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Prescribed-time practical stabilization via majorant systems

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A R T I C L E I N F O

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A B S T R A C T

This paper provides a design methodology of simple control laws to solve the prescribed-time practical stabilization problem for a class of pseudo-linear systems with a single input, also unstable, subject to perturbations, nonlinear and discontinuous, but bounded. The perturbations on the dynamic matrix, such as a ratio of multi-affine polynomials of nonlinear functions or parameters, are not triangular, and a perturbation on the input of multiplicative type is also considered. The stated methodology is based on the majorant systems approach with optimized quadratic Lyapunov functions. The main advantages of the proposed control laws consist in the simplicity of their design and implementation and the reliable performance. Indeed, these control laws depend on a single design parameter, have constant gains, and guarantee the practical convergence with a prescribed relative error, also after a prescribed time. Some examples, also an engineering one, are provided to validate the obtained theoretical results and show the effectiveness and efficiency of the proposed control laws.

1. Introduction

In recent decades, control theory has experienced significant growth due to its importance in the industry. Researchers have been highly attracted to solving stabilization problems for different classes of nonlinear systems. In [1], the exponential global stabilization problem for a broad class of single-input nonlinear systems has been considered for the first time under the assumption that the considered systems should be in an output feedback form or a triangular form with certain growth conditions. This assumption has been commonly adopted in other related works (see e.g., [2-22]).

Over the years, several authors have studied the global stabilization problem for nonlinear systems using different techniques and obtained some valuable results (see e.g., [1-26]). The most researched approach is asymptotic stabilization, where the control objective is to ensure asymptotic convergence of the system states to an equilibrium as time goes to infinity (see e.g., [8-12]).

However, in the practice, a basic issue is to guarantee that the evolution of the system is bounded before a prescribed-time T_f and that after T_f it always belongs to a prescribed neighbourhood of the origin, above all when the life of a system is not finite and/or the considered system does not have any equilibrium point and/or the system is unstable and the measures of some state variables are affected by noises.

In [13-16] a finite-time stabilization approach has been introduced. In some cases, these finite-time convergent control laws exhibit more robustness against perturbations/disturbances, as compared to exponential stability, and ensure rapid transient behavior.

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Moreover, some of the above-mentioned control laws have a disadvantage in terms of their convergence time since they depend on the initial conditions. In detail, by increasing the amplitude of the initial conditions the convergence time increases. As a result, many efforts have been dedicated to finite-time stabilization problems for various classes of nonlinear systems.

To address this drawback, the concept of fixed-time stability has been introduced, aiming to solve the convergence time problem (see e.g., [17-20]). In the fixed-time stability framework, the settling-time function must be uniformly bounded with respect to the initial conditions. However, the convergence time, in many cases, is very conservative with respect to the real stabilization time. Nowadays, there is growing interest in a new concept of stabilization, known as "prescribed-time stabilization" (see e.g., [22-24]). However, it is worth noting that a significant control effort is often required, making its practical implementation unfeasible.

On the other hand, many commonly used control laws are based on such well-known techniques as feedback linearization, inverse model, backstepping, etc. However, these techniques are not always reliable, as the computation time and control signal generation can be significant; moreover, these approaches often turn out be quite complex and experience criticisms from a practical point of view.

Concerning the prescribed-time stabilization problem, many papers available in the literature consider systems with nonlinear perturbations only on the dynamic matrix, of triangular type, and belonging to a symmetrical interval.

Moreover, the control laws proposed, e.g., in [14-19,21-24] have nonlinear and/or time-varying gains, which, in some cases, tend to infinity.

This fact produces practical problems of implementation and, moreover, limits the application of these methods in the presence of unavoidable measurement errors.

Furthermore, the control laws with nonlinear and/or time-varying gains if they are implemented with digital controllers, are subject to unavoidable time-delays, due to the conversion times of the DA and AD converters and to the computation times, which could make the closed-loop system unstable.

Considering the above-mentioned criticisms, this paper provides theoretical results which solves, from an engineering point of view, the global practical stabilization problem with a prescribed relative error for a prescribed time for pseudo-linear systems with a single input, also unstable, subject to perturbations, nonlinear and discontinuous, but bounded. The perturbations on the dynamic matrix, such as a ratio of multi-affine polynomials of nonlinear functions or parameters, are not triangular, and a perturbation on the input of multiplicative type is also considered.

The stated methodology is based on the majorant systems approach with optimized quadratic Lyapunov functions. The main advantages of the proposed control laws consist in the simplicity of their design and implementation and the reliable performance. Indeed, these control laws depend on a single design parameter, whose increase allows to reduce the time of practical stabilization T_f, have constant gains, and guarantee the convergence with a prescribed relative error also after the prescribed time T_f. Finally, some relationships of T_f and the amplitude of the control signal with the single design parameter, which can be useful both to choose the time T_f and size the actuators, are provided.

The paper is organized as follows. The concept of global practical stabilization for a prescribed time and some preliminary results are given in Section 2. The problem statement is presented in Section 3. In Section 4, some control laws for the global practical stabilization in a prescribed time are provided for two different classes of nonlinear systems. Section 5 shows simulation results for several significant examples, also an engineering one. In Section 6, conclusions to this study and the ongoing research are given. Finally, an Appendix provides a Matlab program to compute the optimized Lyapunov functions.

2. Preliminary considerations, notations, and preliminary results

In this section, preliminary considerations, notations, and preliminary theoretical results are provided.

Preliminary considerations

Consider the uncertain nonlinear system $\dot{x} = A(t,x,p,d)x + B(t,x,p,d)u, y = Cx$, where $t \in R$ is the time, $x \in R^n$ is the state, $u \in R^r$ is the input, $y \in R^m$ is the output, $d \in D \subset R^h$ is the disturbance, $p \in \wp \subset R^\mu$ is the vector of uncertainties, $A \in R^{n\times n}$ is the dynamical matrix, $B \in R^{n\times r}$ is the input matrix, and $C \in R^{m\times n}$ is the output matrix of full rank.

A significant problem from a practical point of view is to control, with a law of the type $u(t) = -K(t,x(t))x(t)$, the initial state $x_0 = x(0)$ in a prescribed finite time T_f to the zero state or its sufficiently small neighbourhood, and to keep $x(t)$ at zero or in its neighbourhood also after T_f.

Concerning this, note that if the goal is that $x(T_f)$ is rigorously null, in general, the gain $K(t,x(t))$ has to assume, in the finite interval $[0,T_f]$, values tending to infinity and the measure of the state should not be affected by measurement errors.

About the above matter, consider the simple system $\dot{x}(t) = u(t)$, controlled with the law $u(t) = -k(t)(x(t)+n(t))$, where $n(t)$ is the measurement error. The closed-loop system turns out to be $\dot{x}(t) = -k(t)x(t) - k(t)n(t)$. Assuming that $n(t) = 0$ it is easy to prove that $x(T_f) = e^{-\int_0^{T_f} k(\tau)d\tau}x(0)$. Hence, with a bounded $k(t)$ in the finite interval $[0,T_f]$ $x(T_f)$ cannot be rigorously null. To obtain $x(T_f) = 0$ $k(t)$ has to necessarily assume, in the interval $[0,T_f]$, infinite values. Clearly, if $n(t) \neq 0$ only by chance it would be $x(T_f) = 0$; moreover, the amplitude of the control signal would assume very high values, even infinite.

It is worth noting that if $x(T_f)$ should not be rigorously null, in the hypothesis of an unstable system, the control law should suitably be designed, after T_f, to avoid the divergence of $x(t)$ (see Example 1).

To avoid the above-mentioned drawbacks and the technological ones to implement the control laws with nonlinear and/or time-varying gains tending to infinity it can be required that the control law be designed such that the closed-loop system satisfies the following two conditions (see also Fig. 1):

1) It is globally asymptotically stable.
2) $\forall \delta > 0$ and $\forall x_0 : \| x_0 \| = \delta$ it is $\| x(t) \| / \| x_0 \| \le \varepsilon$, (or $\| y(t) \| / \| x_0 \| \le \varepsilon$), $\forall t \ge T_f$, with $T_f > 0$ and $\varepsilon > 0$ prescribed at will, i.e., the convergence to zero occurs in such a way that after a prescribed time T_f the relative error $\| x(t) - 0 \| / \| x_0 \|$ (or $\| y(t) - 0 \| / \| x_0 \|$) turns out to be always less than a prescribed value ε.

Remark 1. Note that, if the condition 2) is satisfied and the initial state is bounded, as in the practice, e.g., $\| x_0 \| \le \chi_0$, then by choosing $\varepsilon \le \hat{\varepsilon} / \chi_0$, for $t \ge T_f$, it is $\| x(t) \| \le \varepsilon \chi_0 \le \hat{\varepsilon}$. Hence, in such a case, in the hypothesis that the measurement error of $x(t)$ is null, also the absolute error $\| x(t) - 0 \|$ (or $\| y(t) - 0 \|$) can be made small at will after T_f.

Remark 2. Note that the condition 2) originates from the concept of settling time, widely used in the control theory as a quality index of a control system, from some problems of the optimal control (e.g., finite-horizon linear quadratic regulator, also with conditions on $x(T_f)$), from various definitions on uniform ultimate boundedness stability, whose first formulations date back to, e.g., [28-30], and from some practical problems such as, for instance, to empty the tanks of a water system in a prescribed time, or to cool down the environments of a food preservation system in a prescribed time, or also to recover a patient in a prescribed time.

Remark 3. If a quantity $x \in R$ is measured by an instrument with scale range P and accuracy class C_P, the absolute value of the measurement error $|x - x_{measured}|$ with respect to P is very small, but $|x - x_{measured}|$ is not very small. Hence, if $\| x(t) - 0 \|, \forall t \ge T_f$, is small enough with respect to the initial value $\| x_0 \|$ (smaller than the measurement error), from a practical point of view, it is $x(t) = 0$, since the actual value of $x(t)$ is unknown.

It is easy to realize that there is not always a control law of the type $u = -K(x,t)x$ such that the closed-loop system $\dot{x} = (A(t,x,p,d) - B(t,x,p,d)K(t,x(t))x$ satisfies the conditions 1) and 2). Moreover, when such a control law exists it is not easy to determine it and find the best one (e.g., the control law which minimizes the amplitude of the control signal).

In the following, new theoretical results will be stated, which guarantee that the conditions 1) and 2) can always be satisfied for certain classes of uncertain nonlinear systems, and some methods to easily design simple and effective control laws with constant gains guaranteeing the conditions 1) and 2) are provided.

Notations

$$\begin{aligned} &\| x \|_P = \sqrt{x^T P x}, \quad \| x \| = \| x \|_I = \sqrt{x^T x}, \quad C_{P,\rho} = \{x : \| x \|_P = \rho\}, \rho \ge 0 \\ &cond(P) = \lambda_{\max}(P) / \lambda_{\min}(P), \end{aligned} \tag{1}$$

where $P \in R^{n \times n}$ is a symmetric and positive definite (*p.d.*) matrix, x^T is the transpose of $x \in R^n$; $\lambda_{\max}(W)(\lambda_{\min}(W))$ denotes the maximum (minimum) eigenvalue of a square matrix $W \in R^{n \times n}$ with real eigenvalues.

The prescribed relative error ε of $\| x(t) \|$, in the following, will be denoted also with $1/N$.

Preliminary results

Lemma 1. *[31] Consider a symmetric p.d. matrix $P \in R^{n \times n}$, a symmetric matrix $Q(c) \in R^{n \times n}$, continuous with respect to $c \in \mathbb{C}$, where $\mathbb{C}$ is a*

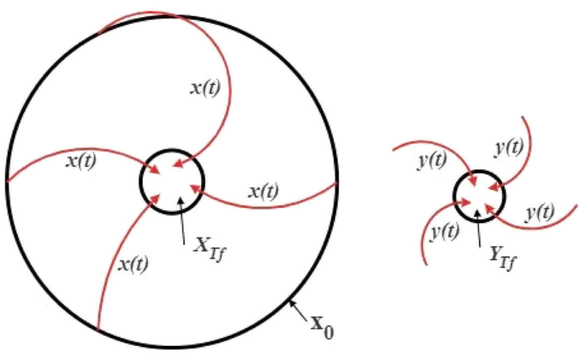

Fig. 1. $X_0 = \{x_0 : \| x_0 \| = \delta\}$, $X_{Tf} = \{x(t) : \| x(t) \| / \| x_0 \| \le \varepsilon, \forall x_0 \in X_0, \forall t \ge T_f\}$ (or $Y_{Tf} = \{y(t) : \| y(t) \| / \| x_0 \| \le \varepsilon, \forall x_0 \in X_0, \forall t \ge T_f\}$).

compact subset of R^{μ}, *a matrix* $B \in R^{n\times r}$ *of rank* r, *a matrix* $C \in R^{m\times n}$ *of rank m, and a vector* $w(t,x,d,p) \in R^{m}$ *such that* $\| w(t,x,d,p) \| \leq \hat{w}$. *Then,* $\forall x \in C_{P,\rho} = \{x \in R^n : x^T P x = \rho^2\}$, $\rho \geq 0$, *the following inequalities hold:*

$$\max_{x\in C_{P,\ \rho},\ c\in\mathbb{C}} x^T Q(c) x \leq \max_{c\in\mathbb{C}} \lambda_{\max}\left(Q(c)P^{-1}\right)\rho^2 \tag{2}$$

$$\max_{x\in C_{P,\ \rho}} x^T P B w(t,x,d,p) \leq \sqrt{\lambda_{max}\left(B^T P B\right)}\,\hat{w}\,\rho \tag{3}$$

$$\| Cx \| \leq \sqrt{\lambda_{\max}\left(CP^{-1}C^T\right)}\,\rho. \tag{4}$$

Lemma 2. *[32] Consider a symmetric p.d. matrix P and a symmetric matrix* $Q(c) \in R^{n\times n}$, *whose elements* q_{ij} *are ratio of a multi-affine polynomials to a non-zero multi-affine polynomials of* $c_1\, c_2 \ldots c_{\mu}$. *If* $c_1\, c_2 \ldots c_{\mu}$ *are bounded functions or parameters,* i.e., *if* $c = [c_1\, c_2 \ldots c_{\mu}]^T \in \mathbb{C} = [c_1^-,\quad c_1^+] \times [c_2^-,\quad c_2^+] \times \cdots \times [c_{\mu}^-,\quad c_{\mu}^+] = [c^-,\quad c^+] \subset R^{\mu}$, *the value of* $\lambda_{\max c\in\mathbb{C}}\left(Q(c)P^{-1}\right)$ *is obtained in one of the* 2^{μ} *vertices* V_c *of* $\mathbb{C}$, i.e., $\lambda_{\max c\in C}\left(Q(c)P^{-1}\right) = \lambda_{\max c_i = c_i^-, c_i^+}\left(Q(c)P^{-1}\right)$.

Lemma 3. *[33] Consider the system*

$$\dot{x} = Ax + Bu, A \in R^{n\times n}, B \in R^{n\times 1}, \tag{5}$$

where (A, B) *is completely reachable, with the control law*

$$u = -Kx, K = ([d_n\, d_{n-1}\, \ldots\, d_1] - [a_n\, a_{n-1}\, \ldots\, a_1])T^{-1}, \tag{6}$$

where

$$T = \left[B\, AB \ldots A^{n-1}B\right] \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}, \lambda^n + a_1\lambda^{n-1} + \ldots a_{n-1}\lambda + a_n = \det(\lambda I - A). \tag{7}$$

Then,

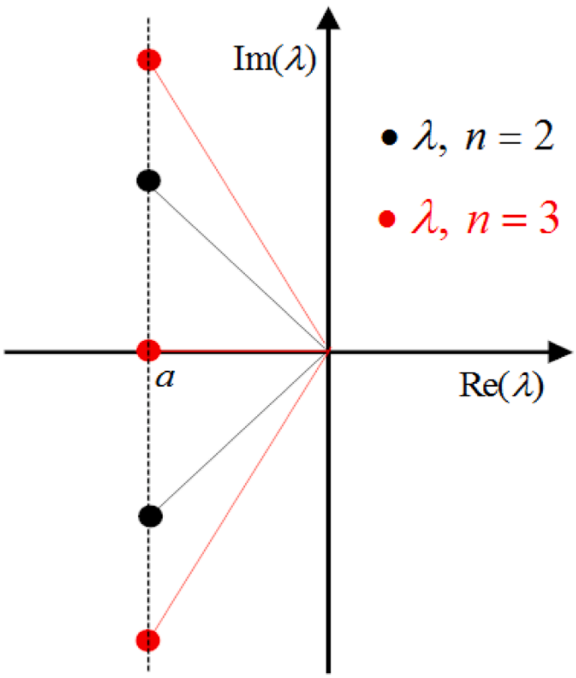

Fig. 2. Eigenvalues (11) for $n = 2, 3$.

$$\det(\lambda I-\widehat{A})=\lambda^n+d_1\lambda^{n-1}+\ldots d_{n-1}\lambda+d_n,\widehat{A}=A-BK \tag{8}$$

$$A_c=T^{-1}\widehat{A}T=\begin{bmatrix}0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\\ -d_n & -d_{n-1} & \cdots & -d_1\end{bmatrix},B_c=T^{-1}B=\begin{bmatrix}0\\ \vdots\\ 0\\ 1\end{bmatrix}. \tag{9}$$

Lemma 4. *[31] Let $A_c\in R^{n\times n}$ be a matrix with distinct eigenvalues $\lambda_i, i=1,2,\ldots,n$, and V be a matrix whose columns are the corresponding eigenvectors. Then,*

$$\lambda_{\max}\left(QP^{-1}\right)/2=\max(real(\lambda_i)),P=(VV^*)^{-1},Q=A_c^TP+PA_c. \tag{10}$$

Lemma 5. *If the eigenvalues of a matrix $A_c\in R^{n\times n}$ are as follows (see Fig. 2):*

$$\lambda_i=-a+j a\tan(\alpha_i),\alpha_i=\frac{n-1+2i}{2n}\pi,i=1,2,\ldots,n,\ \ a\neq 0, \tag{11}$$

and V is a matrix whose columns are the corresponding eigenvectors, then

$$\lambda_{\max}\left(QP^{-1}\right)/2=-a,P=(VV^*)^{-1},Q=A_c^TP+PA_c. \tag{12}$$

Proof. The proof follows from Lemma 4 and (11).

Remark 4. The polynomials $d(\lambda)$ for $n=2,3,4,5$, whose roots λ_i are the ones in (11), are

$$\begin{array}{l} d(\lambda)=\lambda^2+2a\lambda+2a^2 n=2\\ d(\lambda)=\lambda^3+3a\lambda^2+6a^2\lambda+4a^3 n=3\\ d(\lambda)=\lambda^4+4a\lambda^3+12a^2\lambda^2+16a^3\lambda+8a^4 n=4\\ d(\lambda)=\lambda^5+5a\lambda^4+20a^2\lambda^3+40a^3\lambda^2+40a^4\lambda+16a^5 n=5\ . \end{array} \tag{13}$$

Remark 5. There exist several matrices P satisfying (10). To realize this, just compute the eigenvectors matrix V with the Matlab command [V L]=eig(Ac) and, then, consider different normalizations of the columns of V.

Concerning this, the following robustness theorem holds.

Theorem 1. *Consider the matrix*

$$A_n(h,a)=\begin{bmatrix}0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\\ -hd_n(a) & -hd_{n-1}(a) & \cdots & -hd_1(a)\end{bmatrix},h\in[1,\widehat{h}], \tag{14}$$

whose eigenvalues for $h=1$, $a>0$, and $n\geq 2$ are the ones in (11). Suppose that there exists a matrix P_{1n} such that

$$\lambda_{\max}\left(Q_1P_{1n}^{-1}\right)/2=-1,Q_1=A_n^T(h,a)P_{1n}+P_{1n}A_n(h,a),\ \text{for}\ a=1\ \text{and}\ h=1,\widehat{h}. \tag{15}$$

Then,

$$\lambda_{\max}\left(QP^{-1}\right)/2=-a,Q=A_n^T(h,a)P+PA_n(h,a),\forall a>0\ \text{and}\ \forall h\in[1,\widehat{h}], \tag{16}$$

where

$$P=T_aP_{1n}T_a,T_a=diag\left(\left[a^{n-1}a^{n-2}\ldots 1\right]\right). \tag{17}$$

Proof. Take into account that

$$T_aA(h,a)T_a^{-1}=aA(h,1),\quad T_a^{-1}A^T(h,a)T_a=aA^T(h,1),\quad P^{-1}=T_a^{-1}P_{1n}^{-1}T_a^{-1}. \tag{18}$$

Hence,

$$\begin{aligned}&\lambda_{\max}\left(\left(A_n^T(h,a)P+PA_n(h,a)\right)P^{-1}\right)/2=\lambda_{\max}\left(T_a^{-1}\left(\left(A_n^T(h,a)T_aT_a^{-1}P+PT_a^{-1}T_aA_n(h,a)\right)T_a^{-1}P_{1n}^{-1}T_a^{-1}\right)T_a\right)/2\\&=a\lambda_{\max}\left(\left(A_n^T(h,1)P_{1n}+P_{1n}A_n(h,1)\right)P_{1n}^{-1}\right)/2=-a,\end{aligned} \tag{19}$$

from which, taking into account that $A_n(h,1)$ linearly depends on h, by Lemma 2, equality (16) follows.

Remark 6. Some matrices P_{1n} satisfying (15) for $n=1,2,\dots,10$ and $\widehat{h}=100$ can be computed using the Matlab program reported in the Appendix.

Such matrices are obtained with the relation $P_{1n}=\gamma_n\left(\widehat{V}_{1n}\widehat{V}_{1n}^*\right)^{-1}$, where $\widehat{V}_{1n}$ is a matrix whose columns are the eigenvectors of $A_n(1,1)$ suitably normalized with an optimization algorithm, and γ_n is a constant such that $P_{1n}(n,n)=n$.

It is easy to verify that for $n=2,3,4$ (see [27]) and $n=5$ Theorem 1 holds for each $\widehat{h}\geq 1$, i.e., also greater than 100.

For $n\geq 11$, analogous matrices P_{1n} can be computed; if the relation $P_{1n}=\gamma_n\left(V_{1n}V_{1n}^*\right)^{-1}$ is used without appropriately normalizing the eigenvectors matrix V_{1n} then Theorem 1 certainly holds for $\widehat{h}=1$.

In the following, the optimized matrices P_{1n} for $n=2,3,4,5$ are explicitly reported.

$$P_{1n}=\begin{bmatrix}4&2\\2&2\end{bmatrix},\ n=2,\ P_{1n}=\begin{bmatrix}16&16&4\\16&20&6\\4&6&3\end{bmatrix},\ n=3$$

$$P_{1n}=\begin{bmatrix}64&96&48&8\\96&160&88&16\\48&88&56&12\\8&16&12&4\end{bmatrix},\ n=4,\ P_{1n}=\begin{bmatrix}256&512&384&128&16\\512&1088&864&304&40\\384&864&736&280&40\\128&304&280&120&20\\16&40&40&20&5\end{bmatrix},\ n=5. \tag{20}$$

Theorem 2. *Consider the trapezoidal matrix*

$$C=\begin{bmatrix}c_{11}&c_{12}&0&\cdots&0&0\\c_{21}&c_{22}&c_{23}&\cdots&0&0\\c_{31}&c_{32}&c_{33}&\cdots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\c_{n-1,1}&c_{n-1,2}&c_{n-1,3}&\cdots&c_{n-1,n-1}&c_{n-1,n}\\c_{n,1}&c_{n,2}&c_{n,3}&\cdots&c_{n,n-1}&c_{n,n}\end{bmatrix}, \tag{21}$$

and the matrix $T_a=diag\left(\left[a^{n-1}\ a^{n-2}\ \dots\ 1\right]\right)$. Then,

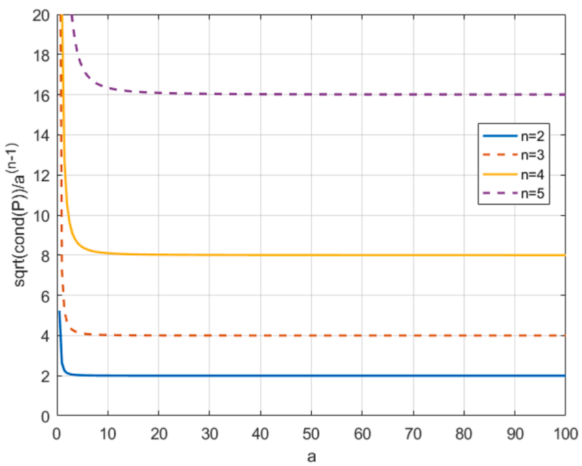

Fig. 3. Behaviours of $\sqrt{cond(P(a))}/a^{(n-1)}$ for $n=2,3,4,5$.

$$T_aCT_a^{-1}=\begin{bmatrix} c_{11} & c_{12}a & 0 & \cdots & 0 & 0 \\ c_{21}/a & c_{22} & c_{23}a & \cdots & 0 & 0 \\ c_{31}/a^2 & c_{32}/a & c_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1,1}/a^{n-2} & c_{n-1,2}/a^{n-3} & c_{n-1,3}c_{n-1,3}/a^{n-4} & \cdots & c_{n-1,n-1} & c_{n-1,n}a \\ c_{n,1}/a^{n-1} & c_{n,2}/a^{n-2} & c_{n,3}/a^{n-3} & \cdots & c_{n,n-1}/a & c_{n,n} \end{bmatrix}. \tag{22}$$

Proof. Note that the non-null element of the *i-th* row of the diagonal matrix T_a is a^{n-i}, while the non-null element of the *j-th* column of the diagonal matrix T_a^{-1} is $1/a^{n-j}$. From this, it follows that the element *ij* of $T_aCT_a^{-1}$ turns out to be $c_{ij}a^{n-i}/a^{n-j}=c_{ij}a^{j-i}$. Hence, the proof follows.

Remark 7. It is worth noting that (22) in Theorem 2 highlights that the weights of the triangular perturbations are inversely proportional to a power of the design parameter a (hence, they reduce as a increases), the ones of the main diagonal are unitary, while the ones of the upper diagonal with respect to the main one increase linearly with a.

Theorem 3. *Let $P\in R^{n\times n}$ be a symmetric p.d. matrix, a matrix $C\in R^{m\times n}$ of rank m, and $f(t)$ be a positive function. If $\| x(t) \|_P\le f(t)\| x_o\|_P$, where $\| x(t) \|_P=\sqrt{x^TPx}$, then*

$$\| Cx(t) \|\le f(t)\sqrt{\lambda_{\max}(CP^{-1}C^T)\lambda_{\max}(P)}\| x_o \| \tag{23}$$

$$\| x(t) \|\le f(t)\sqrt{cond(P)}\| x_o \|, cond(P)=\lambda_{\max}(P)/\lambda_{\min}(P). \tag{24}$$

Proof. The proof of (23) follows from (4), taking into account that $\| x_o\|_P\le\sqrt{\lambda_{\max}(P)}\| x_o \|$. The proof of (24) follows from (23), considering that $\lambda_{\max}(P^{-1})=1/\lambda_{\min}(P)$.

In Fig. 3, the behaviours of $\sqrt{cond(P(a))}/a^{(n-1)}$ for $n=2,3,4,5$ are reported.

For $n=2,3,\dots,10$, it is easy to verify, as it is shown also in Fig. 3 for $n=2,3,4,5$, that for a sufficiently large a holds that

$$\sqrt{cond(P(a))}\cong(2a)^{n-1}. \tag{25}$$

3. Problem statement

First, consider the following system (more general with respect to the ones considered in [14-19,21-24]):

$$\begin{cases} \dot{x}_1=x_2+\sum_{i=1,2}c_{1i}(t,x,p,d)x_i \\ \dot{x}_2=x_3+\sum_{i=1,2,3}c_{2i}(t,x,p,d)x_i \\ \vdots \\ \dot{x}_n=\sum_{i=1,\dots,n}c_{ni}(t,x,p,d)x_i+f(t,x,p,d)u, \end{cases} \tag{26}$$

where $x=[x_1,x_2,\dots,x_n]^T\in R^n$ is the state, $u\in R$ is the control input, $d\in D\subset R^h$ is a disturbance, $p\in\wp\subset R^\mu$ is the vector of uncertain parameters, $f(t,x,p,d)$ is a nonlinearity always different from zero such that there exists a known constant g for which

$$h=fg\in[1,\ h^+],h^+\le\hat{h}, \tag{27}$$

and $c_{ij}(t,x,p,d),\ i=1,\dots,n,\ j=1,\dots,\min\{i+1,n\}$, are generic functions (also discontinuous) satisfying the following condition

$$|c_{ij}(t,x,p,d)|\le\hat{c}_{ij} \tag{28}$$

for some known constants $\hat{c}_{ij}$, or

$$c_{ij}(t,x,p,d)\in\left[c_{ij}^-,c_{ij}^+\right] \tag{29}$$

for some known constants c_{ij}^-,c_{ij}^+.

The system (26) can be rewritten as

$$\dot{x}=(A+C(t,x,p,d))x+Bf(t,x,p,d)u, \tag{30}$$

where

$$A=\begin{bmatrix}0&1&0&\cdots&0&0\\0&0&1&\cdots&0&0\\0&0&0&\cdots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\cdots&0&1\\0&0&0&\cdots&0&0\end{bmatrix}\in R^{n\times n},\ B=\begin{bmatrix}0\\0\\0\\\vdots\\0\\1\end{bmatrix}\in R^{n\times 1}, C=\begin{bmatrix}c_{11}&c_{12}&0&\cdots&0&0\\c_{21}&c_{22}&c_{23}&\cdots&0&0\\c_{31}&c_{32}&c_{33}&\cdots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\c_{n-1,1}&c_{n-1,2}&c_{n-1,3}&\cdots&c_{n-1,n-1}&c_{n-1,n}\\c_{n,1}&c_{n,2}&c_{n,3}&\cdots&c_{n,n-1}&c_{n,n}\end{bmatrix}\in R^{n\times n}. \tag{31}$$

Now, the main objective is to design a control law of the type $u=-Kx$, with a constant $K\in R^{1\times n}$, for the system (30) such that the closed-loop system satisfies the conditions 1) and 2).

4. Control design

A) Consider the system (26) with the simple control law

$$u=-g[d_n(a)d_{n-1}(a)...d_1(a)]x, \tag{32}$$

where $d_n(a), d_{n-1}(a),..., d_1(a)$ are the coefficients of the polynomial $d(\lambda)=\lambda^n+d_1(a)\lambda^{n-1}+...+d_n(a)$, whose roots are the ones given in (11). In particular,

$$\begin{aligned}
&u=-g\left[2a^2 \quad 2a\right]x, \quad n=2\\
&u=-g\left[4a^3 \quad 6a^2 \quad 3a\right]x, \quad n=3\\
&u=-g\left[8a^4 \quad 16a^3 \quad 12a^2 \quad 4a\right]x, \quad n=4\\
&u=-g\left[16a^5 \quad 40a^4 \quad 40a^3 \quad 20a^2 \quad 5a\right]x, \quad n=5\\
&u=-g\left[32a^6 \quad 96\ a^5 \quad 120a^4 \quad 80a^3 \quad 30\ a^2 \quad 6a\right]x, \quad n=6\\
&u=-g\left[64a^7 \quad 224a^6 \quad 336a^5 \quad 280a^4 \quad 140a^3 \quad 42a^2 \quad 7a\right]x, \quad n=7\\
&u=-g\left[128a^8 \quad 512a^7 \quad 896a^6 \quad 896a^5 \quad 560a^4 \quad 224a^3 \quad 56a^2 \quad 8a\right]x, \quad n=8\\
&u=-g\left[256a^9 \quad 1152a^8 \quad 2304a^7 \quad 2688a^6 \quad 2016a^5 \quad 1008a^4 \quad 336a^3 \quad 72a^2 \quad 9a\right]x, \quad n=9\\
&u=-g\left[512a^{10} \quad 2560a^9 \quad 5760a^8 \quad 7680a^7 \quad 6720a^6 \quad 4032a^5 \quad 1680a^4 \quad 480a^3 \quad 90a^2 \quad 10a\right]x, \quad n=10.
\end{aligned} \tag{33}$$

The following main result holds.

Theorem 4. *The evolution of the closed-loop system (26), (32),* i.e.*, of the system*

$$\dot{x}=A_{cn}x=(A_n(h,a)+C)x, \tag{34}$$

where $A_n(h,a)$ is the matrix (14) and $C\in[C^-,C^+]$ is *the matrix (31), can be estimated as follows:*

$$\| x(t)\|\le\sqrt{cond(P)}\| x_0\|e^{-\alpha t}, \tag{35}$$

where P is the matrix (17),

$$\alpha=a-r_1^+a-r_0-\frac{r_1^-}{a}-...-\frac{r_{n-1}^-}{a^{n-1}} \tag{36}$$

$$\begin{aligned}
&r_1^+=\frac{1}{2}\lambda_{\max\limits_{\substack{c_{12}=c_{12}^-,c_{12}^+,...,\\c_{n-1,n}=c_{n-1,n}^-,c_{n-1,n}^+}}}\left(\left(\left(C_1^+\right)^TP_{1n}+P_{1n}C_1^+\right)P_{1n}^{-1}\right), r_0=\frac{1}{2}\lambda_{\max\limits_{c_{ii}=c_{ii}^-,c_{ii}^+,i=1,...,n}}\left(\left(C_0^TP_{1n}+P_{1n}C_0\right)P_{1n}^{-1}\right),...,\\
&r_{n-1}^-=\frac{1}{2}\lambda_{\max\limits_{c_{n1}=c_{1n}^-,c_{n1}^+}}\left(\left(\left(C_{n-1}^-\right)^TP_{1n}+P_{1n}C_{n-1}^-\right)P_{1n}^{-1}\right)
\end{aligned} \tag{37}$$

$$C_1^+=\begin{bmatrix}0&c_{12}&0&\cdots&0\\0&0&c_{23}&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&c_{n-1,n}\\0&0&0&\cdots&0\end{bmatrix}, C_0=\begin{bmatrix}c_{11}&0&0&\cdots&0\\0&c_{22}&0&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&0\\0&0&0&\cdots&c_{nn}\end{bmatrix},...,C_{n-1}^-=\begin{bmatrix}0&0&0&\cdots&0\\0&0&0&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&0\\c_{n1}&0&0&\cdots&0\end{bmatrix}. \tag{38}$$

Proof. By choosing as Lyapunov function the quadratic form $V=x^TPx=\| x\|_P^2=\rho^2$, for x belonging to the generic hyper-circumference $C_{P,\rho}$, its derivative satisfies the inequalities

$$2\rho\dot{\rho} \le \max_{\substack{x\in C_{P,\rho},h=1,h^+\\ c_{ij}=c_{ij}^-,c_{ij}^+}} x^T\left(A_c^TP+PA_c\right)x \le \max_{x\in C_{P,\rho},h=1,h^+} x^T\left(A_n^T(h,a)P+PA_n(h,a)\right)x$$
$$+\max_{\substack{x\in C_{P,\rho}\\ c_{12}=c_{12}^-,c_{12}^+,\dots,\\ c_{n-1,n}=c_{n-1,n}^-,c_{n-1,n}^+}} x^T\left(\left(C_1^+\right)^TP+PC_1^+\right)x+\max_{\substack{x\in C_{P,\rho}\\ c_{ii}=c_{ii}^-,c_{ii}^+,i=1,\dots,n}} x^T\left(C_0^TP+PC_0\right)x+\dots+\max_{\substack{x\in C_{P,\rho},\\ c_{n1}=c_{1n}^-,c_{n1}^+}} x^T\left(\left(C_{n-1}^-\right)^TP+PC_{n-1}^-\right)x, \tag{39}$$

from which, by Lemma 1, it follows that

$$\dot{\rho} \le \frac{1}{2}\max_{C=C^-,C^*,h=1,h^+}\lambda_{\max}\left(\left(A_c^TP+PA_c\right)P^{-1}\right)\rho=\frac{1}{2}\max_{C=C^-,C^*,h=1,h^+}\lambda_{\max}\left(T_a^{-1}\left(A_c^TP+PA_c\right)P^{-1}T_a\right)\rho$$
$$\le\frac{1}{2}\max_{h=1,h^+}\lambda_{\max}\left(T_a^{-1}\left(A_n^T(h,a)P+PA_n(h,a)\right)P^{-1}T_a\right)\rho+\frac{1}{2}\max_{\substack{c_{12}=c_{12}^-,c_{12}^+,\dots,\\ c_{n-1,n}=c_{n-1,n}^-,c_{n-1,n}^+}}\lambda_{\max}\left(T_a^{-1}\left(\left(C_1^+\right)^TP+PC_1^+\right)P^{-1}T_a\right)\rho \tag{40}$$
$$+\frac{1}{2}\max_{c_{ii}=c_{ii}^-,c_{ii}^+,i=1,\dots,n}\left(T_a^{-1}\left(C_0^TP+PC_0\right)P^{-1}T_a\right)\rho+\dots+\frac{1}{2}\max_{c_{n1}=c_{1n}^-,c_{n1}^+}\left(T_a^{-1}\left(\left(C_{n-1}^-\right)^TP+PC_{n-1}^-\right)P^{-1}T_a\right)\rho.$$

From (40) and Theorems 1, 2 holds that

$$\dot{\rho}\le-\alpha\rho,\ \alpha=a-r_1^+a-r_0-\frac{r_1^-}{a}-\dots-\frac{r_{n-1}^-}{a^{n-1}}, \tag{41}$$

from which, by Theorem 3, (35) follows.

If $r_1^+<1$, from (41), for a sufficiently large a, one obtains

$$\alpha\cong\alpha_\infty=\left(1-r_1^+\right)a\,. \tag{42}$$

Remark 8. A value $\alpha_c(a)\ge\alpha$, i.e., less conservative, turns out to be

$$\alpha_c=-\max_{c_{ij}=c_{ij}^-,c_{ij}^+,h=1,h^+}\lambda_{\max}\left(\left(\left(aA_n(h,1)+C_{Ta}\right)^TP_{1n}+P_{1n}\left(aA_n(h,1)+C_{Ta}\right)\right)P_{1n}^{-1}\right)/\,2 \tag{43}$$

where $C_{Ta}=T_aCT_a^{-1}$ is the matrix (22).

In Fig. 4, the behaviours of $\alpha,\alpha_c,\alpha_\infty$ under the hypothesis that $n=2,\ h\in[1,2],c_{11}\in[-2,2],c_{12}\in[-0.7,0.7],c_{21}\in[-3,3],c_{22}\in[-4,4]$ are reported.

It can be concluded that if there exists a value of the parameter a such that $\alpha(a)>0$ ($\alpha_c(a)>0$) (this always occurs if $1-r_1^+\ge\gamma>0$, see (41)), then, $\forall N>0$, e.g., $N=100$, it is $\|x(t)\|<\|x_o\|/N$, $\forall t\ge T_f$, where

$$T_f=\frac{\log\left(N\sqrt{cond(P)}\right)}{\alpha(a)}\quad\left(T_f=\frac{\log\left(N\sqrt{cond(P)}\right)}{\alpha_c(a)}\right). \tag{44}$$

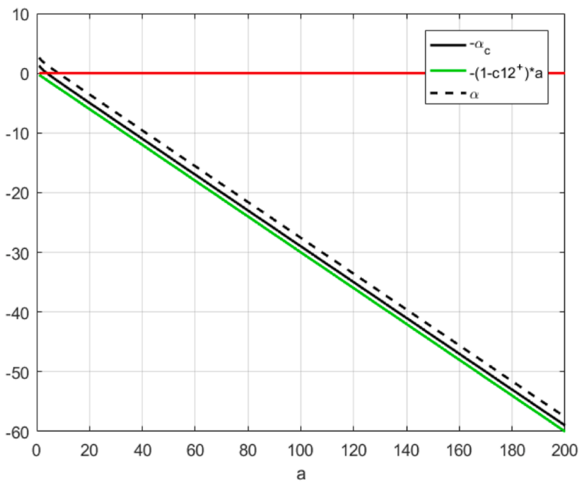

Fig. 4. Behaviours of $\alpha,\alpha_c,\alpha_\infty$ for $n=2,h\in[1,2],c_{11}\in[-2,2],c_{12}\in[-0.7,0.7]$, $c_{21}\in[-3,3],c_{22}\in[-4,4]$.

Remark 9. Condition $1-r_1^+ \geq \gamma > 0$ is surely verified if $c_{i,i+1} = 0, i = 1,2,...,n-1$, i.e., if matrix C is triangular. It is worth noting that no interval $\left[c_{i,i+1}^-, c_{i,i+1}^+\right], i = 1,2,...,n-1,$ can contain the value -1 since for this value the considered system is not completely controllable. Assuming that $|c_{i,i+1}| \leq \widehat{c}_{i,i+1} \geq 0, i = 1,2,...,n-1$, some lower values of r_1^+ are

$$r_1^+ = \begin{cases} \widehat{c}_{12} & n = 2 \\ 1.414\widehat{c}_{12} + 2.236\widehat{c}_{23} & n = 3 \\ 2.449\widehat{c}_{12}+4.472\widehat{c}_{23}+3.742\widehat{c}_{34} & n = 4 \\ 4.472\widehat{c}_{12}+10.099\widehat{c}_{23}+9.592\widehat{c}_{34}+5.477\widehat{c}_{45} & n = 5. \end{cases} \tag{45}$$

Note that if $r_1^+ < 1$ for (25), (41), and (44) when a increases, then T_f decreases and $lim_{a\to\infty} T_f = 0$. Hence, T_f can be chosen at will.

Remark 10. The use of an optimized quadratic Lyapunov function, which guarantees also the condition 1), allows to obtain an upper bound of $|u(t)|$, $\forall t \in [0,\infty)$, and $\forall x_0 \in R^n$. Indeed, using (4) with $c = [d_n(a)\ d_{n-1}(a)\ \dots\ d_1(a)]$ and (33) yields

$$|u(t)| \leq |g|\sqrt{cP^{-1}c^T}\rho = \begin{cases} |g|\sqrt{2}ae^{-\alpha_c t}\rho_0, & n = 2 \\ |g|\sqrt{3}ae^{-\alpha_c t}\rho_0, & n = 3 \\ |g|\sqrt{4}ae^{-\alpha_c t}\rho_0, & n = 4 \\ |g|\sqrt{5}ae^{-\alpha_c t}\rho_0, & n = 5. \end{cases} \tag{46}$$

From (46) it follows that if T_f decreases then $|u(t)|$ increases; hence, (46) can be useful both to choose the time T_f and size the actuators.

Now, the following main theorem can be stated.

Theorem 5. *Using the control law (32), with a sufficiently large a, the system (26) can be globally practically stabilized with a prescribed relative error $1/N$ for a prescribed finite time T_f, if $h^+ \leq \widehat{h}$ and $r_1^+ < 1$, i.e., if the uncertainties $c_{i,i+1}, i = 1,2,...,n-1$, are small enough.*

Proof. The proof follows from Theorem 4, (44), and Remark 9.

B) Consider the system

$$\dot{x} = \left(A_g + C_g(t,x,d,p)\right)x + B_g f(t,x,d,p)u$$
$$A_g = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \in R^{n\times n}, B_g = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in R^{n\times 1}, C_g = \begin{bmatrix} \gamma_{11} & \gamma_{12} & 0 & \cdots & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{n-1,1} & \gamma_{n-1,2} & \gamma_{n-1,3} & \cdots & \gamma_{n-1,n} \\ \gamma_{n1} & \gamma_{n2} & \gamma_{n3} & \cdots & \gamma_{nn} \end{bmatrix}, \tag{47}$$

where a_{ij} are known constants with $a_{i,i+1} \neq 0, i = 1,...,n-1$, and γ_{ij} and f are ratios of multi-affine polynomials to non-zero multi-affine polynomials of bounded functions $\delta = \left[\delta_1(t,x,d,p)\,\delta_2(t,x,d,p)\dots\delta_\mu(t,x,d,p)\right]^T \in D = [\delta_1^-,\ \ \delta_1^+] \times [\delta_2^-,\ \ \delta_2^+] \times \cdots \times [\delta_\mu^-,\ \ \delta_\mu^+] = [\delta^-,\ \ \delta^+] \subset R^\mu$, with $f \neq 0$ such that there exists a known constant g for which $h = fg \in [1, h^+]$.

Remark 11. Note that the class of systems (47) is more general than (26) and many engineering systems belong to it (see, e.g., Example 4 in Section 5).

Since $a_{i,i+1} \neq 0, i = 1,...,n-1$, the pair (A_g, B_g) is completely reachable. Hence, the system (47) can be controlled with the control law

$$u = -gKx, K = ([d_n(a)\ d_{n-1}(a)\ \dots\ d_1(a)] - [a_n\ a_{n-1}\ \dots\ a_1])T^{-1}, \tag{48}$$

where

$$T = \left[B_g \; A_g B_g \; \dots \; A_g^{n-1} B_g \right] \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}, \lambda^n + a_1 \lambda^{n-1} + \dots a_{n-1} \lambda + a_n = \det(\lambda I - A) \tag{49}$$

$$\lambda^n + d_1(a)\lambda^{n-1} + \dots d_{n-1}(a)\lambda + d_n(a) = \prod_{i=1,\dots,n} (\lambda - \lambda_i), \lambda_i = -a + j a \tan(\alpha_i), \alpha_i = \frac{n-1+2i}{2n}\pi, i = 1, 2, \dots, n.$$

Therefore, the closed-loop system turns out to be

$$\dot{x} = \left(A_g - h B_g K + C_g \right) x. \tag{50}$$

By making the change of variables $z = T^{-1}x$, using Lemma 3 and following the analysis of subsection A), it is easy to verify that the evolution $x(t)$ of the system (50) can be estimated as follows:

$$\| x(t) \| \leq \sqrt{cond(P)} \| x_0 \| e^{-\alpha t}, \tag{51}$$

where

$$P = T_i^T T_a P_{1n} T_a T_i, T_i = T^{-1}, T_a = diag\left(\left[a^{n-1} a^{n-2} \dots a 1 \right] \right) \tag{52}$$

$$\alpha(a) = - \max_{\substack{\delta_i = \delta_i^-, \delta_i^+, i=1,\dots,\mu \\ h=1,h^+}} \lambda_{\max} \left(\left(\left(a A_n(h,1) + C_{gTa} \right)^T P_{1n} + P_{1n} \left(a A_n(h,1) + C_{gTa} \right) \right) P_{1n}^{-1} \right) / 2 \tag{53}$$

$$C_{gTa} = T_a \left(T^{-1} \begin{bmatrix} \gamma_{11} & \gamma_{12} & 0 & \cdots & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{n-1,1} & \gamma_{n-1,2} & \gamma_{n-1,3} & \cdots & \gamma_{n-1,n} \\ \gamma_{n1} & \gamma_{n2} & \gamma_{n3} & \cdots & \gamma_{nn} \end{bmatrix} T + (h-1) \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_2 \\ a_1 \end{bmatrix}^T \right) T_a^{-1}. \tag{54}$$

Remark 12. Note that the matrix T and its inverse T^{-1} are lower triangular matrices. Therefore, it follows that if C_g is trapezoidal, C_{gTa} is also trapezoidal; analogously, if C_g is lower triangular, C_{gTa} is also lower triangular.

From the analysis reported in this subsection and Lemma 2, it can be deduced that with the simple control law (48), with a sufficiently large a, the system (47) can be globally practically stabilized with a prescribed relative error $1/N$ for a prescribed time T_f, if $h^+ \leq \hat{h}$ and uncertainties $\gamma_{i,i+1}, i = 1, 2, \dots, n-1$, are small enough (see Examples 3, 4 in Section 5).

Remark 13. Note that if instead of the state x the output $y = [10\dots0]x$ is considered, using (4) and (51), holds that $\| y(t) \| \leq \frac{k}{a^n} \sqrt{cond(P)} \| x_0 \| e^{-\alpha t}$; $k = \sqrt{2}/2, \sqrt{3/8}, \sqrt{5/16}, \dots, n = 2, 3, 4, \dots$. Hence, for a value smaller than a (therefore, also smaller than $|u|$) $y(t)$ practically converges to zero for a prefixed time T_f.

Remark 14. Finally, consider the case where

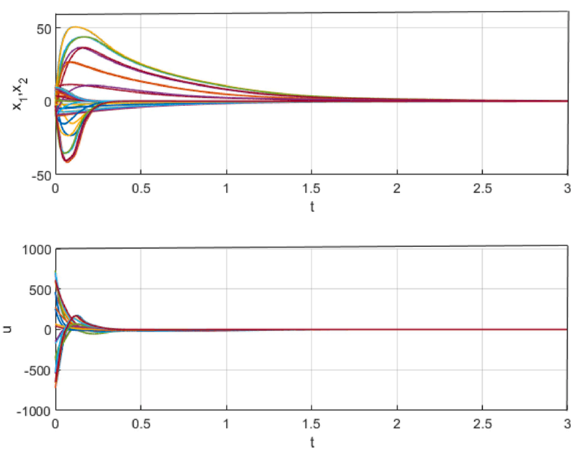

Fig. 5. Time histories of $x(t)$ and $u(t)$, $x_0 : \| x_0 \| = 10$, $a = 6$.

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{23} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \in R^{n\times n}, B=\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} \in R^{n\times 1}, C=\begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{23} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1,1} & c_{n-1,2} & c_{n-1,3} & \cdots & c_{n-1,n} \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nn} \end{bmatrix}. \tag{55}$$

It is worth noting that if the pair (A,B) is completely reachable, then the system (55) can be prescribed time practically stabilizable only for particular values of T_f and N or not stabilizable with the control law (48) (see Example 5 in Section 5).

5. Examples

Some illustrative examples, which consider several systems (also of order greater than two and of engineering type) with various perturbations both on the dynamic matrix (of trapezoidal type) and on the input one, with various random initial conditions, and, in a case, with measurement errors, are provided. They highlight the simplicity, effectiveness, and efficiency of the proposed methodology.

Example 1. Consider the second-order nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 + sin(x_2{}^2)sin(10t)x_1 + 0.7sign(x_1)x_2 \\ \dot{x}_2 &= x_1/(abs(x_2)+1) + x_2/(abs(x_1)+1) + (2+sin(x_1))u, \end{aligned} \tag{56}$$

with the simple control law $u=-g[2a^2\ \ 2a]x, (g=1,$ since $f\ =h\in[1,3])$.

The nonlinear functions $c_{ij}(t,x), i,j=1,2,$ satisfy the following inequalities:

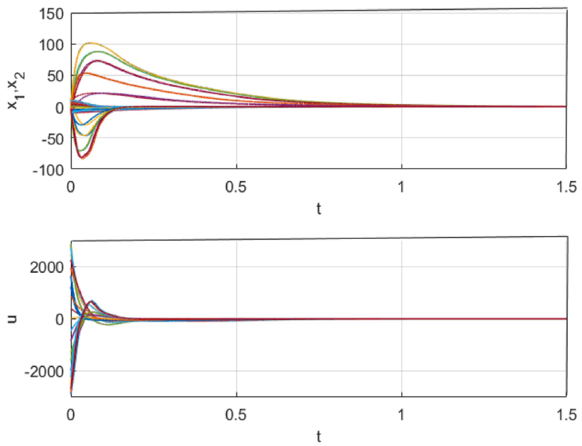

Fig. 6. Time histories of $x(t)$ and $u(t)$, $x_0 : \| x_0 \| = 10$, $a = 12$.

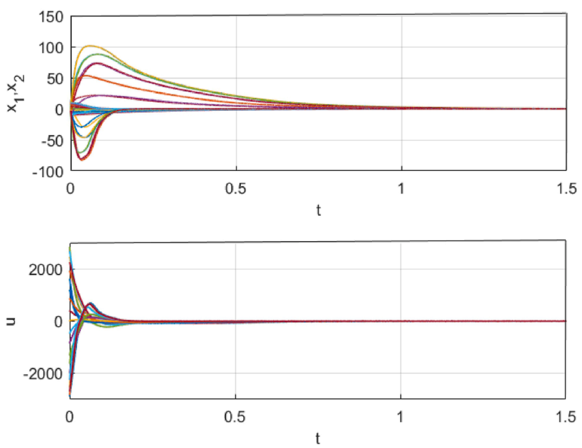

Fig. 7. Time histories of $x(t)$ and $u(t)$, $x_0 : \| x_0 \| = 10$, $a = 12$, and measurement errors of x_i $e_i \in [-0.1, 0.1]$.

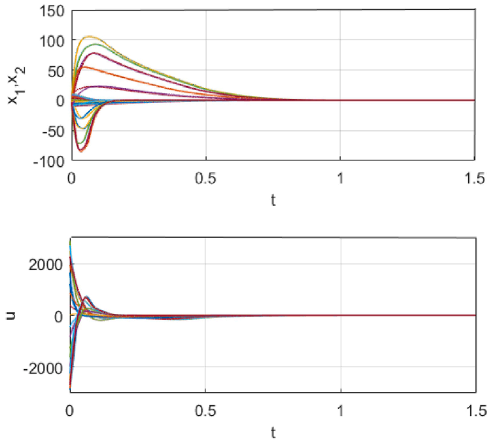

Fig. 8. Time histories of $x(t)$ and $u(t)$, with the control law (58), $x_0 : \| x_0 \| = 10$.

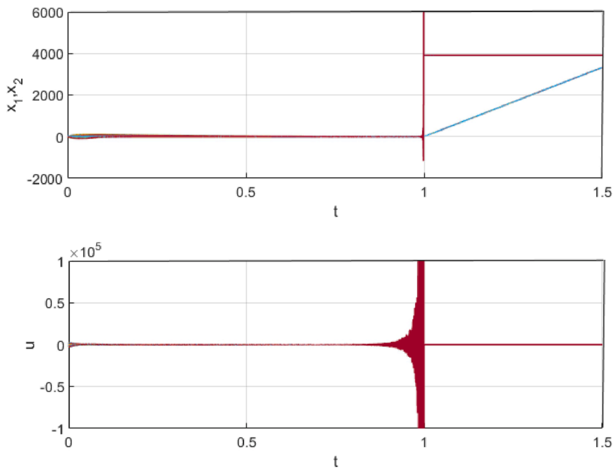

Fig. 9. Time histories of $x(t)$ and $u(t)$, with the control law (58), $x_0 : \| x_0 \| = 10$, and measurement errors of x_i $e_i \in [-0.1, 0.1]$. The actual max(abs (u)) is equal to 5.9962e+08.

$$
\begin{aligned}
&|c_{11}(t,x)x_1| \le 1|x_1|, |c_{12}(x)x_2| \le 0.7|x_2| \\
&|c_{21}(x)x_1| = \left|\frac{1}{|x_2|+1}\right||x_1| \le 1|x_1|, |c_{22}(x)x_2| = \left|\frac{1}{|x_1|+1}\right||x_2| \le 1|x_2|.
\end{aligned}
\tag{57}
$$

Clearly, $\widehat{c}_{ij}$ can be selected as $\widehat{c}_{11} = 1, \widehat{c}_{12} = 0.7, \widehat{c}_{21} = 1, \widehat{c}_{22} = 1$.

In Figs. 5 and 6, the time histories of $x(t)$ and $u(t)$ are reported for some initial conditions $x_0 : \| x_0 \| = 10$, assuming that $a = 6$ and $a = 12$, respectively.

In Fig. 7, the time histories of $x(t)$ and $u(t)$ are reported for some initial conditions $x_0 : \| x_0 \| = 10$, assuming that $a = 12$ and measurement errors e_i of x_i belong to $[-0.1, 0.1]$ (the measurement instruments of x_i are of scale range $P = 100$ and accuracy class $C_p = 0.1$).

Alternatively, using the following control law with gains tending to infinity

$$
u = \begin{cases} -\left[2a^2 \;\; 2a\right]x, a = 12/(1-t), & t < 0.999 \\ 0, & t \ge 0.999 \end{cases},
\tag{58}
$$

the time histories of $x(t)$ and $u(t)$ are reported for some initial conditions $x_0 : \| x_0 \| = 10$ in Fig. 8, while the time histories of $x(t)$ and $u(t)$, in the presence of measurement errors of x_i $e_i \in [-0.1, 0.1]$, and with actuators and measurement instruments without saturations, are shown in Fig. 9.

In Fig. 10, the time histories of $x(t)$ and $u(t)$ are reported for some initial conditions $x_0 : \| x_0 \| = 10$ under the hypothesis that

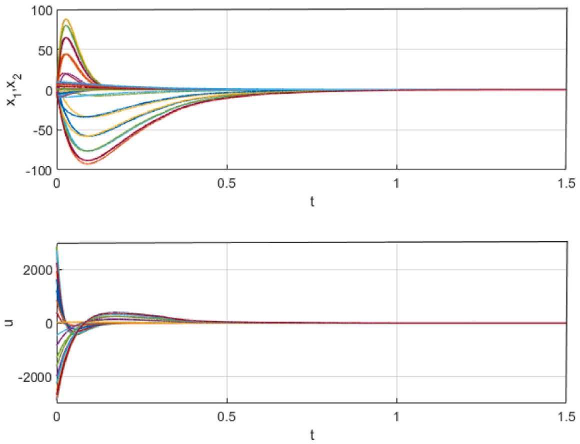

Fig. 10. Time histories of $x(t)$ and $u(t),x_0 : \| x_0 \| = 10$, for the system (59) with $u = -[2a^2 \;\; 2a]x,\;\; a = 12$.

$$\begin{aligned}\dot{x}_1 &= x_2 + x_1 sign(x_2) - 0.7 sign(x_1)x_2 \\ \dot{x}_2 &= x_1 sign(x_2) + x_2 sign(sin(10t)) + (2 - sign(5x_1))u,\end{aligned} \tag{59}$$

with $u = -[2a^2 \;\; 2a]x,\;\; a = 12$.

Example 2. Consider the following third-order system:

$$\begin{aligned}\dot{x}_1 &= x_2 + \frac{x_1 {x_3}^2}{1 + {x_3}^2} + 0.25x_2 \sin({x_2}^2) \\ \dot{x}_2 &= x_3 + x_1 \sin(x_2) + sign(x_3)x_2 \\ \dot{x}_3 &= \frac{x_1}{|x_3| + 1} + \frac{x_2}{1 + {x_1}^2} + 0.9\sin(10t)x_3 + (2 + \cos(x_1))u\end{aligned} \tag{60}$$

with the simple control law $u = -g[4a^3 \;\; 6a^2 \;\; 3a]x$ ($g = 1$, since $f = h \in [1,3]$) and the initial conditions $x_0 : \| x_0 \| = 1$.

Since the nonlinear functions $c_{ij}(t,x), i,j = 1,2,3$, satisfy the following inequalities:

$$\begin{aligned}&|c_{11}(t,x)x_1| = \left|\frac{x_3^2}{1 + x_3^2}\right||x_1| \le 1|x_1|, |c_{12}(t,x)x_2| = |0.25\sin({x_2}^2)||x_2| \le 0.25|x_2|, c_{13}(t,x) = 0 \\ &|c_{21}(t,x)x_1| = |\sin(x_2)||x_1| \le 1|x_1|, |c_{22}(t,x)x_2| = |sign(x_3)||x_2| \le |x_2|, c_{23}(t,x) = 0 \\ &|c_{31}(t,x)x_1| = \left|\frac{1}{1 + |x_3|}\right||x_1| \le 1|x_1|, |c_{32}(t,x)x_2| = \left|\frac{1}{1 + x_1^2}\right||x_2| \le 1|x_2|, |c_{33}(t,x)x_3| = |0.9\sin(10t)||x_3| \le 0.9|x_3|,\end{aligned} \tag{61}$$

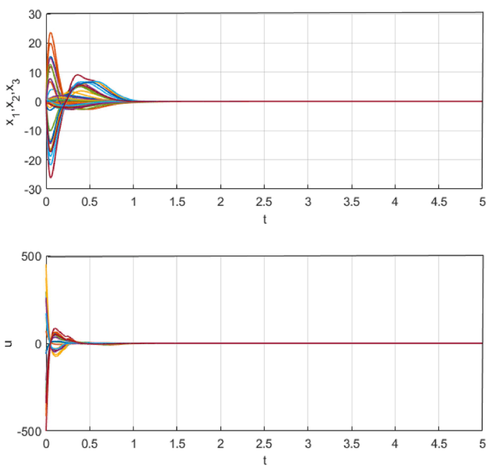

Fig. 11. Time histories of $x(t)$ and $u(t)$, $x_0 : \| x_0 \| = 1$, $a = 5$.

clearly, $\hat{c}_{ij}$ can be selected as $\hat{c}_{11} = 1, \hat{c}_{12} = 0.25, \hat{c}_{13} = 0, \hat{c}_{21} = 1, \hat{c}_{22} = 1, \hat{c}_{23} = 0, \hat{c}_{31} = 1, \hat{c}_{32} = 1, \hat{c}_{33} = 0.9$.

Fig. 11 shows the time histories of $x(t)$ and $u(t)$ for some initial conditions $x_0 = [1\ -1\ 1]$ under the hypothesis that $a = 5$.

Example 3. Consider the system

$$\begin{aligned}
\dot{x}_1 &= x_1 + 2x_2 + c_{11}(t, x_1, x_2, x_3, d, p) + c_{12}(t, x_1, x_2, x_3, d, p) + c_{13}(t, x_1, x_2, x_3, d, p) \\
\dot{x}_2 &= x_1 + 2x_2 + 3x_3 + c_{21}(t, x_1, x_2, x_3, d, p) + c_{22}(t, x_1, x_2, x_3, d, p) + c_{23}(t, x_1, x_2, x_3, d, p) \\
\dot{x}_3 &= 2x_1 + 3x_2 + 4x_3 + f(t, x_1, x_2, x_3, d, p)u + c_{31}(t, x_1, x_2, x_3, d, p) + c_{32}(t, x_1, x_2, x_3, d, p) + c_{33}(t, x_1, x_2, x_3, d, p).
\end{aligned} \tag{62}$$

If

$$\begin{aligned}
&f = h \in [1, 2], c_{11} \in [-1, 1], c_{12} = \in [-0.25, 0.25], c_{13} = 0 \\
&c_{21} \in [-1, 1], c_{22} \in [-1, 1], c_{23} \in [-0.25, 0.25]; c_{31} \in [-1, 1], c_{32} \in [-1, 1], c_{33} \in [-1, 1],
\end{aligned} \tag{63}$$

with the simple control law

$$u = -\left(\left[4a^3\ \ 6a^2\ \ 3a\right] - [-3\ \ 3\ \ -7]\right)\begin{bmatrix} 0.1667 & 0 & 0 \\ 0.1667 & 0.3333 & 0 \\ 0.5000 & 1.000 & 1.000 \end{bmatrix} x, \tag{64}$$

the functions $\alpha(a)$ and $T_f(a)$ are shown in Figs. 12a, b.

For $a = 10, 15, 20, 25, 30, 35, 40, 45, 50$, $N = 100$, the values of T_f are T_f=4.9095, 1.7756, 1.1515, 0.8713, 0.7086, 0.6009, 0.5238, 0.4656, 0.4199, respectively.

In Fig. 13, the responses $x(t)$, for $a = 15$, $x_0 = [1\ -1\ 1]^T$, and for all the maximum and minimum constant values of c_{ij}, are shown.

Remark 15. Note that the actual value of T_f is smaller than the computed one. This occurs because only constant uncertainties are considered and because, as it is well-known, the Lyapunov method, in some cases, is very conservative.

Example 4. Consider the four-room temperature conditioning system in Fig. 14, where $x_i, i = 1, \ldots, 4$, are the overtemperatures of the rooms with respect to the temperature of the external environment, which is supposed constant, G_1, G_2, G_3, G_4 are the dispersion conductances of the rooms towards the external environment, G_{21}, G_{32}, G_{34} are the conductances among the rooms, C_1, C_2, C_3, C_4 are the thermal capacities of the rooms, and the input u is the air conditioner's power.

It is easy to verify that the model of the considered system is given by

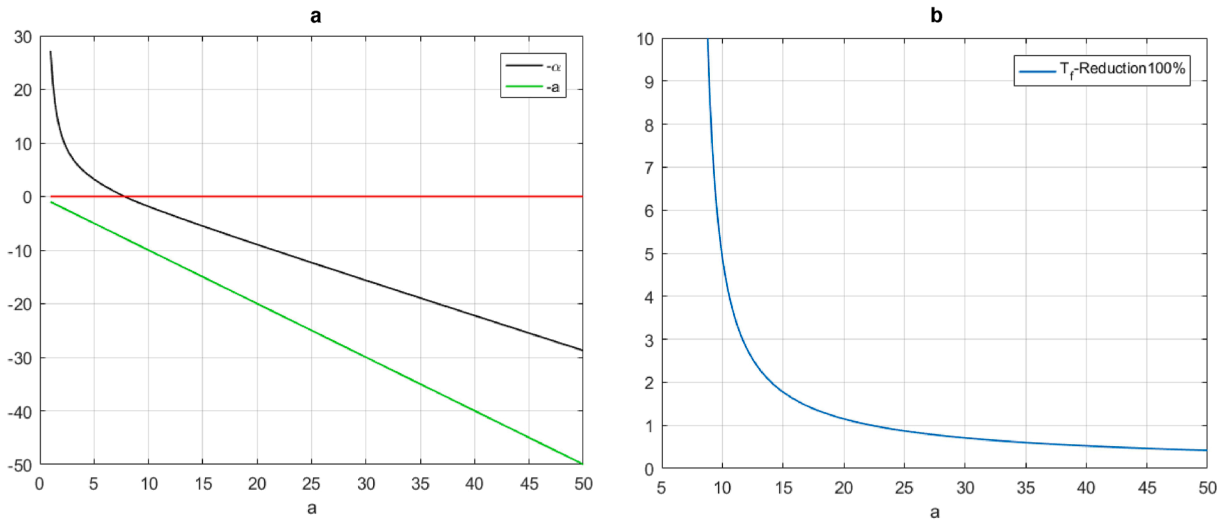

Fig. 12. a, b. $\alpha(a)$ and $T_f(a)$ obtained with the control law (64).

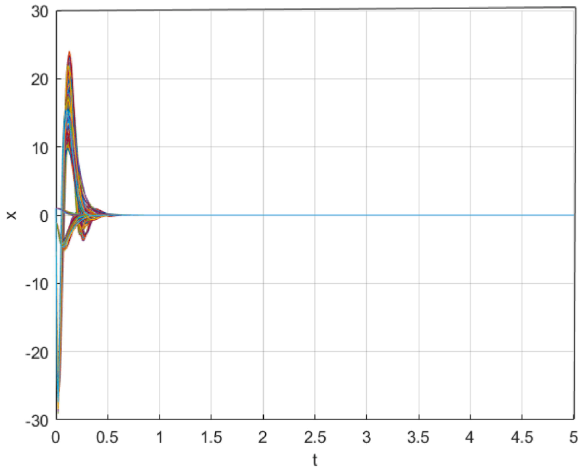

Fig. 13. Some responses for $x_0 = [1 \ -1 \ 1]^T$, $a = 15$.

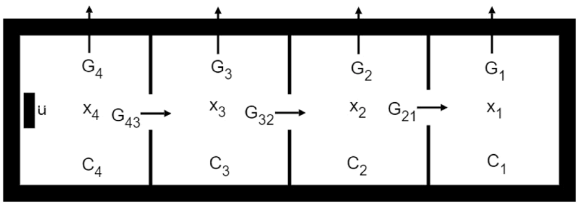

Fig. 14. Four-room temperature conditioning system.

$$\dot{x} = \begin{bmatrix} -(G_1+G_{21})/C_1 & G_{21}/C_1 & 0 & 0 \\ G_{21}/C_2 & -(G_2+G_{21}+G_{33})/C_2 & G_{32}/C_2 & 0 \\ 0 & G_{32}/C_3 & -(G_3+G_{32}+G_{43})/C_3 & G_{43}/C_3 \\ 0 & 0 & G_{43}/C_4 & -(G_4+G_{43})/C_4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/C_4 \end{bmatrix} u \tag{65}$$
$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x.$$

If

$$\begin{aligned} &G_1 = 1.5+\delta_1, G_2 = 1+\delta_2, G_3 = 1+\delta_3, G_4 = 1.5+\delta_4 \\ &G_{21} = 1+\delta_5, G_{32} = 1+\delta_6, G_{43} = 1+\delta_7 \\ &C_1 = 2+\delta_8, C_2 = 2+\delta_9, C_3 = 2+\delta_{10}, C_4 = 2+\delta_{11}, \end{aligned} \tag{66}$$

the system (65) can be rewritten as follows

$$\dot{x} = \left(\begin{bmatrix} -1.25 & 0.5 & 0 & 0 \\ 0.5 & -1.5 & 0.5 & 0 \\ 0 & 0.5 & -1.5 & 0.5 \\ 0 & 0 & 0.5 & -1.25 \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} & 0 & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ 0 & \gamma_{32} & \gamma_{33} & \gamma_{34} \\ 0 & 0 & \gamma_{43} & \gamma_{44} \end{bmatrix} \right) x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} fu, y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x, \tag{67}$$

where

$$\gamma_{11} = -\frac{1.5+\delta_1+1+\delta_5}{2+\delta_8} - (-1.25) = \frac{\delta_1+\delta_5+1.25\delta_8}{2+\delta_8}, \gamma_{12} = \frac{\delta_5-0.5\delta_8}{2+\delta_8}, \dots, f = \frac{1}{2+\delta_{11}}. \tag{68}$$

With the simple control law

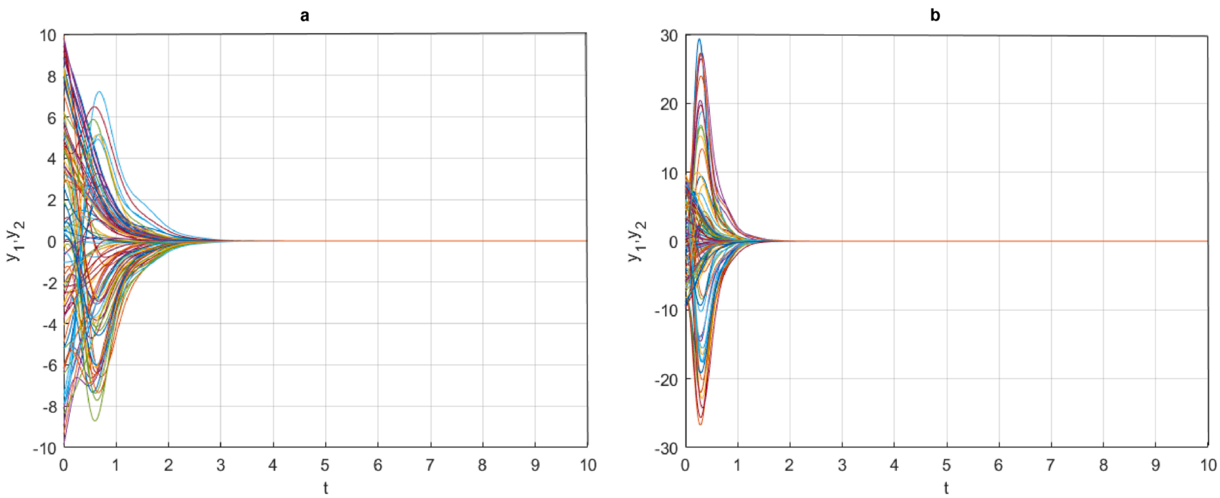

Fig. 15. a, b. Time histories of y for $a = 2.5,\ 5$, respectively.

$$u = -2.2\left(\begin{bmatrix} 8a^4 & 16a^3 & 12a^2 & 4a \end{bmatrix} - \begin{bmatrix} 2.25 & 8.3125 & 10.5625 & 5.5 \end{bmatrix}\right)\begin{bmatrix} 8 & 0 & 0 & 0 \\ -10 & 4 & 0 & 0 \\ 14.5 & -11 & 2 & 0 \\ -23.625 & 24.75 & -8.5 & 1 \end{bmatrix}x \tag{69}$$

the time histories of y for $a = 2.5$ and $a = 5$, initial conditions $x_0 \in \left[\begin{bmatrix} -10 & -10 & -10 \end{bmatrix}^T, \begin{bmatrix} 10 & 10 & 10 \end{bmatrix}^T\right]$, and values of $\delta_1 \in [-0.3, 0.3]$, $\delta_2 \in [-0.2, 0.2]$, $\delta_3 \in [-0.2, 0.2]$, $\delta_4 \in [-0.3, 0.3]$, $\delta_5 \in [-0.1, 0.1]$, $\delta_6 \in [-0.1, 0.1]$, $\delta_7 \in [-0.1, 0.1]$, $\delta_8 \in [-0.2, 0.2]$, $\delta_9 \in [-0.2, 0.2]$, $\delta_{10} \in [-0.2, 0.2]$, $\delta_{11} \in [-0.2, 0.2]$, varied 50 times randomly, are reported in Figs. 15a, b.

Finally, consider the following example, with uncertain nonlinear perturbations for all the elements of A and f, and T, T^{-1} not lower triangular.

Example 5. Consider the system

$$\dot{x} = \begin{bmatrix} 1 + c_{11} & 2 + c_{12} & 1 + c_{13} \\ 1 + c_{21} & 2 + c_{22} & -3 + c_{23} \\ 2 + c_{31} & 3 + c_{32} & -2 + c_{33} \end{bmatrix}x + \begin{bmatrix} 1 \\ -1 \\ 20 \end{bmatrix}fu, \tag{70}$$

where $c_{ij}(t, x, \delta)$ and $f(t, x, \delta)$ are bounded nonlinear functions.

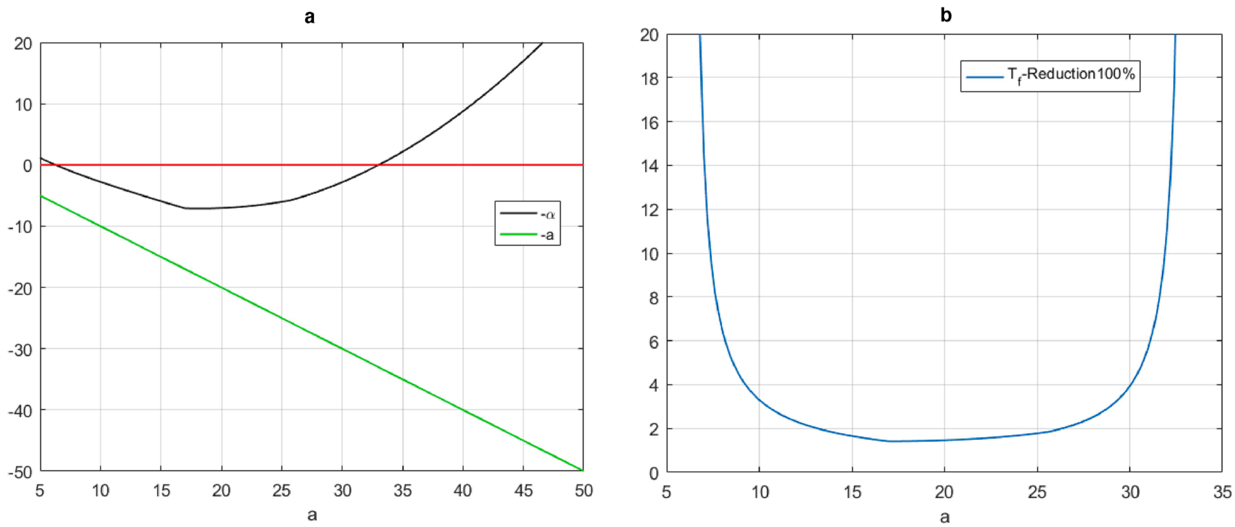

Fig. 16. a, b. Behaviours of $\alpha(a)$ and $T_f(a)$.

If

$$\begin{aligned} &f = h \in [1,2], c_{11} \in [-0.5,0.5], c_{12} \in [-0.1,0.1], c_{13} \in [-0.01,0.01]; c_{21} \in [-0.5,0.5], c_{22} \in [-0.5,0.5] \\ &c_{23} \in [-0.1,0.1],\ \ c_{31} \in [-0.5,0.5], c_{32} \in [-0.5,0.5], c_{33} \in [-0.5,0.5], \end{aligned} \tag{71}$$

with the simple control law

$$u = -\left(\left[4a^3\ \ 6a^2\ \ 3a\right] - [4\ \ 1\ \ -1]\right) \begin{bmatrix} 0.0074 & -0.0025 & 0.0002 \\ -0.0094 & -0.0190 & -0.0005 \\ -0.0293 & -0.0582 & 0.0486 \end{bmatrix} x, \tag{72}$$

the behaviours of $\alpha(a)$, and $T_f(a)$ for $N = 100$, are shown in Figs. 16a, b.

For $a = 10$, 15, 20, 25, 30, the values of T_f are 3.3027, 1.6569, 1.4650, 1.7812, 3.9288.

Note that, in this case, $T_f \geq 1.415$. Hence, T_f cannot be chosen at will.

6. Conclusions and future developments

The main peculiarities and advantages of the proposed methodology can be summarized as follows.

- This paper provides theoretical results which solves, from an engineering point of view, the global practical stabilization problem with a prescribed relative error for a prescribed time for pseudo-linear systems with a single input, also unstable, subject to perturbations, nonlinear and discontinuous, but bounded. The perturbations on the dynamic matrix, such as a ratio of multi-affine polynomials of nonlinear functions or parameters, are not triangular, and a perturbation on the input of multiplicative type is also considered.
- The stated methodology is based on the majorant systems approach with optimized quadratic Lyapunov functions.
- The proposed control laws are simple to design and implement and provide reliable performance.
  Indeed, these control laws depend on a single design parameter, whose increase allows to reduce the time of practical stabilization T_f, have constant gains, and guarantee the convergence with a prescribed relative error also after the prescribed time T_f.
- The provided control laws are explicitly presented for 2, 3, …,10-dimensional systems, which can be extended to the multivariable case.
- Some relationships of T_f and the amplitude of the control signal with the single design parameter, which can be useful both to choose the time T_f and size the actuators, are provided.
- The desired performance of the proposed control laws is verified via numerical simulations, considering several systems (also of order greater than two and of engineering type) with various perturbations both on the dynamic matrix (of trapezoidal type) and the input one, and for various random initial conditions, and in the presence of measurement errors, too.

The ongoing research aims at extending the proposed results in case of unmeasurable states, presence of disturbances and/or additional nonlinearities, measurement errors, and to multi-input multi-output nonlinear systems (number of inputs $r > 1$, and $x \in R^{(n \cdot r) \times 1}$), also in the case where the state is the tracking error.

CRediT authorship contribution statement

Laura Celentano: Writing – review & editing, Writing – original draft, Supervision, Software, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Alison Garza-Alonso:** Writing – original draft, Software. **Michael V. Basin:** Writing – review & editing, Writing – original draft, Supervision, Conceptualization.

Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

In this appendix, the Matlab program to compute the optimized matrices P_{1n} for $n = 2, \ldots, 10$ and $\widehat{h} = 100$ is provided.

```
% OPTIMA P1n, n=2:10, h=1:100

clc, clear all, close all

for n=2:10
 alf=(n-1+2*[1:n])/2/n*pi;p=-ones(1,n)+sqrt(-1)*tan(alf);d=real(poly(p));
 A=[zeros(n-1,1) eye(n-1) ;-d(end:-1:2)];
 [V L]=eig(A);

 ROT=[0.8660  0.8660  0       0       0       0       0       0       0       0
      1.0000  0.6614  0.6614  0       0       0       0       0       0       0
      0.5411  0.5411  0.8950  0.8950  0       0       0       0       0       0
      0.4702  0.4702  0.7137  0.7137  1.0000  0       0       0       0       0
      1.1553  1.1323  1.4831  1.6262  2.4849  2.4945  0       0       0       0
      1.0708  1.4120  1.3346  1.7580  2.1511  2.6867  3.1130  0       0       0
      0.6887  1.4108  0.8220  1.5898  1.7943  1.3210  2.2048  2.3478  0       0
      0.8908  1.0784  1.0446  1.1767  1.4375  1.5314  2.3850  2.2124  2.8705  0
      1.1551  0.7522  0.7794  1.5818  1.1004  1.4640  1.9224  1.7681  2.9109  2.4232];

  RO=diag(ROT(n-1,1:n));
  V=V*RO; P=real(inv(V*V'));
  P=P/P(n,n)*n; if n<=5;P=round(P);end

  clc, n
  Q=A'*P+P*A;
  lM1=max(real(eig(Q*inv(P))/2))
  Ac=A;Ac(end,:)=Ac(end,:)*100;Qc=Ac'*P+P*Ac;
  LM100=max(real(eig(Qc*inv(P))/2))
  pause
end
```

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