\tau_i\right)\right]\left(u_i^+ - \tau_i\right) dx + \\
&+\frac{2}{L}\int_{\Omega^+}\bar g(x_2)\left[\left(u_1^+ - u_2^+\right) - \left(\tau_1 - \tau_2\right)\right]^2 dx \geq 0.
\end{aligned}
\tag{62}
$$

To this aim by assumptions $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)_1$ and $(\mathbf{H}_4)_1$ we can use again a monotonicity argument. Hence for v_1, v_2 in $L^2(\Omega)$, Φ_1, Φ_2 in $(L^2(\Omega \times (a,b)))^2$ and τ_1, τ_2 in $V(\Omega)$ we get

$$
\begin{aligned}
&\sum_{i=1}^{2}\left[\int_{\Omega_\varepsilon} D_i^\varepsilon\left(\nabla u_i^\varepsilon - \Phi_i\right)\left(\nabla u_i^\varepsilon - \Phi_i\right) dx + \int_{\Omega_\varepsilon}\left(h_i(u_i^\varepsilon) - h_i(v_i)\right)\left(u_i^\varepsilon - v_i\right) dx\right] + \\
&+\sum_{i=1}^{2}\varepsilon\int_{\gamma_\varepsilon}\left[\left(g_i(x_2,u_i^\varepsilon) - \bar g(x_2)u_i^\varepsilon\right) - \left(g_i(x_2,\tau_i) - \bar g(x_2)\tau_i\right)\right]\left(u_i^\varepsilon - \tau_i\right) ds \\
&+\varepsilon\int_{\gamma_\varepsilon}\bar g(x_2)\left[\left(u_1^\varepsilon - u_2^\varepsilon\right) - \left(\tau_1 - \tau_2\right)\right]^2 \geq 0.
\end{aligned}
\tag{63}
$$

As in the previous case, let us expand the left hand member of the previous inequality and divide it into four blocks

$$
\begin{aligned}
B1 = &\sum_{i=1}^{2}\left[\int_{\Omega_\varepsilon} D_i^\varepsilon \nabla u_i^\varepsilon \nabla u_i^\varepsilon dx + \int_{\Omega_\varepsilon} h_i(u_i^\varepsilon)u_i^\varepsilon dx\right] + \\
&+\sum_{i=1}^{2}\varepsilon\int_{\gamma_\varepsilon}\left(g_i(x_2,u_i^\varepsilon) - \bar g(x_2)u_i^\varepsilon\right)u_i^\varepsilon\, ds + \varepsilon\int_{\gamma_\varepsilon}\bar g(x_2)\left(u_1^\varepsilon - u_2^\varepsilon\right)^2 ds,
\end{aligned}
$$

$$
\begin{aligned}
B2 = &\sum_{i=1}^{2}\left[\int_{\Omega^-} D_i^\varepsilon\Phi_i\Phi_i dx - \int_{\Omega^-} D_i^\varepsilon\nabla u_i^\varepsilon\Phi_i dx - \int_{\Omega^-} D_i^\varepsilon\Phi_i\nabla u_i^\varepsilon dx\right] + \\
&+\sum_{i=1}^{2}\left[\int_{\Omega^-} h_i(v_i)v_i dx - \int_{\Omega^-} h_i(u_i^\varepsilon)v_i dx - \int_{\Omega^-} h_i(v_i)u_i^\varepsilon dx\right],
\end{aligned}
$$

$$
\begin{aligned}
B3 = &\sum_{i=1}^{2}\left[\int_{\Omega_\varepsilon^+} D_i^\varepsilon\Phi_i\Phi_i dx - \int_{\Omega_\varepsilon^+} D_i^\varepsilon\nabla u_i^\varepsilon\Phi_i dx - \int_{\Omega_\varepsilon^+} D_i^\varepsilon\Phi_i\nabla u_i^\varepsilon dx\right] + \\
&+\sum_{i=1}^{2}\left[\int_{\Omega_\varepsilon^+} h_i(v_i)v_i dx - \int_{\Omega_\varepsilon^+} h_i(u_i^\varepsilon)v_i dx - \int_{\Omega_\varepsilon^+} h_i(v_i)u_i^\varepsilon dx\right],
\end{aligned}
$$

$$
\begin{aligned}
B4 = &\varepsilon\sum_{i=1}^{2}\int_{\gamma_\varepsilon}\left[-\left(g_i(x_2,u_i^\varepsilon) - \bar g(x_2)u_i^\varepsilon\right)\tau_i - \left(g_1(x_2,\tau_i) - \bar g(x_2)\tau_i\right)u_i^\varepsilon\right] ds + \\
&+\varepsilon\sum_{i=1}^{2}\int_{\gamma_\varepsilon}\left(g_i(x_2,\tau_i) - \bar g(x_2)\tau_i\right)\tau_i\, ds \\
&-2\int_{\gamma_\varepsilon}\bar g(x_2)\left(u_1^\varepsilon - u_2^\varepsilon\right)\left(\tau_1 - \tau_2\right) ds + \int_{\gamma_\varepsilon}\bar g(x_2)\left(\tau_1 - \tau_2\right)^2 ds.
\end{aligned}
$$

Since we want to pass to the limit in (63), let us analyze the asymptotic behaviour of each of the previous blocks.

Let us start by block B1. By taking $\varphi_1 = u_1^\varepsilon$ and $\varphi_2 = u_2^\varepsilon$ as test functions in (19) and passing to the limit, by (25) and (26), we get

$$\lim_{\varepsilon\to 0} B1 = \lim_{\varepsilon\to 0}\sum_{i=1}^{2}\left[\int_{\Omega_\varepsilon^+} f_i u_i^\varepsilon\,dx + \int_{\Omega^-} f_i u_i^\varepsilon\,dx\right] =$$
$$= \sum_{i=1}^{2}\left[\frac{b-a}{L}\int_{\Omega^+} f_i u_i^+\,dx + \int_{\Omega^-} f_i u_i^-\,dx\right],$$

since u_1^+ and u_2^+ don't depend on the local variable y_1.

Now, by considering $\varphi_1 = u_1$ and $\varphi_2 = u_2$ in (61) we get

$$\begin{aligned}\lim_{\varepsilon\to 0} B_1 &= \sum_{i=1}^{2}\left[\frac{1}{L}\int_{\Omega^+\times(a,b)}\left(d_{21}^i P_i + d_{22}^i\frac{\partial u_i^+}{\partial x_2}\right)\frac{\partial u_i^+}{\partial x_2}\,dx\,dy_1 + \frac{1}{L}\int_{\Omega^+\times(a,b)}\xi_i u_i^+\,dx\,dy_1\right] + \\ &+ \sum_{i=1}^{2}\left[\int_{\Omega^-} D_i\nabla u_i^-\nabla u_i^-\,dx + \int_{\Omega^-} h_i(u_i^-)u_1^-\,dx\right] \\ &+ \frac{2}{L}\sum_{i=1}^{2}\int_{\Omega^+}\left(\zeta_i - g(x_2)u_i^+\right)u_i^+\,dx + \frac{2}{L}\int_{\Omega^+} g(x_2)\left(u_1^+ - u_2^+\right)^2\,dx.\end{aligned} \tag{64}$$

The limits of the second and the third block are as in (49) and (50) respectively. Now it remains to analyze the asymptotic behaviour of the fourth block. By arguing as in the previous step to deduce (58) and (59), since u_1^+ and u_2^+ don't depend on the local variable y_1, we have

$$\begin{aligned}\lim_{\varepsilon\to 0} B4 = &\frac{2}{L}\sum_{i=1}^{2}\int_{\Omega^+}\left[-\left(\zeta_i - g(x_2)u_i^+\right)\tau_i - \left(g_i(x_2,\tau_i) - g(x_2)\tau_i\right)u_i^+\right]dx + \\ &+\frac{2}{L}\sum_{i=1}^{2}\int_{\Omega^+}\left(g_i(x_2,\tau_i) - g(x_2)\tau_i\right)\tau_i\,dx \\ &-\frac{4}{L}\int_{\Omega^+} g(x_2)\left(u_1^+ - u_2^+\right)\left(\tau_1 - \tau_2\right)dx + \frac{2}{L}\int_{\Omega^+} g(x_2)\left(\tau_1 - \tau_2\right)^2\,dx.\end{aligned} \tag{65}$$

Finally, by collecting (49), (50), (63), (64) and (65) at the limit we obtain (62)

7.3. Step 4: identification of ξ_1, ξ_2, ζ_1 and ζ_2

By taking in (62)

$$\Phi_i = \left(P_i, \frac{\partial u_i^+}{\partial x_2}\right)\chi_{\Omega^+} + \nabla u_i^-\chi_{\Omega^-}, \quad i = 1,2$$
$$\tau_i = u_i, \quad i = 1,2$$

and for any $\lambda > 0$ alternatively

$$v_1 = u_1 - \lambda\psi, \quad v_2 = u_2$$

and

$$v_1 = u_1, \quad v_2 = u_2 - \lambda\psi$$

with $\psi \in C_c^1(\Omega^+)$, since u_1 doesn't depend on y_1, by arguing as in the previous section we obtain again (51) and (52). Finally by taking in (62)

$$\Phi_i = \left(P_i, \frac{\partial u_i^+}{\partial x_2}\right)\chi_{\Omega^+} + \nabla u_i^-\chi_{\Omega^-}, \quad i = 1,2$$
$$v_i = u_i, \quad i = 1,2$$

and for any $\lambda > 0$

$$\tau_1 = u_1 - \lambda\phi, \quad \tau_2 = u_2$$

with $\phi \in C_c^1(\Omega)$, since u_1 and u_2 don't depend on y_1, we get

$$\lambda\int_{\Omega^+}\left(\zeta_1 - g_1(x_2, u_1^+ - \lambda\phi)\right)\phi\,dx - \lambda^2\int_{\Omega^+}\phi^2\,dx \geq 0$$

which easily implies

$$\zeta_1 = g_1(x_2, u_1^+) \quad \text{a. e. in } \Omega^+. \tag{66}$$

Analogously, by taking in (62) the same Φ_i and v_i as before and $\tau_1 = u_1$, $\tau_2 = u_2 - \lambda\phi$ with $\phi \in C_c^1(\Omega)$ and since u_2 doesn't depend on y_1 we get

$$\zeta_2 = g_2(x_2, u_2^+) \quad \text{a. e. in } \Omega^+. \tag{67}$$

Finally, by (54), the limit problem (60) can be written as

$$\sum_{i=1}^{2}\left[\int_{\Omega^+} \mathcal{M}\left(\frac{Det(D_i)}{d_{11}^i}\right)\frac{\partial u_i^+}{\partial x_2}\frac{\partial \varphi_i}{\partial x_2}dx + \int_{\Omega^+} h_i(u_i^+)\varphi_i dx\right]+$$

$$+\sum_{i=1}^{2}\left[\int_{\Omega^-} D_i^- \nabla u_i^- \nabla \varphi_i dx + \int_{\Omega^-} h_i(u_i^-)\varphi_i dx\right]+$$

$$+\frac{2}{L}\int_{\Omega^+}\left(g_1(x_2,u_1^+) - g(x_2)u_2^+\right)\varphi_1 + \frac{2}{L}\int_{\Omega^+}\left(g_2(x_2,u_2^+) - g(x_2)u_1^+\right)\varphi_2,$$

$$= \sum_{i=1}^{2}\left[\frac{(b-a)}{L}\int_{\Omega^+} f_i\varphi_i dx + \int_{\Omega^-} f_i\varphi_i dx\right], \quad \forall(\varphi_1,\varphi_2) \in V(\Omega)\times V(\Omega),$$

which, by (54) is the weak formulation of problem (14). Arguing as in the previous section, due to our assumptions $(\mathbf{H}_1) - (\mathbf{H}_5)$, by classical arguments one can prove that the limit problem (14) admits a unique solution. Hence convergences stated in Proposition 5.1 hold for the entire sequence and this ends the proof.

Data availability

No data was used for the research described in the article.

Acknowledgments

G.C. and C.P. are members of GNAMPA (INDAM).

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