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A COMPACTNESS RESULT FOR ELLIPTIC EQUATIONS WITH SUBQUADRATIC GROWTH IN PERFORATED DOMAINS

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INTRODUCTION

In this paper we study the asymptotic behaviour, as ε tends to zero, of the (unbounded) solutions u_ε of the following problem:

$$\begin{cases} -\operatorname{div}(A^\varepsilon Du_\varepsilon) + H_\varepsilon(x, u_\varepsilon, Du_\varepsilon) = f \quad \text{in } \Omega\backslash T_\varepsilon, \\ (A^\varepsilon Du_\varepsilon)\cdot \nu = 0 \quad \text{on } \partial T_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega, \\ u_\varepsilon \in H^1(\Omega_\varepsilon), \ \ H_\varepsilon(x, u_\varepsilon, Du_\varepsilon) \in L^1(\Omega_\varepsilon), \ \ H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)u_\varepsilon \in L^1(\Omega_\varepsilon), \end{cases} \tag{0.1}$$

and of the (bounded) solutions u_ε of the problem:

$$\begin{cases} -\operatorname{div}(A^\varepsilon Du_\varepsilon) + \gamma u_\varepsilon = H_\varepsilon(x, u_\varepsilon, Du_\varepsilon) + g \quad \text{in } \Omega\backslash T_\varepsilon, \\ (A^\varepsilon Du_\varepsilon)\cdot \nu = 0 \quad \text{on } \partial T_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega, \quad u_\varepsilon \in H^1(\Omega_\varepsilon)\cap L^\infty(\Omega_\varepsilon), \end{cases} \tag{0.2}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $\Omega_\varepsilon = \Omega\backslash T_\varepsilon$ is obtained by removing from Ω a closed subset T_ε ("the holes") of $\mathbb{R}^n$ contained in Ω, A^ε belongs to a family of equibounded and uniformly definite positive matrices, f is in $L^2(\Omega)$, g is in $L^q(\Omega)$, with $q > n/2$, and γ is a positive real. We assume that H_ε is a Caratheodory function defined on $\Omega_\varepsilon \times \mathbb{R} \times \mathbb{R}^n$, with sub-quadratic growth with respect to ξ. Moreover in Problem (0.1) H_ε satisfies the signum property: $H_\varepsilon(x, s, \xi)s \geq 0$.

We work in the framework of the general notion of H^0-convergence, introduced by Briane *et al.* [1] and dealing essentially with the convergence of the solutions of linear problems in perforated domains. When the set of holes is empty, this notion reduces to the H-convergence introduced by Murat and Tartar[2] and in the symmetric case to the G-convergence introduced by De Giorgi and Spagnolo [3].

We prove that there exists a subsequence, still denoted by $\{\varepsilon\}$, a suitable class of extension-operators $\{P_\varepsilon\}_\varepsilon$, a bounded and definite positive matrix A^0, a Caratheodory function H on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ satisfying the same properties of $\{H_\varepsilon\}_\varepsilon$, a positive function χ_0 in $L^\infty(\Omega)$ and u in $H_0^1(\Omega)$ such that

$$\chi_{\Omega_\varepsilon} \rightharpoonup \chi_0 \quad \text{weakly} * \text{ in } L^\infty(\Omega),$$

$$(A^\varepsilon, T_\varepsilon) \xrightarrow{H^0} A^0,$$

$$P_\varepsilon u_\varepsilon \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega),$$

$$[H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)]^\sim \rightharpoonup H(x, u, Du) \quad \text{weakly in } L^1(\Omega),$$

$$-\operatorname{div}(A^\varepsilon \widetilde{Du_\varepsilon}) \rightharpoonup -\operatorname{div}(A^0 Du) \quad \text{weakly in } \mathcal{D}'(\Omega),$$

as ε tends to zero, where $^\sim$ denotes the zero-extension on Ω and u satisfies

$$\begin{cases} -\operatorname{div}(A^0 Du) + H(x, u, Du) = \chi_0 f & \text{in } \Omega, \\ u \in H_0^1(\Omega), \quad H(x, u, Du) \in L^1(\Omega), \quad H(x, u, Du)u \in L^1(\Omega) \end{cases}$$

if u_ε is a solution of (0.1), or

$$\begin{cases} -\operatorname{div}(A^0 Du) + \gamma \chi_0 u = H(x, u, Du) + \chi_0 g & \text{in } \Omega, \\ u \in H_0^1(\Omega) \cap L^\infty(\Omega) \end{cases}$$

if u_ε is a solution of (0.2).

The nonlinear function H is described in terms of the corrector matrices of the linear problem, by adapting to our situation some ideas of [4–6].

We mainly point out the peculiarities of this construction due to the presence of the holes.

Let us recall that (see [1]) in general, the existence of a global corrector requires some additional assumptions on Ω_ε. To avoid that, we use only local correctors: i.e. the existence of a family $\{C_\omega^\varepsilon\}_\varepsilon$ which is a corrector in $\omega \cap \Omega_\varepsilon$, with $\omega \subset\subset \Omega$.

On the other side, in order to describe the limit function H of the nonlinear term H_ε (see Section 3) it is convenient (even if not necessary) to have a corrector which is defined in the whole Ω. To this purpose, in Section 2 we introduce an increasing sequence $\{\omega_h\}_{h \in \mathbb{N}}$ converging to Ω and for any h the corresponding family $\{C_{\omega_h}^\varepsilon\}_\varepsilon$. Then, from the sequence of families $\{C_{\omega_h}^\varepsilon\}_\varepsilon$ we construct a sequence $\{C^\varepsilon\}_\varepsilon$ defined in the whole of Ω and which is a corrector in $\omega \cap \Omega_\varepsilon$, for any compact $\omega \subset\subset \Omega$.

We observe that we are able to prove the corrector result for the family $\{C^\varepsilon\}_\varepsilon$ only in L^q for $q \in \,]1, 2[$ (see (2.20)). This is enough for passing to limit in H_ε, since we are dealing here with a sub-quadratic nonlinearity.

The literature on the nonlinear homogenization problems is very wide. We refer to [7–11] for a detailed bibliography on the subject.

For a fixed domain, the asymptotic behaviour of the solutions of the Dirichlet problem with sub-quadratic growth in the gradient has been studied by Boccardo and Del Vecchio [6], assuming the boundedness in L^∞ of the linear correctors. The corresponding case with quadratic growth in the gradient has been studied by Bensoussan *et al.* [5] assuming the boundedness in L^∞ of the linear correctors and by Bensoussan *et al.* [4] under the natural hypotheses on the linear correctors. An extension to the case of quasi-linear equations with natural growth terms has been studied by Chiado' Piat and De Franceschi [12].

For a domain perforated by ε-periodic holds of size ε, the homogenization of the problem with the homogeneous Neumann boundary condition on the holes, for $A^\varepsilon = A(x/\varepsilon)$, A $]0, 1[^n$-periodic, and with a sub-quadratic growth in the gradient has been studied by Donato and Sgambati [13], assuming A enough smooth. The case with quadratic growth has been treated by Donato *et al.* [14] for bounded solutions and by Cardone and Gaudiello [15] for unbounded solutions.

For a domain perforated by holes of size $r_\varepsilon \ll \varepsilon$, the homogenization of the Dirichlet problem with $A^\varepsilon = I$ and with quadratic growth in the gradient has been studied by Casado Diaz [16] in the case of bounded solutions.

Finally observe that in the case of a periodically perforated domain studied in [14, 15] the homogenization result is based on a Meyers estimate of the corrector independent of ε. This estimate can be proved since the corrector has the special form $C(x/\varepsilon)$. This estimate does not seem to be true in the general case. This is why in this paper we consider only the subquadratic case. For which we are able to avoid any use of the Meyer's estimate.

1. POSITION OF THE PROBLEMS AND MAIN RESULTS

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, and ε the general term of a sequence of positive reals which converges to zero.

For any ε let us introduce a closed subset T_ε ("the holes") of $\mathbb{R}^n$ contained in Ω, the perforated domain

$$\Omega_\varepsilon = \Omega \setminus T_\varepsilon$$

and the space

$$V_\varepsilon = \{v \in H^1(\Omega_\varepsilon) : v_{|\partial\Omega} = 0\}$$

equipped with the H^1-norm.

In the following:

— $M(\alpha, \beta, \Omega)$ denotes, for two positive reals $\alpha < \beta$, the set of the $n \times n$ matrix-valued functions A defined on Ω and satisfying

$$\begin{cases} A \text{ measurable on } \Omega, \\ A(x)\lambda\lambda \geq \alpha|\lambda|^2, \quad |A(x)\lambda| \leq \beta|\lambda| \quad \forall \lambda \in \mathbb{R}^n, \text{ a.e. } x \in \Omega, \end{cases}$$

— χ_E denotes the characteristic function of a subset E of $\mathbb{R}^n$,

— $|E|$ denotes the Lebesgue measure of a Lebesgue-measurable subset E of $\mathbb{R}^n$,

— $\tilde{v}$ (or $[v]^\sim$) denotes the zero extension on Ω of any vector function v defined on Ω_ε,

— ν denotes the unitary external normal vector with respect to Ω_ε.

Recall the following definition introduced by Briane *et al.* [1].

Definition 1.1 [1]. The sequence $\{T_\varepsilon\}_\varepsilon$ is said to be admissible (in Ω) if

$$\text{every } L^\infty \text{ weak } * \text{ limit point of } \{\chi_{\Omega_\varepsilon}\}_\varepsilon \text{ is positive almost everywhere in } \Omega \tag{1.1}$$

and there exist a positive real c_1, independent of ε, and a sequence $\{P_\varepsilon\}_\varepsilon$ of linear extension-operators such that for each ε

$$\begin{cases} P_\varepsilon \in \mathcal{L}(V_\varepsilon, H_0^1(\Omega)), \\ (P_\varepsilon v)_{|\Omega_\varepsilon} = v \quad \forall v \in V_\varepsilon, \\ \|D(P_\varepsilon v)\|_{L^2(\Omega))^n} \leq c_1 \|Dv\|_{(L^2(\Omega_\varepsilon))^n} \quad \forall v \in V_\varepsilon. \end{cases} \tag{1.2}$$

By virtue of (1.2), the Poincaré and Sobolev inequalities hold in V_ε with a constant independent of ε.

In the following we make use of the following assumptions

$$\{T_\varepsilon\}_\varepsilon \text{ is admissible in } \Omega$$

and that there exists a positive real c_2, independent of ε, such that for any ε

$$\|P_\varepsilon v\|_{L^\infty(\Omega)} \le c_2 \|v\|_{L^\infty(\Omega_\varepsilon)} \qquad \forall v \in V_\varepsilon \cap L^\infty(\Omega_\varepsilon). \tag{1.3}$$

Remark 1.2. In the main geometrical situations studied in the literature, the set of the holes is admissible and (1.3) holds. We recall, for example, the case of ε-periodic holes of size r_ε (see [17] for $r_\varepsilon = \varepsilon$ and [18] for $r_\varepsilon \ll \varepsilon$), the case of a nonperiodic spherical lattice (see [1]) and the case of perforated domains with double periodicity (see [19]).

We refer also to [20] for quasi-linear problems in periodically perforated domains.

The following result is proved in [1].

LEMMA 1.3 [1]. Let $\{T_\varepsilon\}_\varepsilon$ be admissible in Ω and $\{P_\varepsilon\}_\varepsilon$ be satisfying (1.2).

Then

$$(\varphi_\varepsilon \rightharpoonup \varphi \text{ weakly in } H_0^1(\Omega)) \Rightarrow (P_\varepsilon(\varphi_{\varepsilon|\Omega_\varepsilon}) \rightharpoonup \varphi \text{ weakly in } H_0^1(\Omega)).$$

If the sequence $\{P_\varepsilon\}_\varepsilon$ satisfies (1.2), we denote by $\{P_\varepsilon^*\}_\varepsilon$ the sequence in $\mathcal{L}(H^{-1}(\Omega), V_\varepsilon')$ defined by

$$P_\varepsilon^* g \colon v \in V_\varepsilon \to \langle g, P_\varepsilon v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \in \mathbb{R}, \qquad g \in H^{-1}(\Omega). \tag{1.4}$$

Recall now the definition of H^0-convergence introduced by Briane *et al.* [1]. The H^0-convergence is an extension to perforated domains of the H-convergence introduced by Murat and Tartar [2].

Definition 1.4 [1]. Let $\{T_\varepsilon\}_\varepsilon$ be admissible in Ω and $\{A^\varepsilon\}_\varepsilon$ be in $M(\alpha, \beta, \Omega)$. The sequence $\{(A^\varepsilon, T_\varepsilon)\}_\varepsilon$ is said to H^0-converge to the matrix A^0 of $M(\alpha', \beta', \Omega)$ (and denoted by $(A^\varepsilon, T_\varepsilon) \xrightarrow{H^0} A^0$) if and only if, for every function g in $H^{-1}(\Omega)$, the solution v_ε of

$$\begin{cases} -\mathrm{div}(A^\varepsilon D v_\varepsilon) = P_\varepsilon^* g & \text{in } \Omega_\varepsilon, \\ (A^\varepsilon D v_\varepsilon) \cdot \nu = 0 & \text{on } \partial T_\varepsilon, \\ v_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

satisfies the weak convergences

$$P_\varepsilon v_\varepsilon \rightharpoonup v \quad \text{weakly in } H_0^1(\Omega), \tag{1.6}$$

$$A^\varepsilon \widetilde{D v_\varepsilon} \rightharpoonup A^0 D v \quad \text{weakly in } (L^2(\Omega))^n, \tag{1.7}$$

where v is the unique solution of the following problem:

$$\begin{cases} -\mathrm{div}(A^0 D v) = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

The definition of H^0-convergence is independent of the sequence $\{P_\varepsilon\}_\varepsilon$ satisfying (1.2) (see [1], Proposition 1.7). Moreover the following compactness result holds.

THEOREM 1.5 [1]. Let $\{T_\varepsilon\}_\varepsilon$ be admissible in Ω and $\{A^\varepsilon\}_\varepsilon$ be in $M(\alpha, \beta, \Omega)$. Then, there exist a subsequence (still denoted by $\{\varepsilon\}$ and a matrix A^0 in $M((\alpha/c_1^2), \beta, \Omega)$, with c_1 given in (1.2), such that $\{(A^\varepsilon, T_\varepsilon)\}_\varepsilon$ H^0-converges to A^0.

Remark 1.6. In all the situations quoted in Remark 1.2 a H^0-convergence result is proved and the H^0-limit is identified.

Consider now, for any ε, a Caratheodory function H_ε defined on $\Omega_\varepsilon \times \mathbb{R} \times \mathbb{R}^n$. In the remainder of the paper, $\{H_\varepsilon\}_\varepsilon$ will satisfy one or more of the following properties:

$$|H_\varepsilon(x, s, \xi)| \leq c_3(1 + |\xi|^p), \tag{1.9}$$

$$|H_\varepsilon(x, s, \xi) - H_\varepsilon(x, s, \bar{\xi})| \leq b_1(|s|)(1 + |\xi|^{p-1} + |\bar{\xi}|^{p-1})|\xi - \bar{\xi}|, \tag{1.10}$$

$$|H_\varepsilon(x, s, \xi) - H_\varepsilon(x, \bar{s}, \xi)| \leq b_2(|s - \bar{s}|)(1 + |\xi|^p), \tag{1.11}$$

$$H_\varepsilon(x, s, \xi)s \geq 0, \tag{1.12}$$

for almost every x in Ω_ε, for any $s, \bar{s}$ in $\mathbb{R}$, for any $\xi, \bar{\xi}$ in $\mathbb{R}^n$ and for any ε, where c_3 is a positive real independent of ε, b_1 and b_2 are continuous increasing functions independent of ε with $b_1(0) \geq 0$, $b_2(0) = 0$ and p is fixed in $]1, 2[$.

Observe that (1.11) together with (1.12) imply

$$|H_\varepsilon(x, s, \xi)| \leq b_2(|s|)(1 + |\xi|^p) \quad \text{a.e. } x \text{ in } \Omega_\varepsilon, \quad \forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.13}$$

for any ε. In fact the Caratheodory assumption together with (1.12) gives

$$H_\varepsilon(x, 0, \xi) = 0 \quad \text{a.e. } x \text{ in } \Omega_\varepsilon, \quad \forall \xi \in \mathbb{R}^n,$$

for any ε. Consequently, by making use of (1.11), inequality (1.13) follows.

Remark 1.7 Let h_ε be a bounded sequence in $L^\infty(\Omega)$, g a bounded Lipschitz continuous function on $\mathbb{R}$ and p in $]1, 2[$. Then

$$H_\varepsilon(x, s, \xi) = h_\varepsilon(x) + g(s)|\xi|^p$$

satisfies conditions (1.9), (1.10) and (1.11).

Moreover, if g is also increasing with $g(0) = 0$, then

$$H_\varepsilon(x, s, \xi) = g(s)|\xi|^p$$

satisfies conditions (1.10), (1.11) and (1.12).

We introduce now some nonlinear equations with subquadratic growth.

(a) Equations with unbounded solutions

Let us consider, for any ε, the following problem:

$$\begin{cases} -\operatorname{div}(A^{\varepsilon}Du_{\varepsilon}) + H_{\varepsilon}(x, u_{\varepsilon}, Du_{\varepsilon}) = f \quad \text{in } \Omega_{\varepsilon}, \\ (A^{\varepsilon}Du_{\varepsilon}) \cdot \nu = 0 \quad \text{on } \partial T_{\varepsilon}, \\ u_{\varepsilon} \in V_{\varepsilon}, \ H_{\varepsilon}(x, u_{\varepsilon}, Du_{\varepsilon}) \in L^1(\Omega_{\varepsilon}), \ H_{\varepsilon}(x, u_{\varepsilon}, Du_{\varepsilon})u_{\varepsilon} \in L^1(\Omega_{\varepsilon}), \end{cases} \tag{1.14}$$

where A^{ε} is in $M(\alpha, \beta, \Omega)$, H_{ε} is a Caratheodory function on $\Omega_{\varepsilon} \times \mathbb{R} \times \mathbb{R}^n$ satisfying (1.10), (1.11), (1.12) and f is in $L^2(\Omega)$.

The variational formulation of (1.14) is given by

$$\begin{cases} \displaystyle\int_{\Omega_{\varepsilon}} A^{\varepsilon}Du_{\varepsilon}Dv \, dx + \int_{\Omega_{\varepsilon}} H_{\varepsilon}(x, u_{\varepsilon}, Du_{\varepsilon})v \, dx = \int_{\Omega_{\varepsilon}} fv \, dx \\ \qquad\qquad \forall v \in V_{\varepsilon} \cap L^{\infty}(\Omega_{\varepsilon}) \text{ and } v = u_{\varepsilon} \\ u_{\varepsilon} \in V_{\varepsilon}, \ H_{\varepsilon}(x, u_{\varepsilon}, Du_{\varepsilon}) \in L^1(\Omega_{\varepsilon}), \ H_{\varepsilon}(x, u_{\varepsilon}, Du_{\varepsilon})u_{\varepsilon} \in L^1(\Omega_{\varepsilon}). \end{cases} \tag{1.15}$$

Making use of the assumption on $\{A^{\varepsilon}\}_{\varepsilon}$, (1.11), (1.12) and following the same outlines as in [21] (see also [22]), it is easy to prove that problem (1.15) admits a solution u_{ε}. Then, taking $v = u_{\varepsilon}$ in (1.15) and making use of the assumption on $\{A^{\varepsilon}\}_{\varepsilon}$ and (1.12), since the Poincaré constant for V_{ε} can be chosen independent of ε, the following a priori estimates hold:

$$\|u_{\varepsilon}\|_{H^1(\Omega_{\varepsilon})} \leq c_4, \tag{1.16}$$

$$\int_{\Omega_{\varepsilon}} H_{\varepsilon}(x, u_{\varepsilon}, Du_{\varepsilon})u_{\varepsilon} \, dx \leq c_4, \tag{1.17}$$

for any ε, where c_4 is a positive real independent of ε.

In Section 4 we prove the following compactness result for problem (1.14).

THEOREM 1.8. Let $\{T_{\varepsilon}\}_{\varepsilon}$ be admissible in Ω, $\{P_{\varepsilon}\}_{\varepsilon}$ be satisfying (1.2) and $\{A^{\varepsilon}\}_{\varepsilon}$ be in $M(\alpha, \beta, \Omega)$. Moreover let, for any ε, H_{ε} a Caratheodory function on $\Omega_{\varepsilon} \times \mathbb{R} \times \mathbb{R}^n$ satisfying (1.10), (1.11), (1.12) and u_{ε} be a solution of (1.14) with f in $L^2(\Omega)$.

Then there exist a subsequence (still denoted by $\{\varepsilon\}$), a positive function χ_0 in $L^{\infty}(\Omega)$, a matrix A^0 in $M((\alpha/c_1^2), \beta, \Omega)$, with c_1 given in (1.2), a Caratheodory function H on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ satisfying, up to a multiplicative constant, (1.10), (1.11), (1.12) in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and a function u in $H_0^1(\Omega)$ (χ_0, A^0, H and u depend on the subsequence of $\{\varepsilon\}$) such that

$$\chi_{\Omega_{\varepsilon}} \rightharpoonup \chi_0 \quad \text{weakly} * \text{ in } L^{\infty}(\Omega), \tag{1.18}$$

$$(A^{\varepsilon}, T_{\varepsilon}) \overset{H^0}{\rightharpoonup} A^0, \tag{1.19}$$

$$\begin{cases} P_{\varepsilon}u_{\varepsilon} \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega), \\ P_{\varepsilon}u_{\varepsilon} \to u \quad \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega, \end{cases} \tag{1.20}$$

$$[H_{\varepsilon}(x, u_{\varepsilon}, Du_{\varepsilon})]^{\sim} \rightharpoonup H(x, u, Du) \quad \text{weakly in } L^1(\Omega), \tag{1.21}$$

$$-\operatorname{div}(A^{\varepsilon}\widetilde{Du_{\varepsilon}}) \rightharpoonup -\operatorname{div}(A^0Du) \quad \text{weakly in } \mathcal{D}'(\Omega), \tag{1.22}$$

as ε tends to zero and

$$\begin{cases} -\mathrm{div}(A^0 Du) + H(x, u, Du) = \chi_0 f & \text{in } \Omega, \\ u \in H_0^1(\Omega), \quad H(x, u, Du) \in L^1(\Omega), \quad H(x, u, Du)u \in L^1(\Omega). \end{cases} \tag{1.23}$$

(b) Equations with bounded solutions

Consider, for any ε, the following problem:

$$\begin{cases} -\mathrm{div}(A^\varepsilon Du_\varepsilon) + \gamma u_\varepsilon = H_\varepsilon(x, u_\varepsilon, Du_\varepsilon) + g & \text{in } \Omega_\varepsilon, \\ (A^\varepsilon Du_\varepsilon) \cdot \nu = 0 \quad \text{on } \partial T_\varepsilon, \\ u_\varepsilon \in V_\varepsilon \cap L^\infty(\Omega_\varepsilon), \end{cases} \tag{1.24}$$

where A^ε is in $M(\alpha, \beta, \Omega)$, H_ε is a Caratheodory function on $\Omega_\varepsilon \times \mathbb{R} \times \mathbb{R}^n$ satisfying (1.9), (1.10), (1.11), g is in $L^q(\Omega)$, with $q > n/2$, and γ is a positive real.

The variational formulation of (1.24) is given by

$$\begin{cases} \displaystyle\int_{\Omega_\varepsilon} A^\varepsilon Du_\varepsilon Dv \,\mathrm{d}x + \gamma \int_{\Omega_\varepsilon} u_\varepsilon v \,\mathrm{d}x = \int_{\Omega_\varepsilon} H_\varepsilon(x, u_\varepsilon, Du_\varepsilon) v \,\mathrm{d}x + \int_{\Omega_\varepsilon} gv \,\mathrm{d}x \\ \hfill \forall v \in V_\varepsilon \cap L^\infty(\Omega_\varepsilon) \\ u_\varepsilon \in V_\varepsilon \cap L^\infty(\Omega_\varepsilon). \end{cases} \tag{1.25}$$

Arguing as in [23] (Theorem 2.1) and using the assumptions on $\{A^\varepsilon\}_\varepsilon$ and (1.9), one can prove that problem (1.24) has a solution. Moreover, adapting the proof of Theorem 1 of [24] to our case (this is possible since the Poincaré and Sobolev embedding constants for V_ε can be chosen independent of ε) we have that, if $\{u_\varepsilon\}_\varepsilon$ is a sequence of solutions of (1.24), the following a priori estimates hold:

$$\|u_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} \le c_5, \qquad \|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le c_5 \tag{1.26}$$

for any ε, where c_5 is a real independent of ε.

In Section 5 we prove the following compactness result for problem (1.24).

THEOREM 1.9. Let $\{T_\varepsilon\}_\varepsilon$ be admissible in Ω, $\{P_\varepsilon\}_\varepsilon$ be satisfying (1.2) and (1.3) and $\{A^\varepsilon\}_\varepsilon$ be in $M(\alpha, \beta, \Omega)$. Moreover let, for any ε, H_ε be a Caratheodory function on $\Omega_\varepsilon \times \mathbb{R} \times \mathbb{R}^n$ satisfying (1.9), (1.10), (1.11) and u_ε be a solution of (1.24) with γ a positive real.

Then there exists a subsequence (still denoted by $\{\varepsilon\}$), a positive function χ_0 in $L^\infty(\Omega)$, a matrix A^0 in $M((\alpha/c_1^2), \beta, \Omega)$, with c_1 given in (1.2), a Caratheodory function H on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ satisfying, up to a multiplicative constant, (1.9), (1.10), (1.11) in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and a function u in $H_0^1(\Omega) \cap L^\infty(\Omega)$ (χ_0, A^0, H and u depend on the subsequence of $\{\varepsilon\}$) such that

$$\chi_{\Omega_\varepsilon} \rightharpoonup \chi_0 \quad \text{weakly} * \text{ in } L^\infty(\Omega), \tag{1.27}$$

$$(A^\varepsilon, T_\varepsilon) \xrightarrow{H^0} A^0, \tag{1.28}$$

$$\begin{cases} P_\varepsilon u_\varepsilon \rightharpoonup u & \text{weakly in } H_0^1(\Omega) \text{ and weakly} * \text{ in } L^\infty(\Omega), \\ P_\varepsilon u_\varepsilon \to u & \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega, \end{cases} \tag{1.29}$$

$$[H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)]^\sim \rightharpoonup H(x, u, Du) \quad \text{weakly in } L^1(\Omega), \tag{1.30}$$

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)u_\varepsilon\,dx = \int_\Omega H(x, u, Du)u\,dx, \tag{1.31}$$

$$-\mathrm{div}(A^\varepsilon\widetilde{Du_\varepsilon}) \rightharpoonup -\mathrm{div}(A^0 Du) \quad \text{weakly in } \mathcal{D}'(\Omega) \tag{1.32}$$

as ε tends to zero and

$$\begin{cases} -\mathrm{div}(A^0 Du) + \gamma\chi_0 u = H(x, u, Du) + \chi_0 g & \text{in } \Omega,\\ u \in H_0^1(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{1.33}$$

To prove Theorems 1.8 and 1.9, we first state some properties about the corrector for the corresponding linear problem. Moreover we show that the linear corrector is also a corrector for nonlinear problems (1.14) and (1.24). Then we construct the limit H of the nonlinear terms H_ε. Finally we pass to the limit in (1.14) and (1.24).

2. CORRECTORS FOR THE LINEAR PROBLEM

This section is essentially devoted to describe some properties of the correctors for the linear problem associated to (1.14) and (1.24).

Let $\{T_\varepsilon\}_\varepsilon$ be admissible in Ω, $\{P_\varepsilon\}_\varepsilon$ be satisfying (1.2) and $\{A^\varepsilon\}_\varepsilon$ be in $M(\alpha, \beta, \Omega)$. Moreover assume that $\{(A^\varepsilon, T_\varepsilon)\}_\varepsilon$ H^0-converges to A^0.

Following [1] (see also [2, 8, 14, 25]), let us introduce the local correctors for the linear problem (1.5). Let us recall that (see [1]) a global corrector exists only under additional assumptions on Ω_ε.

Let $\{\omega_h\}_{h\in\mathbb{N}}$ be a sequence of open subsets of $\mathbb{R}^n$ with smooth boundary and $\{\varphi_h\}_{h\in\mathbb{N}}$ be a sequence of functions defined on Ω such that

$$\begin{cases} \omega_h \subset \omega_{h+1} \subset\subset \Omega & \forall h \in \mathbb{N},\\ \bigcup_{h\in\mathbb{N}} \omega_h = \Omega, & \\ \varphi_h \in C_0^\infty(\Omega) & \forall h \in \mathbb{N},\\ \varphi_h = 1 \text{ in } \omega_h & \forall h \in \mathbb{N}. \end{cases} \tag{2.1}$$

For any h in $\mathbb{N}$, let us introduce the family $\{C_h^\varepsilon\}_\varepsilon$ in $(L^2(\Omega))^{n^2}$ defined by

$$C_h^\varepsilon e_i = D(P_\varepsilon w_{h,i}^\varepsilon) \qquad \text{a.e. in } \Omega,\ i = 1, \dots, n, \tag{2.2}$$

where $\{e_1, \dots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$ and, for any h in $\mathbb{N}$, for any ε and for $i = 1, \dots, n$, $w_{h,i}^\varepsilon$ is the unique solution of the following problem:

$$\begin{cases} -\mathrm{div}(A^\varepsilon Dw_{h,i}^\varepsilon) = P_\varepsilon^*(-\mathrm{div}(A^0 D(\varphi_h x_i))) & \text{in } \Omega_\varepsilon,\\ (A^\varepsilon Dw_{h,i}^\varepsilon)\cdot \nu = 0 & \text{on } \partial T_\varepsilon,\\ w_{h,i}^\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

with P_ε^* defined by (1.4).

Since $\{(A^\varepsilon, T_\varepsilon)\}_\varepsilon$ H^0-converges to A^0 and φ_h is equal to 1 in ω_h, it follows that

$$\begin{cases} \|C_h^\varepsilon\|_{(L^2(\Omega))^{n^2}} \le c_6(h)\\ \text{for any } \varepsilon, \text{ where } c_6(h) \text{ is a real independent of } \varepsilon \text{ but dependent on } h, \end{cases} \tag{2.4}$$

$$\begin{cases} P_\varepsilon w^\varepsilon_{h,i} \rightharpoonup \varphi_h x_i & \text{weakly in } H^1_0(\Omega), \\ P_\varepsilon w^\varepsilon_{h,i} \rightharpoonup x_i & \text{weakly in } H^1(\omega_h), \end{cases} \tag{2.5}$$

$$\chi_{\Omega_\varepsilon} A^\varepsilon C^\varepsilon_h \rightharpoonup A^0 \quad \text{weakly in } (L^2(\omega_h))^{n^2}, \tag{2.6}$$

for any h in $\mathbb{N}$, for $i = 1, \ldots, n$ and as ε tends to zero.

The following theorem is implicitly contained in [1] and its proof makes use of standard arguments. For the reader's convenience we give here the full details of the proof.

THEOREM 2.1. Let $\{\omega_h\}_{h\in\mathbb{N}}$ and $\{\varphi_h\}_{h\in\mathbb{N}}$ be satisfying (2.1), $\{T_\varepsilon\}_\varepsilon$ be admissible in Ω, $\{P_\varepsilon\}_\varepsilon$ be satisfying (1.2) and $\{A^\varepsilon\}_\varepsilon$ be in $M(\alpha, \beta, \Omega)$. Assume that $\{(A^\varepsilon, T_\varepsilon)\}_\varepsilon$ H^0-converges to $A^0 \in M(\alpha', \beta, \Omega)$. Let v_ε and v be the solutions of (1.5) and (1.8), respectively, and let, for any h in $\mathbb{N}$, $\{C^\varepsilon_h\}_\varepsilon$ be the sequence in $(L^2(\Omega))^{n^2}$ associated to ω_h, φ_h, $\{P_\varepsilon\}_\varepsilon$, $\{A^\varepsilon\}_\varepsilon$ and A^0 by means of Definition (2.2).

Then it results

$$\chi_{\Omega_\varepsilon} A^\varepsilon(C^\varepsilon_h \xi)(C^\varepsilon_h \bar\xi) \rightharpoonup A^0 \xi \bar\xi \quad \text{weakly in } \mathfrak{D}'(\omega_h), \quad \forall (\xi, \bar\xi) \in \mathbb{R}^{2n}, \tag{2.7}$$

$$\chi_{\Omega_\varepsilon} A^\varepsilon((C^\varepsilon_h - C^\varepsilon_{h+k})\xi)((C^\varepsilon_h - C^\varepsilon_{h+k})\bar\xi) \rightharpoonup 0 \quad \text{weakly in } \mathfrak{D}'(\omega_h), \quad \forall (\xi, \bar\xi) \in \mathbb{R}^{2n}, \tag{2.8}$$

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} A^\varepsilon((C^\varepsilon_h - C^\varepsilon_{h+k})\phi)((C^\varepsilon_h - C^\varepsilon_{h+k})\phi)\, dx = 0 \quad \forall \phi \in (C^\infty_0(\omega_h))^n, \tag{2.9}$$

$$\begin{cases} \limsup\limits_{\varepsilon\to 0} \|Dv_\varepsilon - C^\varepsilon_h \phi\|_{(L^2(\Omega_\varepsilon))^n} \leq c_7 \|Dv - \phi\|_{(L^2(\Omega))^n} \quad \forall \phi \in (C^\infty_0(\omega_h))^n, \\ \text{where } c_7 \text{ is a real independent of } h \text{ and } \phi, \end{cases} \tag{2.10}$$

for any h and k in $\mathbb{N}$, as ε tends to zero.

Proof. Let us prove (2.7). Fix h in $\mathbb{N}$ and $\xi = (\xi_1, \ldots, \xi_n)$, $\bar\xi = (\bar\xi_1, \ldots, \bar\xi_n)$ in $\mathbb{R}^n$. Then, recalling Definition (2.2), choosing $\xi_i \bar\xi_j \phi w^\varepsilon_{h,j}$ with ϕ in $C^\infty_0(\omega_h)$ as test function in (2.3) and making use of (2.6), (2.5) and Lemma 1.3, it results

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_\Omega \phi \chi_{\Omega_\varepsilon} A^\varepsilon(C^\varepsilon_h \xi)(C^\varepsilon_h \bar\xi)\, dx \\
&= \sum_{i,j=1}^n \lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \phi \xi_i \bar\xi_j A^\varepsilon D w^\varepsilon_{h,i} D w^\varepsilon_{h,j}\, dx \\
&= \sum_{i,j=1}^n \lim_{\varepsilon\to 0} \left(\int_{\Omega_\varepsilon} A^\varepsilon D w^\varepsilon_{h,i} D(\xi_i \bar\xi_j \phi w^\varepsilon_{h,j})\, dx - \int_\Omega P_\varepsilon w^\varepsilon_{h,j} (\chi_{\Omega_\varepsilon} A^\varepsilon C^\varepsilon_h e_i) D(\xi_i \bar\xi_j \phi)\, dx \right) \\
&= \sum_{i,j=1}^n \lim_{\varepsilon\to 0} \int_\Omega A^0 D(\varphi_h x_i) D P_\varepsilon(\xi_i \bar\xi_j \phi w^\varepsilon_{h,j})\, dx - \sum_{i,j=1}^n \int_\Omega x_j A^0 e_i D(\xi_i \bar\xi_j \phi)\, dx \\
&= \sum_{i,j=1}^n \left(\int_\Omega A^0 e_i D(\xi_i \bar\xi_j \phi x_j)\, dx - \int_\Omega x_j A^0 e_i D(\xi_i \bar\xi_j \phi)\, dx \right) \\
&= \int_\Omega \phi A^0 \xi \bar\xi\, dx \qquad \forall \phi \in C^\infty_0(\omega_h),
\end{aligned} \tag{2.11}$$

i.e. (2.7).

Observe now that from (2.1) and (2.5) it follows that $P_\varepsilon(w^\varepsilon_{h,j} - w^\varepsilon_{h+k,j})$ converges to zero in $L^2(\omega_h)$, as ε goes to zero. Then convergence (2.8) follows by arguing as in (2.11). In fact, for any fixed h, k in $\mathbb{N}$ and $\xi = (\xi_1, \dots, \xi_n)$, $\bar{\xi} = (\bar{\xi}_1, \dots, \bar{\xi}_n)$ in $\mathbb{R}^n$, it results

$$\lim_{\varepsilon\to 0}\int_\Omega \phi\chi_{\Omega_\varepsilon}A^\varepsilon((C^\varepsilon_h - C^\varepsilon_{h+k})\xi)((C^\varepsilon_h - C^\varepsilon_{h+k})\bar{\xi})\,dx$$

$$= \sum_{i,j=1}^n \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \phi\xi_i\bar{\xi}_j A^\varepsilon D(w^\varepsilon_{h,i} - w^\varepsilon_{h+k,i})D(w^\varepsilon_{h,j} - w^\varepsilon_{h+k,j})\,dx$$

$$= \sum_{i,j=1}^n \lim_{\varepsilon\to 0}\left(\int_{\Omega_\varepsilon} A^\varepsilon D(w^\varepsilon_{h,i} - w^\varepsilon_{h+k,i})D(\xi_i\bar{\xi}_j\phi(w^\varepsilon_{h,j} - w^\varepsilon_{h+k,j}))\,dx\right.$$

$$\left. - \int_\Omega P_\varepsilon(w^\varepsilon_{h,j} - w^\varepsilon_{h+k,j})(\chi_{\Omega_\varepsilon}A^\varepsilon(C^\varepsilon_h - C^\varepsilon_{h+k})e_i)D(\xi_i\bar{\xi}_j\phi)\,dx\right)$$

$$= \sum_{i,j=1}^n \lim_{\varepsilon\to 0}\int_\Omega A^0 D(\varphi_h x_i - \varphi_{h+k}x_i)DP_\varepsilon(\xi_i\bar{\xi}_j\phi(w^\varepsilon_{h,j} - w^\varepsilon_{h+k,j}))\,dx = 0$$

$$\forall \phi \in C^\infty_0(\omega_h).$$

The same arguments give (2.9).

Finally let us prove (2.10). Fix h in $\mathbb{N}$ and $\phi = (\phi_1, \dots, \phi_n)$ in $(C^\infty_0(\omega_h))^n$. From the assumption on $\{A^\varepsilon\}_\varepsilon$ it follows that

$$\alpha\|Dv_\varepsilon - C^\varepsilon_h\phi\|^2_{(L^2(\Omega_\varepsilon))^n} \le \int_{\Omega_\varepsilon} A^\varepsilon(Dv_\varepsilon - C^\varepsilon_h\phi)(Dv_\varepsilon - C^\varepsilon_h\phi)\,dx$$

$$= \int_{\Omega_\varepsilon} A^\varepsilon Dv_\varepsilon Dv_\varepsilon\,dx - \int_{\Omega_\varepsilon} A^\varepsilon Dv_\varepsilon(C^\varepsilon_h\phi)\,dx$$

$$- \int_{\Omega_\varepsilon} A^\varepsilon(C^\varepsilon_h\phi)Dv_\varepsilon\,dx + \int_{\Omega_\varepsilon} A^\varepsilon(C^\varepsilon_h\phi)(C^\varepsilon_h\phi)\,dx, \tag{2.12}$$

for any ε. Choosing v_ε as test function in (1.5) and making use of the H^0-convergence of $\{(A^\varepsilon, T_\varepsilon)\}_\varepsilon$ to A^0, it results

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} A^\varepsilon Dv_\varepsilon Dv_\varepsilon\,dx = \lim_{\varepsilon\to 0}\langle g, P_\varepsilon v_\varepsilon\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}$$

$$= \langle g, v\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}$$

$$= \int_\Omega A^0 Dv Dv\,dx. \tag{2.13}$$

Recalling Definition (2.2), choosing $\phi_i w^\varepsilon_{h,i}$ as test function in (1.5) and making use of (2.5), Lemma 1.3 and the H^0-convergence of $\{(A^\varepsilon, T_\varepsilon)\}_\varepsilon$ to A^0, it results

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} A^\varepsilon Dv_\varepsilon (C_h^\varepsilon \phi)\,dx$$

$$= \sum_{i=1}^{n} \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \phi_i A^\varepsilon Dv_\varepsilon Dw_{h,i}^\varepsilon\,dx$$

$$= \sum_{i=1}^{n} \left(\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} A^\varepsilon Dv_\varepsilon D(\phi_i w_{h,i}^\varepsilon)\,dx - \lim_{\varepsilon\to 0}\int_{\Omega} P_\varepsilon w_{h,i}^\varepsilon A^\varepsilon \widetilde{Dv_\varepsilon} D\phi_i\,dx\right)$$

$$= \sum_{i=1}^{n} \left(\lim_{\varepsilon\to 0}\langle g, P_\varepsilon(\phi_i w_{h,i}^\varepsilon)\rangle - \int_{\Omega} x_i A^0 Dv D\phi_i\,dx\right)$$

$$= \sum_{i=1}^{n} \left(\langle g, \phi_i x_i\rangle - \int_{\Omega} x_i A^0 Dv D\phi_i\,dx\right),$$

from which, choosing $\phi_i x_i$ as test function in (1.8), it follows that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} A^\varepsilon Dv_\varepsilon (C_h^\varepsilon \phi)\,dx = \int_{\Omega} A^0 Dv\phi\,dx. \tag{2.14}$$

Recalling Definition (2.2), choosing $\phi_i v_\varepsilon$ as test function in (2.3) and making use of the H^0-convergence of $\{(A^\varepsilon, T_\varepsilon)\}_\varepsilon$ to A^0, (2.6) and Lemma 1.3, it results

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} A^\varepsilon (C_h^\varepsilon \phi) Dv_\varepsilon\,dx$$

$$= \sum_{i=1}^{n} \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \phi_i A^\varepsilon Dw_{h,i}^\varepsilon Dv_\varepsilon\,dx$$

$$= \sum_{i=1}^{n} \lim_{\varepsilon\to 0}\left(\int_{\Omega_\varepsilon} A^\varepsilon Dw_{h,i}^\varepsilon D(\phi_i v_\varepsilon)\,dx - \int_{\Omega} P_\varepsilon v_\varepsilon(\chi_{\Omega_\varepsilon} A^\varepsilon C_h^\varepsilon e_i) D\phi_i\,dx\right)$$

$$= \sum_{i=1}^{n} \left(\lim_{\varepsilon\to 0}\int_{\Omega} A^0 D(\varphi_h x_i) DP_\varepsilon(\phi_i v_\varepsilon)\,dx - \int_{\Omega} vA^0 e_i D\phi_i\,dx\right)$$

$$= \sum_{i=1}^{n} \left(\int_{\Omega} A^0 e_i D(\phi_i v)\,dx - \int_{\Omega} vA^0 e_i D\phi_i\,dx\right)$$

$$= \int_{\Omega} A^0 \phi Dv\,dx. \tag{2.15}$$

Arguing as in (2.11), it is easy to prove that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} A^\varepsilon (C_h^\varepsilon \phi)(C_h^\varepsilon \phi)\,dx = \int_{\Omega} A^0 \phi\phi\,dx. \tag{2.16}$$

Then combining (2.12) with (2.13), (2.14), (2.15) and (2.16) it follows that

$$\limsup_{\varepsilon \to 0} \|Dv_\varepsilon - C_h^\varepsilon \phi\|_{(L^2(\Omega_\varepsilon))^n} \le \left(\frac{1}{\alpha} \int_\Omega A^0(Dv - \phi)(Dv - \phi)\,\mathrm{d}x\right)^{1/2}$$

which implies (2.10), being A^0 in $M(\alpha', \beta, \Omega)$. ■

Define now, $\{C^\varepsilon\}_\varepsilon$ in $(L^2_{\mathrm{loc}}(\Omega))^{n^2}$ by

$$C^\varepsilon = C_h^\varepsilon \quad \text{a.e. in } \omega_h - \overline{\omega_{h-1}}, \quad h \in \mathbb{N}, \tag{2.17}$$

where $\omega_0 = \emptyset$, $\{\omega_h\}_{h \in \mathbb{N}}$ satisfies (2.1) and $\{C_h^\varepsilon\}_{h \in \mathbb{N}}$ is given in (2.2).

From (2.1) and (2.4) it follows that

$$\begin{cases} \|C^\varepsilon\|_{(L^2(\omega))^{n^2}} \le c_\omega, \\ \text{for any } \omega \subset\subset \Omega \text{ and for any } \varepsilon, \\ \text{where } c_\omega \text{ is a real independent of } \varepsilon, \text{ but dependent on } \omega. \end{cases} \tag{2.18}$$

Theorem 2.1 allows to prove the following result.

THEOREM 2.2. Let $\{T_\varepsilon\}_\varepsilon$ be admissible in Ω and $\{A^\varepsilon\}_\varepsilon$ be in $M(\alpha, \beta, \Omega)$. Assume that $\{(A^\varepsilon, T_\varepsilon)\}_\varepsilon$ H^0-converges to $A^0 \in M(\alpha', \beta, \Omega)$. Let v_ε and v be the solutions of (1.5) and (1.8) respectively, $\{C^\varepsilon\}_\varepsilon$ be defined by (2.17) and q in $]1, 2[$.

Then it results

$$\begin{cases} \limsup_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |C^\varepsilon \xi|^q \varphi \,\mathrm{d}x \le c_8 |\xi|^q \int_\Omega \varphi \,\mathrm{d}x \quad \forall \xi \in \mathbb{R}^n, \forall \varphi \in C_0^\infty(\Omega), \varphi \ge 0 \text{ in } \Omega, \\ \text{where } c_8 \text{ is a real independent of } \varphi \text{ and } \xi; \end{cases} \tag{2.19}$$

$$\begin{cases} \limsup_{\varepsilon \to 0} \|Dv_\varepsilon - C^\varepsilon \phi\|_{(L^q(\Omega_\varepsilon))^n} \le c_9 \|Dv - \phi\|_{(L^2(\Omega))^n} \quad \forall \phi \in (C_0^\infty(\Omega))^n, \\ \text{where } c_9 \text{ is a real independent of } \phi. \end{cases} \tag{2.20}$$

Proof. Let us prove (2.19).

Let $\{\omega_h\}_{h \in \mathbb{N}}$ be a sequence satisfying (2.1). Let ξ be given in $\mathbb{R}^n$, φ in $C_0^\infty(\Omega)$ with $\varphi \ge 0$ and fix l in $\mathbb{N}$ such that $\varphi \in C_0^\infty(\omega_l)$. Then, for any ε, it results

$$\int_{\Omega_\varepsilon} |C^\varepsilon \xi|^q \varphi \,\mathrm{d}x \le 2^{q-1} \left(\int_{\Omega_\varepsilon} |C_l^\varepsilon \xi|^q \varphi \,\mathrm{d}x + \int_{\Omega_\varepsilon} |(C^\varepsilon - C_l^\varepsilon)\xi|^q \varphi \,\mathrm{d}x \right) \tag{2.21}$$

where C_l^ε is defined by (2.2).

Hölder inequality, convergence (2.7) and the assumption on $\{A^\varepsilon\}_\varepsilon$ and A^0 imply that

$$\begin{aligned}\limsup_{\varepsilon\to 0}\int_{\Omega_\varepsilon}|C_l^\varepsilon\xi|^q\varphi\,dx &\le \left(\int_{\omega_l}\varphi\,dx\right)^{(2-q)/2}\limsup_{\varepsilon\to 0}\left(\int_{\Omega_\varepsilon}|C_l^\varepsilon\xi|^2\varphi\,dx\right)^{q/2}\\ &\le \left(\int_{\omega_l}\varphi\,dx\right)^{(2-q)/2}\limsup_{\varepsilon\to 0}\left(\frac{1}{\alpha}\int_{\omega_l}\chi_{\Omega_\varepsilon}A^\varepsilon(C_l^\varepsilon\xi)(C_l^\varepsilon\xi)\varphi\,dx\right)^{q/2}\\ &= \left(\int_{\omega_l}\varphi\,dx\right)^{(2-q)/2}\left(\frac{1}{\alpha}\int_{\omega_l}A^0\xi\xi\varphi\,dx\right)^{q/2}\\ &\le \left(\frac{\beta}{\alpha}\right)^{q/2}|\xi|^q\int_\Omega\varphi\,dx. \end{aligned}\tag{2.22}$$

On the other hand let φ_η^h be a function in $C_0^\infty(\omega_h)$, for $\eta>0$ and $h=1,\dots,l$, such that $\varphi_\eta^h\ge 0$ and

$$\|\varphi_\eta^h-\varphi\|_{L^{2/(2-q)}(\Omega_h)}<\eta. \tag{2.23}$$

Then, from the Hölder inequality, (2.23), (2.18), the assumption on $\{A^\varepsilon\}_\varepsilon$ and convergence (2.8), it results

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\int_{\Omega_\varepsilon}|(C^\varepsilon-C_l^\varepsilon)\xi|^q\varphi\,dx\\ &\le\sum_{h=1}^{l}\limsup_{\varepsilon\to 0}\int_{\Omega_\varepsilon\cap\omega_h}|(C_h^\varepsilon-C_l^\varepsilon)\xi|^q\varphi\,dx\\ &\le\sum_{h=1}^{l}\left[\limsup_{\varepsilon\to 0}\int_{\omega_h}|(C_h^\varepsilon-C_l^\varepsilon)\xi|^q(\varphi-\varphi_\eta^h)\,dx\right.\\ &\qquad\left.+\limsup_{\varepsilon\to 0}\int_{\Omega_\varepsilon\cap\omega_h}|(C_h^\varepsilon-C_l^\varepsilon)\xi|^q\varphi_\eta^h\,dx\right]\\ &\le\sum_{h=1}^{l}\left[\eta\limsup_{\varepsilon\to 0}\|(C_h^\varepsilon-C_l^\varepsilon)\xi\|^q_{(L^2(\omega_h))^n}\right.\\ &\qquad\left.+\left(\int_{\omega_h}\varphi_\eta^h\,dx\right)^{(2-q)/2}\limsup_{\varepsilon\to 0}\left(\int_{\Omega_\varepsilon\cap\omega_h}|(C_h^\varepsilon-C_l^\varepsilon)\xi|^2\varphi_\eta^h\,dx\right)^{q/2}\right]\\ &\le\eta|\xi|^q\sum_{h=1}^{l}(2c_6(h))^q+\sum_{h=1}^{l}\left[\left(\int_{\omega_h}\varphi_\eta^h\,dx\right)^{(2-q)/2}\right.\\ &\qquad\left.\cdot\limsup_{\varepsilon\to 0}\left(\frac{1}{\alpha}\int_{\omega_h}\chi_{\Omega_\varepsilon}A^\varepsilon((C_h^\varepsilon-C_l^\varepsilon)\xi)((C_h^\varepsilon-C_l^\varepsilon)\xi)\varphi_\eta^h\,dx\right)^{q/2}\right]\\ &=\eta|\xi|^q\sum_{h=1}^{l}(2c_6(h))^q. \end{aligned}\tag{2.24}$$

By virtue of the arbitrariety of $\eta > 0$, (2.24) implies

$$\limsup_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |(C^\varepsilon - C_l^\varepsilon)\xi|^q \varphi \, dx = 0. \tag{2.25}$$

Finally, combining (2.21) with (2.22) and (2.25), we obtain (2.19) with

$$c_8 = 2^{q-1}\left(\frac{\beta}{\alpha}\right)^{q/2}.$$

Let us prove (2.20). Let ϕ chosen in $(C_0^\infty(\Omega))^n$ and fix l in $\mathbb{N}$ such that $\phi \in (C_0^\infty(\omega_l))^n$. Then, for any ε, it results

$$\|Dv_\varepsilon - C^\varepsilon \phi\|_{(L^q(\Omega_\varepsilon))^n} \le \|Dv_\varepsilon - C_l^\varepsilon \phi\|_{(L^q(\Omega_\varepsilon))^n} + \|(C^\varepsilon - C_l^\varepsilon)\phi\|_{(L^q(\Omega_\varepsilon))^n}, \tag{2.26}$$

where C_l^ε is defined by (2.2).

From (2.10) it follows that

$$\limsup_{\varepsilon \to 0} \|Dv_\varepsilon - C_l^\varepsilon \phi\|_{(L^q(\Omega_\varepsilon))^n} \le |\Omega|^{(2-q)/2} c_7 \|Dv - \phi\|_{(L^2(\Omega))^n}. \tag{2.27}$$

By arguing as in (2.24) (but by making use of (2.9) instead of (2.8)), it results

$$\limsup_{\varepsilon \to 0} \|(C^\varepsilon - C_l^\varepsilon)\phi\|_{(L^q(\Omega_\varepsilon))^n} = 0. \tag{2.28}$$

Finally, by combining (2.26) with (2.27) and (2.28), we obtain (2.20) with

$$c_9 = c_7 |\Omega|^{(2-q)/2}. \quad \blacksquare$$

3. CONSTRUCTION OF THE LIMIT H

Let us recall the following known Lemma (see [5] for a proof).

LEMMA 3.1. Let $\{g_\varepsilon\}_\varepsilon$ be a sequence of functions which converges weakly in $L^1(\Omega)$ to a function g_0 and let $\{t_\varepsilon\}_\varepsilon$ be a sequence of equibounded and measurable functions which converges almost pointwise in Ω to a function t_0 as ε tends to zero. Then

$$\lim_{\varepsilon \to 0} \int_\Omega g_\varepsilon t_\varepsilon \, dx = \int_\Omega g_0 t_0 \, dx.$$

Arguing as in [4] Proposition 2.1, in the following proposition we construct the nonlinearity H by means of the linear corrector.

PROPOSITION 3.2. Let $\{T_\varepsilon\}_\varepsilon$ be admissible in Ω and $\{A^\varepsilon\}_\varepsilon$ be in $M(\alpha, \beta, \Omega)$. Assume that $\{(A^\varepsilon, T_\varepsilon)\}_\varepsilon$ H^0-converges to $A^0 \in M(\alpha', \beta, \Omega)$ and let $\{C^\varepsilon\}_\varepsilon$ be defined by (2.17). Moreover let $\{H_\varepsilon\}_\varepsilon$ be a sequence of Caratheodory functions on $\Omega_\varepsilon \times \mathbb{R} \times \mathbb{R}^n$ satisfying (1.10)–(1.12) (resp. (1.9)–(1.11)).

Then there exist a subsequence (still denoted by $\{\varepsilon\}$ and a Caratheodory function H on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ (dependent on the subsequence) satisfying (1.10), (1.11) and (1.12) (resp. (1.9), (1.10) and (1.11), up to a multiplicative real c_{10}, such that, as ε tends to zero,

$$[H_\varepsilon(x, s, C^\varepsilon\xi)]^\sim \rightharpoonup H(x, s, \xi) \quad \text{weakly in } L^1_{\text{loc}}(\Omega) \quad \forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^n \tag{3.1}$$

and

$$[H_\varepsilon(x, \varphi_\varepsilon, C^\varepsilon\phi)]^\sim \rightharpoonup H(x, \varphi_0, \phi) \quad \text{weakly in } L^1_{\text{loc}}(\Omega) \tag{3.2}$$

for any ϕ in $(L^\infty(\Omega))^n$ and for any sequence $\{\varphi_\varepsilon\}_\varepsilon$ in $L^\infty(\Omega)$ such that

$$\begin{cases} \varphi_\varepsilon \to \varphi_0 \quad \text{a.e. in } \Omega, \\ \|\varphi_\varepsilon\|_{L^\infty(\Omega)} \le c_{11} \end{cases} \tag{3.3}$$

with c_{11} real independent of ε.

Remark 3.3. Let us point out that in the case of periodic holes [14, 15] as well as in the case of a fixed domain [4], convergences analogous to (3.1) and (3.2) hold in $L^1(\Omega)$. Here we have only the local convergences (3.1) and (3.2), due to the fact that the corrector family $\{C^\varepsilon\}_\varepsilon$ is bounded only locally in $(L^2(\Omega))^{n^2}$ (see (2.18)). This is specific of our geometrical framework.

Proof of Proposition 3.2. This Proposition will be proved in the case where (1.10)-(1.12) are satisfied. The proof for the case where (1.9)-(1.11) hold can be done in a similar way.

Step 1. Let us construct H and let us prove (3.1). Let p be given by (1.10)-(1.12) and $\eta = 2/p$. From (1.13) and (2.18) it follows that, for any $\omega \subset\subset \Omega$ and (s, ξ) in $Q \times Q^n$,

$$\begin{aligned} \int_\omega |[H_\varepsilon(x, s, C^\varepsilon\xi)]^\sim|^\eta \, dx &\le (b_2(|s|))^\eta \int_\omega (1 + |C^\varepsilon\xi|^p)^\eta \, dx \\ &\le (b_2(|s|))^\eta 2^{\eta-1} \int_\omega 1 + |C^\varepsilon\xi|^2 \, dx \\ &\le (b_2(|s|))^\eta 2^{\eta-1}(|\Omega| + c_\omega^2|\xi|^2) \end{aligned}$$

for any ε. Consequently, since $\eta > 1$, there exists a subsequence (still denoted by $\{\varepsilon\}$) and a function $H(\cdot, s, \xi)$ in $L^1_{\text{loc}}(\Omega)$ such that

$$[H_\varepsilon(x, s, C^\varepsilon\xi)]^\sim \rightharpoonup H(x, s, \xi) \quad \text{weakly in } L^1_{\text{loc}}(\Omega), \quad \forall (s, \xi) \in Q \times Q^n \tag{3.4}$$

as ε tends to zero.

Let us prove that

$$|H(x, s, \xi) - H(x, s, \bar{\xi})| \le c_8(c_8 + 1)b_1(|s|)(1 + |\xi|^{p-1} + |\bar{\xi}|^{p-1})|\xi - \bar{\xi}| \quad \text{a.e. } x \text{ in } \Omega, \quad \forall s \in A, \ \forall \xi, \bar{\xi} \in Q^n. \tag{3.5}$$

Fix s in Q, ξ, $\bar{\xi}$ in Q^n and let φ be in $C_0^\infty(\Omega)$ with $\varphi \geq 0$. Then (3.4), (1.10), the weak lower-semicontinuity of the L^1-norm, the Hölder inequality and (2.19) of Theorem 2.2 imply

$$\begin{aligned}
&\int_\Omega |H(x,s,\xi) - H(x,s,\bar{\xi})|\varphi\,\mathrm{d}x \\
&\leq \liminf_{\varepsilon\to 0} \int_\Omega |[H_\varepsilon(x,s,C^\varepsilon\xi)]^\sim - [H_\varepsilon(x,s,C^\varepsilon\bar{\xi})]^\sim|\varphi\,\mathrm{d}x \\
&\leq b_1(|s|)\liminf_{\varepsilon\to 0}\int_{\Omega_\varepsilon} (1 + |C^\varepsilon\xi|^{p-1} + |C^\varepsilon\bar{\xi}|^{p-1})|C^\varepsilon(\xi - \bar{\xi})|\varphi\,\mathrm{d}x \\
&\leq b_1(|s|)\limsup_{\varepsilon\to 0}\left(\int_\Omega \chi_{\Omega_\varepsilon}|C^\varepsilon(\xi-\bar{\xi})|^p\varphi\,\mathrm{d}x\right)^{1/p}\left[\left(\int_\Omega \varphi\,\mathrm{d}x\right)^{(p-1)/p}\right. \\
&\quad \left. + \limsup_{\varepsilon\to 0}\left(\int_\Omega \chi_{\Omega_\varepsilon}|C^\varepsilon\xi|^p\varphi\,\mathrm{d}x\right)^{(p+1)/p} + \limsup_{\varepsilon\to 0}\left(\int_\Omega \chi_{\Omega_\varepsilon}|C^\varepsilon\bar{\xi}|^p\varphi\,\mathrm{d}x\right)^{(p-1)/p}\right] \\
&\leq b_1(|s|)c_8|\xi - \bar{\xi}|\left(\int_\Omega \varphi\,\mathrm{d}x\right)^{1/p} \\
&\quad \cdot\left[\left(\int_\Omega \varphi\,\mathrm{d}x\right)^{(p-1)/p} + c_8|\xi|^{p-1}\left(\int_\Omega \varphi\,\mathrm{d}x\right)^{(p-1)/p} + c_8|\bar{\xi}|^{p-1}\left(\int_\Omega \varphi\,\mathrm{d}x\right)^{(p-1)/p}\right] \\
&\leq c_8(1 + c_8)b_1(|s|)|\xi - \bar{\xi}|(1 + |\xi|^{p-1} + |\bar{\xi}|^{p-1})\int_\Omega \varphi\,\mathrm{d}x
\end{aligned}$$

from which (3.5) follows, since φ is any positive function in $C_0^\infty(\Omega)$.

Similarly we obtain

$$|H(x,s,\xi) - H(x,\bar{s},\xi)| \leq (1 + c_8)b_2(|s - \bar{s}|)(1 + |\xi|^p) \quad \text{a.e. } x \text{ in } \Omega,\ \forall s, \bar{s} \in Q,\ \forall \xi \in Q^n. \tag{3.6}$$

By virtue of (3.5) and (3.6) there exists a negligible subset ω of Ω such that for any x in $\Omega\backslash\omega$, $H(x,\cdot,\cdot)$ is continuous in $Q \times Q^n$. In fact, up to a multiplicative constant, it results

$$|H(x,s,\xi) - H(x,\bar{s},\bar{\xi})| \leq b_1(|s|)(1 + |\xi|^{p-1} + |\bar{\xi}|^{p-1})|\xi - \bar{\xi}| + b_2(|s - \bar{s}|)(1 + |\bar{\xi}|^p) \quad \text{a.e. } x \text{ in } \Omega,\ \forall s, \bar{s} \in Q,\ \forall \xi, \bar{\xi} \in Q^n.$$

This permits us to define $H(x,s,\xi)$, for any x in $\Omega\backslash\omega$ and for any (s,ξ) in $\mathbb{R} \times \mathbb{R}^n$, as the pointwise limit in $\mathbb{R}$ of any Cauchy sequence $\{H(x,s_n,\xi_n)\}_{n\in N}$ such that $\{(s_n,\xi_n)\}_{n\in N}$ is a sequence in $Q \times Q^n$ converging to (s,ξ). This definition is well-posed and H, so defined, is a Caratheodory function satisfying, up to a multiplicative constant,

$$|H(x,s,\xi) - H(x,s,\bar{\xi})| \leq b_1(|s|)(1 + |\xi|^{p-1} + |\bar{\xi}|^{p-1})|\xi - \bar{\xi}| \quad \text{a.e. } x \text{ in } \Omega,\ \forall s \in \mathbb{R},\ \forall \xi, \bar{\xi} \in \mathbb{R}^n, \tag{3.7}$$

$$|H(x,s,\xi) - H(x,\bar{s},\xi)| \leq b_2(|s - \bar{s}|)(1 + |\xi|^p) \quad \text{a.e. } x \text{ in } \Omega,\ \forall s, \bar{s} \in \mathbb{R},\ \forall \xi \in \mathbb{R}^n. \tag{3.8}$$

Obviously also H satisfies the signum property:

$$H(x, s, \xi)s \geq 0, \quad \text{a.e. } x \text{ in } \Omega, \quad \forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^n. \tag{3.9}$$

Moreover by approximating (s, ξ) in $\mathbb{R} \times \mathbb{R}^n$ with a sequence $\{(s_n, \xi_n)\}_{n \in N}$ in $Q \times Q^n$ and making use of (1.10), (1.11), (3.4), (3.7) and (3.8) we obtain (3.1).

Step 2. Let us prove (3.2).

Fix $\omega \subset\subset \Omega$. By approximating, as in [4], a function φ of $L^\infty(\omega)$ with a sequence of simple functions in the $L^\infty(\omega)$-norm and making use of (1.11), (3.8), (2.18) and (3.1), it results, for any φ in $L^\infty(\omega)$ and ξ in $\mathbb{R}^n$,

$$[H_\varepsilon(x, \varphi, C^\varepsilon\xi)]^\sim \rightharpoonup H(x, \varphi, \xi) \quad \text{weakly in } L^1(\omega) \tag{3.10}$$

as ε tends to zero.

Let now $\{\varphi_\varepsilon\}_\varepsilon$ and φ_0 be as in (3.3). Let us prove that, for any ξ in $\mathbb{R}^n$,

$$[H_\varepsilon(x, \varphi_\varepsilon, C^\varepsilon\xi)]^\sim - [H_\varepsilon(x, \varphi_0, C^\varepsilon\xi)]^\sim \to 0 \quad \text{strongly in } L^1(\omega) \tag{3.11}$$

as ε tends to zero.

Fix ξ in $\mathbb{R}^n$. The estimate used to prove (3.4) shows that, up to a subsequence,

$$\{|C^\varepsilon\xi|^p\}_\varepsilon \text{ is weakly compact in } L^1(\omega). \tag{3.12}$$

On the other hand, assumption (1.11) implies that,

$$\int_\omega |[H_\varepsilon(x, \varphi_\varepsilon, C^\varepsilon\xi)]^\sim - [H_\varepsilon(x, \varphi_0, C^\varepsilon\xi)]^\sim| \, dx$$

$$\leq \int_\omega |b_2(|\varphi_\varepsilon - \varphi_0|)|(1 + |C^\varepsilon\xi|^p) \, dx \tag{3.13}$$

for any ε. Then, passing to the limit, as ε tends to zero, in (3.13) and making use of (3.3), (3.12) and Lemma 3.1, we obtain (3.11).

Combining (3.10) with (3.11) we have that, for any $\{\varphi_\varepsilon\}_\varepsilon$ and φ_0 satisfying (3.3) and for any ξ in $\mathbb{R}^n$,

$$[H_\varepsilon(x, \varphi_\varepsilon, C^\varepsilon\xi)]^\sim \rightharpoonup H(x, \varphi_0, \xi) \quad \text{weakly in } L^1(\omega) \tag{3.14}$$

as ε tends to zero. Finally we approximate, as in [4], a function ϕ of $(L^\infty(\omega))^n$ by a sequence of simple functions in the $(L^\infty(\omega))^n$-norm. Then (3.2) follows from (1.10), (3.7), (2.18) and (3.14). ■

4. PROOF OF THEOREM 1.8

First we extend the corrector result (2.20) to the nonlinear problem (1.14). To this purpose we give some preliminary results.

Let $\{T_\varepsilon\}_\varepsilon$ be admissible in Ω, $\{P_\varepsilon\}_\varepsilon$ be satisfying (1.2), c_1 be given in (1.2), u_ε be a solution of problem (1.14) with A^ε in $M(\alpha, \beta, \Omega)$, H_ε satisfying (1.11), (1.12) and f in $L^2(\Omega)$. Then, by virtue of (1.1), (1.2), (1.16) and Theorem 1.5 there exists a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$), a positive function χ_0 in $L^\infty(\Omega)$, u in $H_0^1(\Omega)$ and A^0 in $M((\alpha/c_1^2), \beta, \Omega)$ (χ_0, u and A^0 depend on the subsequence) which satisfy (1.18), (1.19) and (1.20).

Let now v_ε be the unique solution of the following auxiliary linear problem:

$$\begin{cases} -\mathrm{div}(A^\varepsilon Dv_\varepsilon) = P_\varepsilon^*(-\mathrm{div}(A^0 Du)) & \text{in } \Omega_\varepsilon, \\ (A^\varepsilon Dv_\varepsilon)\cdot \nu = 0 & \text{on } \partial T_\varepsilon, \\ v_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where A^0 is defined by (1.19), u by (1.20) and P_ε^* by (1.4). Then

$$P_\varepsilon v_\varepsilon \rightharpoonup u \quad \text{weakly in } H_0^1(\Omega), \tag{4.2}$$

as ε tends to zero for a subsequence satisfying (1.19).

Let us prove the following corrector result for problem (1.14).

PROPOSITION 4.1. Let $\{T_\varepsilon\}_\varepsilon$ be admissible in Ω. Let u_ε be a solution of problem (1.14) with A^ε in $M(\alpha, \beta, \Omega)$, H_ε satisfying (1.11), (1.12) and f in $L^2(\Omega)$. Moreover let u be defined by (1.20), A^0 by (1.19), $\{C^\varepsilon\}_\varepsilon$ by (2.17) and c_9 be given in (2.20).

Then it results

$$\limsup_{\varepsilon\to 0} \|Du_\varepsilon - C^\varepsilon\phi\|_{(L^q(\Omega_\varepsilon))^n} \le c_9\|Du - \phi\|_{(L^q(\Omega))^n} \qquad \forall \phi \in (C_0^\infty(\Omega))^n,$$

for any subsequence $\{\varepsilon\}$ satisfying (1.18) (1.19) and (1.20) and q in $]1,2[$.

In order to prove this Proposition we need some preliminary results. The following lemmas are an adaptation of some techniques of [6] (see also [13]) to the case of perforated domains.

LEMMA 4.2. Let u_ε be a solution of problem (1.14) with A^ε in $M(\alpha, \beta, \Omega)$. H_ε satisfying (1.11), (1.12) and f in $L^2(\Omega)$. Then $\{[H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)]^-\}_\varepsilon$ is weakly compact in $L^1(\Omega)$.

Proof. From (1.12), (1.13), (1.16), (1.17) and Hölder inequality it follows that

$$\begin{aligned} &\int_E |[H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)]^-|\,dx \\ &= \int_{E\cap\{|u_\varepsilon|\le t\}} |H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)|\,dx + \int_{E\cap\{|u_\varepsilon|>t\}} |H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)|\,dx \\ &\le \int_{E\cap\{|u_\varepsilon|\le t\}} b_2(|u_\varepsilon|)(1 + |Du_\varepsilon|^p)\,dx + \int_{E\cap\{|u_\varepsilon|>t\}} \frac{H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)}{|u_\varepsilon|} u_\varepsilon\,dx \\ &\le b_2(t)(|E| + c_4^p|E|^{(2-p)/2}) + \frac{1}{t}c_4, \end{aligned}$$

for any $t > 0$, for any ε and for any E measurable subset of Ω. Consequently, by virtue of the Dunford–Pettis Theorem, the compactness result holds. ■

Let, for any k in $(0, +\infty)$, T_k be the truncation function given by

$$T_k: s \in \mathbb{R} \to \max\{-k, \min\{s, k\}\} \in [-k, k].$$

Making use of (1.18), (1.20) and (4.2), it is easy to prove that, for any k in $(0, +\infty)$,

$$\begin{cases} T_k(P_\varepsilon u_\varepsilon - P_\varepsilon v_\varepsilon) \rightharpoonup 0 & \text{weakly in } H_0^1(\Omega), \\ T_k(P_\varepsilon u_\varepsilon - P_\varepsilon v_\varepsilon) \to 0 & \text{strongly in } L^2(\Omega), \\ T_k(P_\varepsilon u_\varepsilon - P_\varepsilon v_\varepsilon) \to 0 & \text{a.e. in } \Omega, \end{cases} \tag{4.3}$$

as ε tends to zero in a subsequence satisfying (1.18), (1.19) and (1.20) (see also [13]).

LEMMA 4.3. Let $\{T_\varepsilon\}_\varepsilon$ be admissible in Ω and $\{P_\varepsilon\}_\varepsilon$ be satisfying (1.2). Let u_ε be a solution of problem (1.14) with A^ε in $M(\alpha, \beta, \Omega)$, H_ε satisfying (1.11), (1.12) and f in $L^2(\Omega)$. Let v_ε be the solution of problem (4.1) with A^0 defined by (1.19), u by (1.20) and P_ε^* by (1.4).
Then it results

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} A^\varepsilon D(u_\varepsilon - v_\varepsilon) D(T_k(u_\varepsilon - v_\varepsilon))\, dx = 0$$

for any subsequence $\{\varepsilon\}$ satisfying (1.18), (1.19), (1.20) and for any k in $(0, +\infty)$.

Proof. Subtracting (4.1) from (1.14) first and choosing then $T_k(u_\varepsilon - v_\varepsilon) \in V_\varepsilon \cap L^\infty(\Omega_\varepsilon)$ as test function in the difference, it results

$$\begin{aligned} &\int_{\Omega_\varepsilon} A^\varepsilon D(u_\varepsilon - v_\varepsilon) D(T_k(u_\varepsilon - v_\varepsilon))\, dx \\ &\quad = -\int_\Omega [H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)]^{\sim} T_k(P_\varepsilon u_\varepsilon - P_\varepsilon v_\varepsilon)\, dx \\ &\quad\quad + \int_\Omega \chi_{\Omega_\varepsilon} f T_k(P_\varepsilon u_\varepsilon - P_\varepsilon v_\varepsilon)\, dx - \int_\Omega A^0 Du DP_\varepsilon(T_k(u_\varepsilon - v_\varepsilon))\, dx. \end{aligned} \tag{4.4}$$

The result follows by passing to the limit in (4.4), as ε tends to zero for a subsequence satisfying (1.18), (1.19), (1.20) and applying Lemmas 1.3, 3.1, 4.2 and convergences (4.3). ■

By making use of Lemma 4.3, we obtain the following result.

PROPOSITION 4.4. Assume the same hypotheses of Lemma 4.3. Then it results

$$\lim_{\varepsilon \to 0} \|D(u_\varepsilon - v_\varepsilon)\|_{(L^q(\Omega_\varepsilon))^n} = 0$$

for any subsequence $\{\varepsilon\}$ satisfying (1.18), (1.19), (1.20) and for any q in $]1, 2[$.

Proof. Fix q in $]1,2[$ and a subsequence $\{\varepsilon\}$ satisfying (1.18), (1.19) and (1.20). For a given k in $(0,+\infty)$, Hölder inequality and (1.2) imply

$$\begin{aligned}
&\int_{\Omega_\varepsilon} |D(u_\varepsilon - v_\varepsilon)|^q \, dx \\
&= \int_{\Omega_\varepsilon \cap \{|u_\varepsilon - v_\varepsilon| \le k\}} |D(u_\varepsilon - v_\varepsilon)|^q \, dx + \int_{\Omega_\varepsilon \cap \{|u_\varepsilon - v_\varepsilon| > k\}} |D(u_\varepsilon - v_\varepsilon)|^q \, dx \\
&\le \left(\int_{\Omega_\varepsilon \cap \{|u_\varepsilon - v_\varepsilon| \le k\}} |D(u_\varepsilon - v_\varepsilon)|^2 \, dx \right)^{q/2} |\Omega|^{(2-q)/2} \\
&\quad + \left(\int_\Omega |D(P_\varepsilon u_\varepsilon - P_\varepsilon v_\varepsilon)|^2 \, dx \right)^{q/2} |\{x \in \Omega : |P_\varepsilon u_\varepsilon - P_\varepsilon v_\varepsilon| > k\}|^{(2-q)/2}
\end{aligned} \tag{4.5}$$

for any ε.

On the other hand, from the assumption on $\{A^\varepsilon\}_\varepsilon$ and Lemma 4.3 it follows that

$$\begin{aligned}
0 &\le \limsup_{\varepsilon \to 0} \int_{\Omega_\varepsilon \cap \{|u_\varepsilon - v_\varepsilon| \le k\}} |D(u_\varepsilon - v_\varepsilon)|^2 \, dx \\
&\le \frac{1}{\alpha} \limsup_{\varepsilon \to 0} \int_{\Omega_\varepsilon \cap \{|u_\varepsilon - v_\varepsilon| \le k\}} A^\varepsilon D(u_\varepsilon - v_\varepsilon) D(u_\varepsilon - v_\varepsilon) \, dx \\
&= \frac{1}{\alpha} \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} A^\varepsilon D(u_\varepsilon - v_\varepsilon) D(T_k(u_\varepsilon - v_\varepsilon)) \, dx = 0.
\end{aligned} \tag{4.6}$$

Moreover (1.20) and (4.2) imply that

$$\int_\Omega |D(P_\varepsilon u_\varepsilon - P_\varepsilon v_\varepsilon)|^2 \, dx \le c_{12} \tag{4.7}$$

for any ε, where c_{12} is a real independent of ε, and

$$\lim_{\varepsilon \to 0} |\{x \in \Omega : |P_\varepsilon u_\varepsilon - P_\varepsilon v_\varepsilon| > k\}| = 0. \tag{4.8}$$

Then combining (4.5) with (4.6), (4.7) and (4.8), the result is proved. ■

Proof of Proposition 4.1. The result follows immediately by applying estimate (2.20) of Theorem 2.2 to problem (4.1) and by making use of Proposition 4.4. ■

Finally we can prove Theorem 1.8.

Proof of Theorem 1.8. Here we adapt some ideas of the proof of Theorem 4.1 in [4] (see also [15]).

As observed at the beginning of this section, there exist a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$), a positive function χ_0 in $L^\infty(\Omega)$, u in $H_0^1(\Omega)$ and A^0 in $M((\alpha/c_1^2), \beta, \Omega)$ which satisfy (1.18), (1.19) and (1.20).

Let $\{C^\varepsilon\}_\varepsilon$ be defined by (2.17) and A^0 by (2.2). Then, by virtue of Proposition 3.2, there exists a subsequence of the previous (still denoted by $\{\varepsilon\}$) and a function H on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ (depending on the subsequence) satisfying (1.10), (1.11) and (1.12) in

$\Omega \times \mathbb{R} \times \mathbb{R}^n$, up to a multiplicative real c_{10}, such that for any k in $(0, +\infty)$

$$[H_\varepsilon(x, T_k(P_\varepsilon u_\varepsilon), C^\varepsilon \phi)]^\sim \rightharpoonup H(x, T_k u, \phi) \quad \text{weakly in } L^1_{\text{loc}}(\Omega) \quad \forall \phi \in (C_0^\infty(\Omega))^n \tag{4.9}$$

as ε tends to zero. In the remainder of this proof, $\{\varepsilon\}$ denotes a subsequence satisfying (1.18), (1.19), (1.20) and (4.9).

Step 1. Let us prove that, for any k in $(0, +\infty)$,

$$[H_\varepsilon(x, T_k(P_\varepsilon u_\varepsilon), Du_\varepsilon)]^\sim \rightharpoonup H(x, T_k u, Du) \quad \text{weakly in } L^1(\Omega) \tag{4.10}$$

as ε tends to zero.

Fix k in $(0, +\infty)$. First observe that

$$\{[H_\varepsilon(x, T_k(P_\varepsilon u_\varepsilon), Du_\varepsilon)]^\sim\}_\varepsilon \text{ is weakly compact in } L^1(\Omega). \tag{4.11}$$

In fact, for $q = 2/p$, from (1.13) and (1.16) it follows that

$$\int_\Omega |[H_\varepsilon(x, T_k(P_\varepsilon u_\varepsilon), Du_\varepsilon)]^\sim|^q \, dx \le b_2^q(k) \int_{\Omega_\varepsilon} (1 + |Du_\varepsilon|^p)^q \, dx$$

$$\le b_2^q(k) 2^{q-1} \int_{\Omega_\varepsilon} 1 + |Du_\varepsilon|^2 \, dx \le b_2^q(k) 2^{q-1}(|\Omega| + c_4^2)$$

for any ε. Consequently, since $q > 1$, (4.11) holds.

By virtue of (4.11), for obtaining (4.10) it is enough to prove that, for any k in $(0, +\infty)$,

$$[H_\varepsilon(x, T_k(P_\varepsilon u_\varepsilon), Du_\varepsilon)]^\sim \rightharpoonup H(x, T_k u, Du) \quad \text{weakly in } D'(\Omega) \tag{4.12}$$

as ε tends to zero.

To do that, fix φ in $C_0^\infty(\Omega)$. For $\eta > 0$ let ϕ_η be in $(C_0^\infty(\Omega))^n$ such that

$$\|\phi_\eta - Du\|_{(L^2(\Omega))^n} < \eta. \tag{4.13}$$

Then, by virtue of assumption (1.10) of H_ε and property (3.7) of H, it results

$$\begin{aligned}
&\left| \int_\Omega ([H_\varepsilon(x, T_k(P_\varepsilon u_\varepsilon), Du_\varepsilon)]^\sim - H(x, T_k u, Du))\varphi \, dx \right| \\
&\le \int_{\Omega_\varepsilon} |H_\varepsilon(x, T_k u_\varepsilon, Du_\varepsilon) - H_\varepsilon(x, T_k u_\varepsilon, C^\varepsilon \phi_\eta)| |\varphi| \, dx \\
&\quad + \left| \int_\Omega ([H_\varepsilon(x, T_k(P_\varepsilon u_\varepsilon), C^\varepsilon \phi_\eta)]^\sim - H(x, T_k u, \phi_\eta))\varphi \, dx \right| \\
&\quad + \int_\Omega |H(x, T_k u, \phi_\eta) - H(x, T_k u, Du)| |\varphi| \, dx \\
&\le b_1(k) \|\varphi\|_{L^\infty(\Omega)} \int_{\Omega_\varepsilon} (1 + |Du_\varepsilon|^{p-1} + |C^\varepsilon \phi_\eta|^{p-1}) |u_\varepsilon - C^\varepsilon \phi_\eta| \, dx \\
&\quad + \left| \int_\Omega ([H_\varepsilon(x, T_k(P_\varepsilon u_\varepsilon), C^\varepsilon \phi_\eta)]^\sim - H(x, T_k u, \phi_\eta))\varphi \, dx \right| \\
&\quad + c_{10} b_1(k) \|\varphi\|_{L^\infty(\Omega)} \int_\Omega (3 + |\phi_\eta| + |Du|) |\phi_\eta - Du| \, dx.
\end{aligned} \tag{4.14}$$

for any ε (since $|\xi|^{p-1} \le 1 + |\xi|$). On the other hand, from (1.16), Hölder inequality, Proposition 4.1 and (4.13) it follows that

$$\limsup_{\varepsilon\to 0} \int_{\Omega_\varepsilon} (1 + |Du_\varepsilon|^{p-1} + |C^\varepsilon\phi_\eta|^{p-1})|Du_\varepsilon - C^\varepsilon\phi_\eta|\,dx$$

$$\le \limsup_{\varepsilon\to 0}[(|\Omega|^{(p-1)/p} + \|Du_\varepsilon\|^{p-1}_{(L^p(\Omega_\varepsilon))^n} + \|C^\varepsilon\phi_\eta\|^{p-1}_{(L^p(\Omega_\varepsilon))^n})\|Du_\varepsilon - C^\varepsilon\phi_\eta\|_{(L^p(\Omega_\varepsilon))^n}]$$

$$\le \limsup_{\varepsilon\to 0}[(|\Omega|^{(p-1)/p} + 2 + 2\|Du_\varepsilon\|_{(L^p(\Omega_\varepsilon))^n} + \|Du_\varepsilon - C^\varepsilon\phi_\eta\|_{(L^p(\Omega_\varepsilon))^n})$$

$$\cdot \|Du_\varepsilon - C^\varepsilon\phi_\eta\|_{(L^p(\Omega_\varepsilon))^n}]$$

$$\le (|\Omega|^{(p-1)/p} + 2 + 2|\Omega|^{(2-p)/2p}c_4 + c_9\|Du - \phi_\eta\|_{(L^2(\Omega))^n})c_9\|Du - \phi_\eta\|_{(L^2(\Omega))^n}$$

$$\le (|\Omega|^{(p-1)/p} + 2 + 2|\Omega|^{(2-p)/2p}c_4 + c_9\eta)c_9\eta, \tag{4.15}$$

$$\int_\Omega (3 + |\phi_\eta| + |Du|)|\phi_\eta - Du|\,dx$$

$$\le (3|\Omega|^{1/2} + \|\phi_\eta - Du\|_{(L^2(\Omega))^n} + 2\|Du\|_{(L^2(\Omega))^n})\|\phi_\eta - Du\|_{(L^2(\Omega))^n}$$

$$\le (3|\Omega|^{1/2} + \eta + 2\|Du\|_{(L^2(\Omega))^n})\eta. \tag{4.16}$$

Hence (4.9), (4.14), (4.15) and (4.16) imply that, up to a multiplicative constant independent of η,

$$\limsup_{\varepsilon\to 0}\left|\int_\Omega ([H_\varepsilon(x, T_k(P_\varepsilon u_\varepsilon), Du_\varepsilon)]^{\sim} - H(x, T_k u, Du))\varphi\,dx\right| < \eta.$$

The arbitrariety of η gives

$$\lim_{\varepsilon\to 0}\left|\int_\Omega ([H_\varepsilon(x, T_k(P_\varepsilon u_\varepsilon), Du_\varepsilon)]^{\sim} - H(x, T_k u, Du))\phi\,dx\right| = 0.$$

Convergence (4.12) follows then from the arbitrariety of φ in $C_0^\infty(\Omega)$.

Step 2. In this step let us prove that

$$H(x, u, Du)u \in L^1(\Omega), \tag{4.17}$$

$$H(x, u, Du) \in L^1(\Omega). \tag{4.18}$$

Let us prove first (4.17). From (1.12) and (1.17) it follows that

$$\int_\Omega [H_\varepsilon(x, T_k(P_\varepsilon u_\varepsilon), Du_\varepsilon)]^{\sim} T_k(P_\varepsilon u_\varepsilon)\chi_{\{x\in\Omega : |P_\varepsilon u_\varepsilon| < k\}}\,dx$$

$$= \int_{\{x\in\Omega : |P_\varepsilon u_\varepsilon| < k\}} [H_\varepsilon(x, P_\varepsilon u_\varepsilon, Du_\varepsilon)]^{\sim} P_\varepsilon u_\varepsilon\,dx$$

$$\le \int_\Omega [H_\varepsilon(x, P_\varepsilon u_\varepsilon, Du_\varepsilon)]^{\sim} P_\varepsilon u_\varepsilon\,dx \le c_4 \tag{4.19}$$

for any ε and for any k in $(0, +\infty)$. On the other hand (1.20) implies that

$$\chi_{\{x \in \Omega : |P_\varepsilon u_\varepsilon| < k\}} \to \chi_{\{x \in \Omega : |u| < k\}} \quad \text{a.e. in } \Omega, \tag{4.20}$$

and

$$T_k(P_\varepsilon u_\varepsilon) \to T_k(u) \quad \text{a.e. in } \Omega,$$

as ε tends to zero, for almost every k in $(0, +\infty)$. Then passing to the limit, as ε tends to zero, in (4.19) and making use of Lemma 3.1 and (4.10) it results

$$\int_\Omega H(x, u, Du) u \chi_{\{x \in \Omega : |u| < k\}} \, dx = \int_\Omega H(x, T_k(u), Du) T_k(u) \chi_{\{x \in \Omega : |u| < k\}} \, dx \le c_4 \tag{4.21}$$

for a.e. k in $(0, +\infty)$. Consequently, passing to the limit in (4.21) as k tends to infinity, by virtue of signum property of H and Beppo Levi Theorem, we obtain (4.17).

Let us prove now (4.18). Since, for a fixed k

$$|H(x, u, Du)| = \left| H(x, u, Du) \chi_{\{x \in \Omega : |u| < k\}} + \frac{1}{u} H(x, u, Du) u \chi_{\{x \in \Omega : |u| \ge k\}} \right|$$

$$\le |H(x, T_k(u), Du)| \chi_{\{x \in \Omega : |u| < k\}} + \frac{1}{k} H(x, u, Du) u,$$

the statement (4.18) follows from (4.10) and (4.17).

Step 3. Let us prove (1.21).

Let φ be in $L^\infty(\Omega)$. We have that

$$\int_\Omega [H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)]^\sim \varphi \, dx = \int_\Omega [H_\varepsilon(x, P_\varepsilon u_\varepsilon, Du_\varepsilon)]^\sim \varphi \, dx = \int_{\{x \in \Omega : |P_\varepsilon u_\varepsilon| < k\}} [H_\varepsilon(x, T_k(P_\varepsilon u_\varepsilon), Du_\varepsilon)]^\sim \varphi \, dx + \int_{\{x \in \Omega : |P_\varepsilon u_\varepsilon| \ge k\}} [H_\varepsilon(x, P_\varepsilon u_\varepsilon, Du_\varepsilon)]^\sim \varphi \, dx, \tag{4.22}$$

$$\int_\Omega H(x, u, Du) \varphi \, dx = \int_{\{x \in \Omega : |u| < k\}} H(x, T_k(u), Du) \varphi \, dx + \int_{\{x \in \Omega : |u| \ge k\}} H(x, u, Du) \varphi \, dx \tag{4.23}$$

for any ε and for any k in $(0, +\infty)$.

On the other hand from Lemma 3.1, (4.10) and (4.20) it follows that

$$\lim_{\varepsilon\to 0}\int_{\{x\in\Omega:|P_\varepsilon u_\varepsilon|<k\}}[H_\varepsilon(x,T_k(P_\varepsilon u_\varepsilon),Du_\varepsilon)]^{\sim}\varphi\,\mathrm{d}x = \int_{\{x\in\Omega:|u|<k\}}H(x,u,Du)\varphi\,\mathrm{d}x \tag{4.24}$$

for a.e. k in $(0,+\infty)$. Moreover (1.12) and (1.17) imply that

$$\int_{\{x\in\Omega:|P_\varepsilon u_\varepsilon|\geq k\}}|H_\varepsilon(x,P_\varepsilon u_\varepsilon,Du_\varepsilon)||\varphi|\,\mathrm{d}x \leq \|\varphi\|_{L^\infty(\Omega)}\frac{c_4}{k} \tag{4.25}$$

for any ε and for any k in $(0,+\infty)$ and (4.18) provides that

$$\lim_{k\to+\infty}\int_{\{x\in\Omega:|u|\geq k\}}H(x,u,Du)\varphi\,\mathrm{d}x = 0. \tag{4.26}$$

Then passing to the limit in (4.22), first as ε tends to zero and then as k goes to infinity, and using (4.23)–(4.26) we obtain convergence (1.21).

Step 4. To prove (1.22) observe that

$$\begin{aligned}-\mathrm{div}(A^\varepsilon\widetilde{Du_\varepsilon}) = {}& -\mathrm{div}(A^\varepsilon[(Du_\varepsilon - Dv_\varepsilon)]^{\sim}) + (-\mathrm{div}(A^\varepsilon\widetilde{Dv_\varepsilon} - P_\varepsilon^*(-\mathrm{div}(A^0Du)))\\ &+ P_\varepsilon^*(-\mathrm{div}(A^0Du)),\end{aligned} \tag{4.27}$$

for any ε, where v_ε is the solution of (4.1).

On the other hand from the assumption on $\{A^\varepsilon\}_\varepsilon$, Hölder inequality and Proposition 4.4 it follows that, for a fixed q in $]1,2[$,

$$\begin{aligned}\lim_{\varepsilon\to 0}\left|\int_\Omega A^\varepsilon[(Du_\varepsilon - Dv_\varepsilon)]^{\sim}D\varphi\,\mathrm{d}x\right| &\leq \beta\lim_{\varepsilon\to 0}\|Du_\varepsilon - Dv_\varepsilon\|_{(L^q(\Omega_\varepsilon))^n}\|D\varphi\|_{(L^{q/(q-1)}(\Omega))^n}\\ &= 0 \qquad \forall\varphi\in C_0^\infty(\Omega).\end{aligned} \tag{4.28}$$

Moreover it is easy to verify that

$$-\mathrm{div}(A^\varepsilon\widetilde{Dv_\varepsilon}) - P_\varepsilon^*(-\mathrm{div}(A^0Du)) = 0 \qquad \text{in } H^{-1}(\Omega) \tag{4.29}$$

for any ε and that from Lemma 1.3

$$P_\varepsilon^*(-\mathrm{div}(A^0Du)) \to -\mathrm{div}(A^0Du) \quad \text{strongly in } H^{-1}(\Omega) \tag{4.30}$$

as ε tends to zero. Then, combining (4.27) with (4.28), (4.29) and (4.30), convergence (1.22) holds.

Step 5. Now passing to the limit, as ε tends to zero, in (1.15) and making use of (1.18), (1.21) and (1.22) it results

$$\int_\Omega A^0DuDv\,\mathrm{d}x + \int_\Omega H(x,u,Du)v\,\mathrm{d}x = \int_\Omega \chi_0 fv \qquad \forall v\in C_0^\infty(\Omega). \tag{4.31}$$

In conclusion (4.31) holds true for all v in $H_0^1(\Omega)\cap L^\infty(\Omega)$ and also for $v=u$, by virtue of a result of Brezis–Browder (see [26]). Then u is a solution of (1.23). ■

5. PROOF OF THEOREM 1.9

The proof of Theorem 1.9 is given in the same spirit of the proof of Theorem 1.8. We just point out the main differences.

First, as in Section 4, we extend (2.20) to the nonlinear problem (1.24).

Let $\{T_\varepsilon\}_\varepsilon$ be admissible in Ω, $\{P_\varepsilon\}_\varepsilon$ be satisfying (1.2) and (1.3), c_1 be given in (1.2), u_ε be a solution of problem (1.24) with A^ε in $M(\alpha, \beta, \Omega)$, H_ε satisfying (1.9) and γ a positive real. Then, by virtue of (1.1), (1.2), (1.3), (1.26) and Theorem 1.5 there exists a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$), a positive function χ_0 in $L^\infty(\Omega)$, u in $H_0^1(\Omega) \cap L^\infty(\Omega)$ and A^0 in $M(\alpha/c_1^2, \beta, \Omega)$ (χ_0, u and A^0 depend on the subsequence) which satisfy (1.27), (1.28) and (1.29).

Let now v_ε be the unique solution of the following auxiliary linear problem:

$$\begin{cases} -\operatorname{div}(A^\varepsilon D v_\varepsilon) = P_\varepsilon^*(-\operatorname{div}(A^0 Du)) & \text{in } \Omega_\varepsilon, \\ (A^\varepsilon D v_\varepsilon) \cdot \nu = 0 & \text{on } \partial T_\varepsilon, \\ v_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where A^0 is defined by (1.28), u by (1.29) and P_ε^* by (1.4).

The following L^1 weak compactness result holds.

LEMMA 5.1. Let u_ε be a solution of problem (1.24) with A^ε in $M(\alpha, \beta, \Omega)$, H_ε satisfying (1.9) and γ a positive real.

Then $\{[H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)]^-\}_\varepsilon$ is weakly compact in $L^1(\Omega)$.

Proof. Set $q = 2/p$. From (1.9) and (1.26) it follows that

$$\begin{aligned} \int_\Omega |[H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)]^-|^q \, dx &\le c_3^q \int_{\Omega_\varepsilon} (1 + |Du_\varepsilon|^p)^q \, dx \\ &\le c_3^q 2^{q-1} \int_{\Omega_\varepsilon} 1 + |Du_\varepsilon|^2 \, dx \\ &\le c_3^q 2^{q-1}(|\Omega| + c_5^2), \end{aligned}$$

for any ε, which gives the compactness result, since $q > 1$. ■

Applying Lemma 5.1 and following the same outlines as that of the proof of Proposition 4.4, it is easy to prove the following result.

PROPOSITION 5.2. Let $\{T_\varepsilon\}_\varepsilon$ be admissible in Ω and $\{P_\varepsilon\}_\varepsilon$ be satisfying (1.2) and (1.3). Let u_ε be a solution of problem (1.24) with A^ε in $M(\alpha, \beta, \Omega)$, H_ε satisfying (1.9) and γ a positive real. Let v_ε be the solution of problem (5.1) with A^0 defined by (1.28), u by (1.27) and P_ε^* by (1.4).

Then it results

$$\lim_{\varepsilon \to 0} \|D(u_\varepsilon - v_\varepsilon)\|_{(L^q(\Omega_\varepsilon))^n} = 0$$

for any subsequence $\{\varepsilon\}$ satisfying (1.27), (1.28), (1.29) and for any q in $]1, 2[$.

From (2.20) of Theorem 2.2 and Proposition 5.2 it follows immediately.

COROLLARY 5.3. Let $\{T_\varepsilon\}_\varepsilon$ be admissible in Ω. Let u_ε be a solution of problem (1.24) with A^ε in $M(\alpha, \beta, \Omega)$, H_ε satisfying (1.9) and γ a positive real. Moreover let u be defined by (1.29), A^0 by (1.28), $\{C^\varepsilon\}_\varepsilon$ by (2.17) and c_9 be given in (2.20).

Then it results

$$\limsup_{\varepsilon\to 0} \|Du_\varepsilon - C^\varepsilon\phi\|_{(L^q(\Omega_\varepsilon))^n} \le c_9\|Du - \phi\|_{(L^2(\Omega))^n} \qquad \forall\phi \in (C_0^\infty(\omega))^n$$

for any subsequence $\{\varepsilon\}$ satisfying (1.27), (1.28), (1.29) and for any q in $]1, 2[$.

Now to prove Theorem 1.9 observe that, by virtue of Proposition 3.2, there exists a subsequence of the previous (still denoted by $\{\varepsilon\}$) and a function H on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ (depending on the subsequence), satisfying in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ (1.9), (1.10), (1.11) up to a multiplicative real c_{10}, such that

$$[H_\varepsilon(x, P_\varepsilon u_\varepsilon, C^\varepsilon\phi)]^\sim \rightharpoonup H(x, u, \phi) \quad \text{weakly in } L^1_{\text{loc}}(\Omega), \quad \forall\phi \in (C_0^\infty(\Omega))^n \tag{5.2}$$

as ε tends to zero.

Arguing as in the proof of (4.10) and using (1.10), (1.29), (5.2), the properties satisfied by H and Corollary 5.3 we obtain (1.30).

Further (1.31) is a consequence of convergences (1.29), (1.30) and Lemma 3.1.

Arguing as in the proof of Theorem 1.8 and making use of Proposition 5.2 we deduce (1.32). Finally, as in Section 4, (1.33) follows from (1.27), (1.29), (1.30) and (1.32).

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