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# THE ONE-DIMENSIONAL DIRAC EQUATION WITH CONCENTRATED NONLINEARITY*

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**Abstract.** We define and study the Cauchy problem for a one-dimensional (1-D) nonlinear Dirac equation with nonlinearities concentrated at one point. Global well-posedness is provided and conservation laws for mass and energy are shown. Several examples, including nonlinear Gesztesy–Šeba models and the concentrated versions of the Bragg resonance and 1-D Soler (also known as massive Gross–Neveu) type models, all within the scope of the present paper, are given. The key point of the proof consists in the reduction of the original equation to a nonlinear integral equation for an auxiliary, space-independent variable.

**Key words.** nonlinear Dirac equation, well-posedness, point interactions

**AMS subject classifications.** 35Q41, 35A01, 35B25

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## 1. Introduction.

Interest in the nonlinear Dirac equation can be traced back to the paper [39], in which a soluble nonlinear quantum field model in 1+1 space-time dimensions for self-interacting fermions was introduced. Other well-known quantum field theoretic examples are given in [38], again describing a self-interacting electron in 3+1 space-time dimensions, and later in [23], this time describing a model related to quantum chromodynamics. However, the nonlinear Dirac equation also appears as an effective equation in condensed matter physics, here describing localization effects for solutions of the nonlinear Schrödinger or Gross–Pitaevskii equation in small periodic potentials (see, e.g., [20] and the monograph [33] for an extended description and bibliography). Relevant applications are in photonic crystals and in Bose–Einstein condensates, where a two-dimensional (2-D) nonlinear Dirac equation plays the role of an effective equation governing the evolution of wavepackets spectrally concentrated near the Dirac points of a periodic optical lattice (see [1, 15, 18, 20] and references therein). Inspired by the above models, the rigorous analysis of the Dirac equation with general nonlinearities is now a major subject. As regards well-posedness results, we only mention some of the relevant papers, being [8, 13, 16, 20, 30, 36]. Specifically, for the one-dimensional (1-D) case, results relating to the global well-posedness in the Sobolev space $H^1(\mathbb{R})$ for several types of nonlinearities are known. For a review about the global well-posedness of the nonlinear Dirac equation in one space dimension, see [34]. For the especially relevant cases of Thirring and Gross–Neveu models, see also [25, 26].

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In this paper we define and solve the Cauchy problem for a Dirac-type equation with concentrated nonlinearity. By this we mean that the nonlinearity is space-dependent and acts at a single point in space. Models of this type are popular in physics in the case of the Schrödinger equation (see, e.g., [9, 17, 31]), and there is also a growing amount of literature of a mathematical nature relating to their well-posedness [2, 3, 5] (see also [24]), orbital and asymptotic stability of standing waves (see [4]), and approximation through smooth space-dependent nonlinearities [10, 11, 12] (see also [27]). Work on the wave equation with concentrated nonlinearities in three dimensions began with [32], and using similar techniques the Klein–Gordon equation has recently been treated [29]. To the knowledge of present authors, there is currently no comparable activity related to the Dirac equation and this paper is possibly a first contribution to the subject (see, however, the interesting paper [28] in which a model that represents a regularization of a concentrated nonlinearity is considered). To introduce the problem in the simplest way (details are given in the following sections), we consider the Dirac operator

$$D_m\Psi := -i\hbar\, c\,\sigma_1\frac{d\Psi}{dx} + m\,c^2\sigma_3\Psi,$$

where $\sigma_1$ and $\sigma_3$ are a suitable choice of Pauli matrices. The nonlinear Dirac equation with a space-dependent nonlinearity is given by

$$i\hbar\,\frac{\partial}{\partial t}\Psi = D_m\Psi + V\,g(\Psi),$$

where $V = V(x)$. In this paper we would ideally treat the case where $V \to \delta_y$ weakly. This limit procedure can be consistently pursued in the case of the nonlinear Schrödinger equation, and yields to a well-defined, nontrivial, and nonlinear dynamics (see [10, 11] and references therein). The corresponding three-dimensional model has also been studied mainly from a mathematical point of view (see [2, 3, 4, 12]). The same constructive analysis could be attempted for the Dirac equation, but here we make use of a more abstract approach which has the virtue of complete generality. The starting point is the construction of linear singular perturbations of the Dirac operator, well known for a long time (see [6, 7, 14, 19]). The idea is to restrict the free Dirac operator $D_m$ to regular functions out of the point $y$, obtaining a symmetric, non-self-adjoint operator. The self-adjoint extensions of this operator give rise to a unitary dynamics. Among them there is of course the Dirac operator itself, but many others exist which differ for the singular behavior at the point $y$. They are parametrized through a singular boundary condition embodied in the domain of the extended operator

$$\{\Psi \in H^1(\mathbb{R}\backslash\{y\})\otimes\mathbb{C}^2 : \quad ic\,\sigma_1[\Psi]_y = A\underline{q}\},$$

where the two-component vector $\underline{q} := \frac{1}{2}(\Psi(y^+) + \Psi(y^-))$ is the mean value, and $[\Psi]_y := \Psi(y^+) - \Psi(y^-)$ is the jump of the spinor $\Psi$ at $y$, while $A$ is any $2\times 2$ Hermitian matrix. The case $A = 0$ gives of course the free Dirac operator on the line, while in all other cases there are singularities at the point $y$, because the jump of $\Psi$ is nontrivial. It is easy to see that one ends up with the evolution described by the distributional equation

$$i\hbar\frac{d}{dt}\Psi(t) = D_m\Psi + i\hbar\,c\,\sigma_1[\Psi]_y\delta_y$$

with $\Psi$ belonging to the above domain.

To define a nonlinear dynamics, we let the matrix $A$ be dependent on the function $\underline{q}$, arriving at a nonlinear operator $H_A^{nl}$ with a domain characterized by a nonlinear boundary condition at the point $y$:

$$D(H_A^{nl}) = \left\{\Psi \in \mathcal{H} : \Psi \in H^1(\mathbb{R}\backslash\{y\}) \otimes \mathbb{C}^2,\, ic\,\sigma_1[\Psi]_y = A(\underline{q})\underline{q}\right\},$$

where the matrix $A(\underline{z})$ is Hermitian for all $\underline{z} \in \mathbb{C}^2$.

Under a technical condition (see Assumption 3.2), we will show the following well-posedness result (see Theorem 3.5 for the precise statement).

For any $\Psi_\circ \in D(H_A^{nl})$, there exists a unique, global-in-time solution $\Psi(t)$ of the Cauchy problem

$$\begin{cases} i\hbar \dfrac{d}{dt}\Psi(t) = H_A^{nl}\Psi(t) = D_m\Psi + \hbar A(\underline{q})\underline{q}\,\delta_y, \\ \Psi(0) = \Psi_\circ \in D(H_A^{nl}). \end{cases}$$

A relevant fact about the proof of the main theorem is that, recasting the initial value problem in integral form through the Duhamel formula, $\Psi(t)$ turns out to depend on the solution of a nonlinear integral equation (giving the evolution of the function $\underline{q}$; see (3.4)), which rules the behavior of the system. Once the solution of this nonlinear integral equation is guaranteed, a representation formula for the solution of the Cauchy problem (which seems to be new even in the linear case when a point interaction is present) allows one to close the proof of the theorem. Assumption 3.2 on $A(\underline{z})$ is needed to treat the existence and uniqueness of the solution of (3.4), and the rest of the proof consists in assuring the stated regularity properties of the solution.

To conclude the introduction, we now give a brief outline of the content of the various sections of the paper.

In the preliminary section 2, in order to keep the presentation self-contained, we recall the definition of the 1-D Dirac equation with a linear point interaction. Here we also provide a new representation formula for the solution of the linear Cauchy problem (see Proposition 2.1).

Section 3 is the core of the paper. The definition of the Dirac operator perturbed by a concentrated nonlinearity is given and it is shown how to split the nonlinear flow into the sum of the free flow plus a part containing only $\underline{q}$ (depending on the total initial datum) which satisfies a nonlinear integral equation. It is then shown that (3.4) in the stated hypotheses admits a unique solution and the main theorem is proved. The section ends with the proof of three complementary but relevant properties. The independence of the global well-posedness results on the special representation of the algebra of the Dirac matrices employed is proven (see Remark 3.7), and the mass (or $L^2$-norm; see Theorem 3.8) and energy conservation laws (see Theorem 3.9) are shown. As regards the energy, in order to obtain conservation one has to restrict the admissible matrix fields $A(\underline{z})$ by imposing the constraint $A(\underline{z}) = \mathcal{A}(\bar{\underline{z}}, \underline{z}) = \mathcal{A}(\underline{z}, \bar{\underline{z}})$.

In section 4, several examples are given, to compare our point-like nonlinear models to the ones found in literature. Among others, the nonlinear versions of the Gesztesy–Šeba models and the concentrated nonlinearities mimicking the 1-D Soler-type and Bragg resonance models are treated. Finally, in Appendix A, the representation formula for the free Dirac evolution in 1-D is recalled and the $H^1$-regularity in time of the evaluation at the singularity of the free part of the evolution is proved; see Proposition A.1.

Throughout the paper we will use the following notation.

- The inner product between two vector-valued functions in $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ is denoted by $\langle \Psi, \Phi \rangle$, and it is antilinear in the first argument. The corresponding norm is simply denoted by $\|\Psi\|$.
- The inner product between two vectors in $\mathbb{C}^2$ is denoted by $\langle \underline{z}, \underline{\xi} \rangle_{\mathbb{C}^2}$, and it is antilinear in the first argument. The corresponding norm is simply denoted by $|\underline{z}|$, not to be mistaken for the usual absolute value which is denoted in the same way.
- $C$ denotes a generic positive constant whose value may change from line to line.

**2. The Cauchy problem for the Dirac equation with point interactions.** Let $D_m : \mathcal{S}'(\mathbb{R}) \otimes \mathbb{C}^2 \to \mathcal{S}'(\mathbb{R}) \otimes \mathbb{C}^2$ be the differential operator

$$D_m \Psi := -i\hbar\, c\, \sigma_1 \frac{d\Psi}{dx} + m\, c^2 \sigma_3 \Psi$$

corresponding to the free 1-D Dirac operator with mass $m \geq 0$. Here $\hbar$ is Planck's constant, $c$ is the light velocity, $\mathcal{S}'(\mathbb{R})$ denotes the space of tempered distribution, $\Psi = \binom{\psi_1}{\psi_2}$, and $\sigma_1$ and $\sigma_3$ are first and third among the three Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

On the Hilbert space $L^2(\mathbb{R}) \otimes \mathbb{C}^2$, the linear operator

$$H : D(H) \subset L^2(\mathbb{R}) \otimes \mathbb{C}^2 \to L^2(\mathbb{R}) \otimes \mathbb{C}^2, \qquad H\Psi := D_m \Psi$$

with domain $D(H) = H^1(\mathbb{R}) \otimes \mathbb{C}^2$ is self-adjoint, where $H^1(\mathbb{R})$ denotes the Sobolev space of square integrable functions with square integrable first-order distributional derivatives.

Now we recall the construction of the self-adjoint singular perturbations of $H$ formally corresponding to the addition of a $\delta$-type potential (see, e.g., [6, 7, 14, 19]).

Given $y \in \mathbb{R}$, let $H_-$ and $H_+$ be the free Dirac operators on $L^2(-\infty, y) \otimes \mathbb{C}^2$ and $L^2(y, +\infty) \otimes \mathbb{C}^2$ with domains $D(H_-) = H^1(-\infty, y) \otimes \mathbb{C}^2$ and $D(H_+) = H^1(y, +\infty) \otimes \mathbb{C}^2$, respectively. Denoting by $S$ the restriction of $H$ to the domain $D(S) := \{\Psi \in H^1(\mathbb{R}) : \Psi(y) = 0\}$, one has that $S$ is closed symmetric and has defect indices $(2, 2)$ and adjoint $S^* = H_- \oplus H_+$. In order to define self-adjoint extensions of $S$, we consider the Hermitian $2 \times 2$ matrices

$$A = \begin{bmatrix} \alpha_1 & \gamma \\ \bar{\gamma} & \alpha_2 \end{bmatrix}, \qquad \alpha_1, \alpha_2 \in \mathbb{R},\ \gamma \in \mathbb{C}.$$

Then one gets a self-adjoint operator $H_A$ on $L^2(\mathbb{R}) \otimes \mathbb{C}^2$ by restricting $H_- \oplus H_+$ to the domain

$$D(H_A) = \left\{ \Psi \in H^1(\mathbb{R}\backslash\{y\}) \otimes \mathbb{C}^2 : ic\, \sigma_1 [\Psi]_y = A\underline{q} \right\}, \tag{2.1}$$

where $H^1(\mathbb{R}\backslash\{y\}) := H^1(-\infty, y) \oplus H^1(y, +\infty)$,

$$[\Psi]_y = \begin{pmatrix} [\psi_1]_y \\ [\psi_2]_y \end{pmatrix} := \Psi(y^+) - \Psi(y^-)$$

denotes the jump of $\Psi$ at the point $y$, and

$$\underline{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} := \frac{1}{2}\left(\Psi(y^+) + \Psi(y^-)\right)$$

denotes the mean value of $\Psi$ at the point $y$. The case $A = 0$ gives the free Dirac operator $H$. By using distributional derivatives, one has

$$H_A\Psi = D_m\Psi + i\hbar\, c\,\sigma_1[\Psi]_y\delta_y = D_m\Psi + \hbar A\underline{q}\,\delta_y. \tag{2.2}$$

The domain and the action of $H_A$ can be described in an alternative way as follows (for simplicity of exposition we consider only the case where $m > 0$; a similar description holds also in the $m = 0$ case). Let $G$ denote the solution of $-D_mG = \delta_y \otimes \mathbb{1}$, i.e.,

$$G(x) = -\frac{1}{2\hbar c}\, e^{-\frac{mc}{\hbar}|x-y|}(i\,\mathrm{sgn}(x-y)\sigma_1 + \sigma_3).$$

Then, since

$$i\hbar c\,\sigma_1[G\underline{\xi}]_y = \underline{\xi}$$

and $\Psi \in H^1(\mathbb{R}\backslash\{y\}) \otimes \mathbb{C}^2$ belongs to $H^1(\mathbb{R}) \otimes \mathbb{C}^2$ if and only if $[\Psi]_y = 0$, one gets

$$H^1(\mathbb{R}\backslash\{y\}) \otimes \mathbb{C}^2 = \{\Psi = \Phi + G\underline{\xi},\ \Phi \in H^1(\mathbb{R}) \otimes \mathbb{C}^2,\ \xi \in \mathbb{C}^2\},$$

$$S^*\Psi = H\Phi,$$

and so, since

$$\frac{1}{2}\left(G\underline{\xi}(y^+) + G\underline{\xi}(y^-)\right) = -\frac{\sigma_3\underline{\xi}}{2\hbar c}, \tag{2.3}$$

the self-adjoint extension $H_A$ can be equivalently defined as

$$D(H_A) = \left\{\Psi = \Phi + G\underline{\xi},\ \Phi \in H^1(\mathbb{R}) \otimes \mathbb{C}^2,\ \xi \in \mathbb{C}^2, \left(\mathbb{1} + \frac{1}{2c}A\sigma_3\right)\underline{\xi} = \hbar A\Phi(y)\right\}, \tag{2.4}$$

$$H_A\Psi = H\Phi.$$

We now consider the Cauchy problem

$$\begin{cases} i\hbar\dfrac{d}{dt}\Psi(t) = H_A\Psi(t), \\ \Psi(0) = \Psi_\circ. \end{cases} \tag{2.5}$$

Since $H_A$ is self-adjoint, such a Cauchy problem is well-posed for any $\Psi_\circ \in L^2(\mathbb{R})\otimes\mathbb{C}^2$ by Stone's theorem. In the following proposition, we give a representation formula for the solution of problem (2.5) in the case $\Psi_\circ \in D(H_A)$. For simplicity of exposition we only consider the case $t \geq 0$; a similar representation holds for $t \leq 0$.

PROPOSITION 2.1. *Let $\Psi_\circ \in D(H_A)$. Then for any $t \geq 0$ the solution $\Psi(t) = e^{-\frac{i}{\hbar}tH_A}\Psi_\circ$ of the Cauchy problem* (2.5) *is given by*

$$\begin{aligned}\Psi(x,t) = \Psi^f(x,t) &- \frac{i}{2c}\theta\left(t - \frac{|x-y|}{c}\right)\begin{bmatrix} 1 & \mathrm{sgn}(x-y) \\ \mathrm{sgn}(x-y) & 1 \end{bmatrix} A\underline{q}\left(t - \frac{|x-y|}{c}\right) \\ &- i\theta\left(t - \frac{|x-y|}{c}\right)\int_0^{t-\frac{|x-y|}{c}} ds\, K(x-y, t-s)A\underline{q}(s),\end{aligned} \tag{2.6}$$

*where* $\Psi^f(t) := e^{-\frac{i}{\hbar}tH}\Psi_\circ$ *and* $\underline{q}(t)$ *is the solution of the integral equation*

$$\underline{q}(t) = \Psi^f(y,t) - \frac{i}{2c}A\underline{q}(t) - i\int_0^t ds\, K(0,t-s)A\underline{q}(s). \tag{2.7}$$

*Here* $\theta$ *denotes Heaviside's step function, and the matrix-valued kernel* $K$ *is defined by*

$$K(x,t) = -\frac{mc}{2\hbar}\left(i\sigma_3 J_0\left(\frac{m\,c}{\hbar}\sqrt{(ct)^2-x^2}\right) + (ct\mathbb{1} + x\sigma_1)\,\frac{J_1\left(\frac{m\,c}{\hbar}\sqrt{(ct)^2-x^2}\right)}{\sqrt{(ct)^2-x^2}}\right)$$

*with* $J_k$ *denoting the Bessel function of order* $k$*.*

*Proof.* Recall that $t \mapsto e^{-\frac{i}{\hbar}tH_A}$ is a strongly continuous unitary group and, moreover, note that the maps on $H^1(\mathbb{R}\backslash\{y\})$ to $\mathbb{C}$,

$$f \mapsto \lim_{x\to y^\pm} f(x) \equiv f(y^\pm), \qquad f \in H^1(\mathbb{R}\backslash\{y\}),$$

are continuous. Then the map $t \mapsto \underline{q}(t)$, with $\underline{q}(t)$ defined as

$$\underline{q}(t) := \frac{1}{2}(\Psi(y^+,t) + \Psi(y^-,t)), \tag{2.8}$$

is continuous as well.

The relation (2.2) leads us to consider the distributional Cauchy problem

$$\begin{cases} i\hbar\dfrac{d}{dt}\Psi(t) = D_m\Psi(t) + \hbar A\underline{q}(t)\delta_y, \\ \Psi(0) = \Psi_\circ, \end{cases} \tag{2.9}$$

where $\underline{q}(t)$ is defined by (2.8). Then

$$\Psi(t) = \Psi^f(t) + \Psi^\delta(t),$$

where $\Psi^f(t) = e^{-\frac{i}{\hbar}tH}\Psi_\circ$, and $\Psi^\delta(t)$ solves (2.9) with zero initial conditions. By Duhamel's formula,

$$\Psi^\delta(t) = -i\int_0^t ds\, e^{-\frac{i}{\hbar}(t-s)H}Aq(s)\delta_y.$$

Let us notice that, by (A.2), the group of evolution $\exp(-\frac{i}{\hbar}tH)$ continuously maps $\mathcal{S}(\mathbb{R})\otimes\mathbb{C}^2$ in $\mathcal{S}(\mathbb{R})\otimes\mathbb{C}^2$ and so it extends by duality to a group of evolution (which we denote by the same symbol) on $\mathcal{S}'(\mathbb{R})\otimes\mathbb{C}^2$ to $\mathcal{S}'(\mathbb{R})\otimes\mathbb{C}^2$. Using the definition of the unitary group $e^{-\frac{i}{\hbar}tH}$ (see (A.3)), we get

$$\begin{aligned}\Psi^\delta(t) = &-\frac{i}{2}\int_0^t ds\left((\mathbb{1}+\sigma_1)Aq(s)\,\delta_{y+c(t-s)} + (\mathbb{1}-\sigma_1)Aq(s)\,\delta_{y-c(t-s)}\right) \\ &- i\int_0^t ds\int_{-c(t-s)}^{c(t-s)} d\xi\, K(\xi,t-s)Aq(s)\,\delta_{y+\xi}\,.\end{aligned}$$

Exploiting the Dirac delta distributions in the integrals (recalling that $t \geq 0$), we get

$$\begin{aligned}\Psi^\delta(x,t) = &-\frac{i}{2c}\theta\left(t-\frac{|x-y|}{c}\right)\begin{bmatrix}1 & \mathrm{sgn}(x-y) \\ \mathrm{sgn}(x-y) & 1\end{bmatrix}Aq\left(t-\frac{|x-y|}{c}\right) \\ &- i\theta\left(t-\frac{|x-y|}{c}\right)\int_0^{t-\frac{|x-y|}{c}} ds\, K(x-y,t-s)Aq(s).\end{aligned} \tag{2.10}$$

Therefore, we have proved that the solution of the Cauchy problem (2.5) satisfies the identity (2.6) with $\underline{q}(t)$ defined by (2.8).

To prove that $\underline{q}(t)$ satisfies the identity (2.7), we note that, by (2.6), one has

$$\Psi(y_\pm,t) = \Psi^f(y,t) - \frac{i}{2c}(\mathbb{1}\pm\sigma_1)Aq(t) - i\int_0^t ds\, K(0,t-s)Aq(s). \qquad (2.11)$$ □

*Remark* 2.2. Note that from (2.6) one can see that $\Psi(t)$ satisfies the boundary conditions in (2.1) for any $t>0$. Indeed, by (2.11), one has that

$$[\Psi(t)]_y = -\frac{i}{2c}((\mathbb{1}+\sigma_1)-(\mathbb{1}-\sigma_1))A\underline{q}(t) = -\frac{i}{c}\,\sigma_1 A\underline{q}(t).$$

*Remark* 2.3. Despite the presence of the $\theta$-function on the right hand side (r.h.s.) of (2.6), the function $\Psi(x,t)$ is continuous in $x=y\pm ct$. In fact, from formula (A.3), it follows that, for any $t>0$, $\Psi^f(t)$ is discontinuous in $x=y\pm ct$ and

$$[\Psi^f(t)]_{y\pm ct} = \frac{1}{2}[(\mathbb{1}\pm\sigma_1)\Psi_\circ]_y = \frac{1}{2}\begin{pmatrix}[\psi_1^\circ]_y \pm [\psi_2^\circ]_y\\ \pm[\psi_1^\circ]_y + [\psi_2^\circ]_y\end{pmatrix}. \qquad (2.12)$$

On the other hand, since

$$\lim_{x\to(y\pm ct)^\mp}\theta\left(t-\frac{|x-y|}{c}\right)=1;\qquad \lim_{x\to(y\pm ct)^\pm}\theta\left(t-\frac{|x-y|}{c}\right)=0,$$

one has that

$$\left[\theta\left(t-\tfrac{|x-y|}{c}\right)\right]_{x=y\pm ct} = \lim\nolimits_{x\to(y\pm ct)^+}\theta\left(t-\tfrac{|x-y|}{c}\right) - \lim\nolimits_{x\to(y\pm ct)^-}\theta\left(t-\tfrac{|x-y|}{c}\right) = \mp 1.$$

Then, (2.10) gives

(2.13)
$$[\Psi^\delta(t)]_{y\pm ct} = \pm\frac{i}{2c}(\mathbb{1}\pm\sigma_1)Aq(0) = \mp\frac{1}{2}(\mathbb{1}\pm\sigma_1)\begin{pmatrix}[\psi_2^\circ]_y\\ [\psi_1^\circ]_y\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}[\psi_1^\circ]_y \pm [\psi_2^\circ]_y\\ \pm[\psi_1^\circ]_y + [\psi_2^\circ]_y\end{pmatrix},$$

where in the second equality we used the fact that $\Psi_\circ \in D(H_A)$. Equations (2.12) and (2.13) give $[\Psi(t)]_{y\pm ct} = [\Psi^f(t)]_{y\pm ct} + [\Psi^\delta(t)]_{y\pm ct} = 0$.

*Remark* 2.4. From (A.3), it follows that, for any $t>0$, $\Psi^f(y,t)$ is continuous in $t$, and

$$\Psi^f(y,0) \equiv \lim_{t\to 0}\Psi^f(y,t) = \frac{1}{2}\big((\mathbb{1}+\sigma_1)\Psi_\circ(y^-) + (\mathbb{1}-\sigma_1)\Psi_\circ(y^+)\big).$$

This implies that

$$\Psi^f(y,0) - \frac{i}{2c}A\underline{q}(0) = \frac{1}{2}\big((\mathbb{1}+\sigma_1)\Psi_\circ(y^-) + (\mathbb{1}-\sigma_1)\Psi_\circ(y^+)\big) + \frac{1}{2}\sigma_1[\Psi_\circ]_y = \underline{q}(0),$$

which is in agreement with the fact that $\underline{q}(t)$ satisfies (2.7).

**3. The Cauchy problem for the Dirac equation with concentrated nonlinearity.** Now we define a Dirac operator $H_A^{nl}$ with concentrated nonlinearity such that the coupling between the jump and the mean value of the spinor function is given by a nonlinear relation. To this aim we define the nonlinear domain

$$D(H_A^{nl}) = \left\{\Psi \in \mathcal{H} : \Psi \in H^1(\mathbb{R}\backslash\{y\}) \otimes \mathbb{C}^2,\, ic\,\sigma_1[\Psi]_y = A(\underline{q})\underline{q}\right\},$$

where $\mathbb{C}^2 \ni \underline{z} \mapsto A(\underline{z})$ is a matrix-valued function such that $A(\underline{z})$ is self-adjoint for all $\underline{z}$; $H_A^{nl}$ is then defined as the restriction of $S^* = H_- \oplus H_+$ to $D(H_A^{nl})$, so that

$$H_A^{nl} : D(H_A^{nl}) \subset L^2(\mathbb{R}) \otimes \mathbb{C}^2 \to L^2(\mathbb{R}) \otimes \mathbb{C}^2, \quad H_A^{nl}\Psi = D_m\Psi + \hbar A(\underline{q})\underline{q}\,\delta_y. \tag{3.1}$$

*Remark* 3.1. We use the notation $H_A^{nl}$ just for convenience; indeed, the nonlinear operator $H_A^{nl}$ depends on the function $\underline{z} \mapsto A(\underline{z})\underline{z}$ and not only on $A(\underline{z})$: there could be two different matrices $A_1(\underline{z})$ and $A_2(\underline{z})$ such that $A_1(\underline{z})\underline{z} = A_2(\underline{z})\underline{z}$. Clearly, $H_A^{nl} = H_A$ whenever $A$ is $\underline{z}$-independent.

In order to solve the nonlinear Cauchy problem

$$\begin{cases} i\hbar\dfrac{d}{dt}\Psi(t) = H_A^{nl}\Psi(t), \\ \Psi(0) = \Psi_\circ, \end{cases} \tag{3.2}$$

we first of all write an equivalent formulation mimicking the Dirac flow in the representation formula given in Proposition 2.1.

Take $\Psi_\circ \in D(H_A^{nl})$ and set $\Psi^f(t) = e^{-\frac{i}{\hbar}tH}\Psi_\circ$. For $t \geq 0$, the nonlinear Dirac flow $U_t^{nl}$ is defined by $U_t^{nl}\Psi_\circ := \Psi(t)$, where

(3.3)

$$\begin{aligned}\Psi(x,t) := \Psi^f(x,t) - \frac{i}{2c}\theta\!\left(t - \frac{|x-y|}{c}\right)\begin{bmatrix} 1 & \operatorname{sgn}(x-y) \\ \operatorname{sgn}(x-y) & 1\end{bmatrix}(A(\underline{q})\underline{q})\left(t - \frac{|x-y|}{c}\right) \\ - i\theta\left(t - \frac{|x-y|}{c}\right)\int_0^{t-\frac{|x-y|}{c}} ds\, K(x-y, t-s)(A(\underline{q})\underline{q})(s),\end{aligned}$$

where $\underline{q}(t)$ is the solution of the nonlinear integral equation

$$\underline{q}(t) = \Psi^f(y,t) - \frac{i}{2c}\,(A(\underline{q})\underline{q})(t) - i\int_0^t ds\, K(0, t-s)(A(\underline{q})\underline{q})(s), \tag{3.4}$$

and $(A(\underline{q})\underline{q})(t)$ is shorthand notation for $A(\underline{q}(t))\underline{q}(t)$.

The first step towards proving the well-posedness of problem (3.2) is to show that, for any $T > 0$, (3.4) admits a unique (sufficiently regular) solution for $t \in [0,T]$. This is achieved in Lemma 3.4 below.

In the proof of Lemma 3.4, we need the map

$$F_A : \mathbb{C}^2 \to \mathbb{C}^2, \quad F_A(\underline{z}) := \underline{z} + \frac{i}{2c}\,A(\underline{z})\underline{z}$$

to be locally bi-Lipschitz continuous. Therefore, we make the following assumption on the matrix-valued function $A(\underline{z})$.

ASSUMPTION 3.2. *The map $\underline{z} \mapsto A(\underline{z})$ from $\mathbb{C}^2$ to the space of $2 \times 2$ self-adjoint matrices is such that $F_A$ is a $C^1$-diffeomorphism as a map from $\mathbb{R}^4$ to itself.*

By Hadamard's global inverse function theorem (see, e.g., [21, 22] and references therein), a $C^1$ map $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ is a $C^1$-diffeomorphism if and only if its Jacobian determinant never vanishes and $\|\Phi(\underline{x})\| \to +\infty$ as $\|\underline{x}\| \to \infty$. Since the complex map $F_A$ can be equivalently seen as a map from $\mathbb{R}^4$ to $\mathbb{R}^4$, such a global inverse function theorem applies to $F_A$ as well. By $|F_A(\underline{z})|^2 = |\underline{z}|^2 + |A(\underline{z})\underline{z}|^2/(4c^2)$, it follows that $|F_A(\underline{z})| \to +\infty$ whenever $|\underline{z}| \to +\infty$. Hence, Assumption 3.2 is equivalent to the following assumption.

ASSUMPTION 3.3. *The map $\underline{z} \mapsto A(\underline{z})$ from $\mathbb{C}^2$ to the space of $2\times 2$ self-adjoint matrices is such that $F_A$ is $C^1(\mathbb{R}^4, \mathbb{R}^4)$ and its Jacobian determinant never vanishes.*

We are now ready to state our first results that concern the well-posedness of the equation for $\underline{q}(t)$.

LEMMA 3.4. *Let $A(\underline{z})$ be such that Assumption 3.2 is satisfied. Then, for any $\Psi_\circ \in D(H_A^{nl})$ and $T > 0$, there exists a unique solution $\underline{q} \in H^1(0,T) \otimes \mathbb{C}^2$ of (3.4).*

*Proof.* At first we prove that there exists a unique solution $\underline{q} \in C[0,T] \otimes \mathbb{C}^2$. We equivalently show that, for fixed $T > 0$ and $\Psi_\circ \in D(H_A^{nl})$, there exists $\bar{t} > 0$ for which the following holds true: for all $k \geq 0$ such that $k\bar{t} \leq T$, suppose that there exists a unique solution $\underline{q}^k \in C[0, k\bar{t}\,] \otimes \mathbb{C}^2$ of (3.4) (no assumption in the case $k = 0$); then (3.4) has an unique solution $\underline{q}^{k+1} \in C[0,(k+1)\bar{t}\,] \otimes \mathbb{C}^2$.

To begin with, we show that if $\underline{q}(t)$ solves (3.4) for $t \in [0,T]$, then there exists a positive constant $C_1$ such that

$$\sup_{t\in[0,T]} |(A(\underline{q})\underline{q})(t)| \leq C_1. \tag{3.5}$$

To prove this claim recall that, by Remark 2.4, $\sup_{t\in[0,T]} |\Psi^f(y,t)| \leq C$, and that $\sup_{t\in[0,T]} |K(0,t)| \leq C$ for some positive constant $C$. Hence, by (3.4), and using the fact that $A(\underline{z})$ is self-adjoint, we get that for all $t \in [0,T]$ the following inequality holds true:

$$|(A(\underline{q})\underline{q})(t)| \leq 2c \left| \left( \mathbb{1} + \frac{i}{2c} A(\underline{q}(t)) \right) \underline{q}(t) \right| \leq C \left( 1 + \int_0^t ds\, |(A(\underline{q})\underline{q})(s)| \right).$$

Then the bound (3.5) follows by Grönwall's inequality.

Next, let us pose $t_k := k\bar{t}$. For $t \in [t_k, t_k + \bar{t}\,]$, the solution of (3.4) satisfies the identity

$$\left( \mathbb{1} + \frac{i}{2c} A(\underline{q}(t)) \right) \underline{q}(t) = \Psi^f(y,t) - i \int_0^{t_k} ds\, K(0,t-s)(A(\underline{q}^k)\underline{q}^k)(s) - i \int_{t_k}^t ds\, K(0,t-s)(A(\underline{q})\underline{q})(s). \tag{3.6}$$

We set

$$f_k(t) = \Psi^f(y,t) - i \int_0^{t_k} ds\, K(0,t-s)(A(\underline{q}^k)\underline{q}^k)(s)$$

and

$$I_k(\underline{q})(t) = -i \int_{t_k}^t ds\, K(0,t-s)(A(\underline{q})\underline{q})(s),$$

and rewrite (3.6) as

$$\underline{q}(t) = F_A^{-1}(f_k(t) + I_k(\underline{q})(t)),$$

where, by Assumption 3.2, $F_A^{-1}$ exists and is a $C^1$ map from $\mathbb{C}^2$ to $\mathbb{C}^2$.

Since $\Psi^f(y,t)$ and $K(0,t)$ are bounded, and by the bound (3.5), it follows that

$$\sup_{t\in[0,T]} |f_k(t)| \le C(1 + TC_1) \equiv R_1.$$

Let $B_k(R) := \{\underline{g} \in C[t_k, t_k + \bar{t}\,] \otimes \mathbb{C}^2 : \sup_{t\in[t_k,t_k+\bar{t}\,]} |\underline{g}(t)| \le R\}$.

For any $\underline{g} \in B_k(2R_1)$ and $\bar{t} \le t_1 = R_1(C \sup_{|\underline{z}|\le 2R_1} |A(\underline{z})\underline{z}|)^{-1}$ independent on $k$, we have that

$$\sup_{t\in[t_k,t_k+\bar{t}\,]} |f_k(t) + I_k(\underline{g})(t)| \le R_1 + \bar{t}C \sup_{|\underline{z}|\le 2R_1} |A(\underline{z})\underline{z}| \le 2R_1.$$

Define the map

$$G_k(\underline{g}) := F_A^{-1}(f_k + I_k(\underline{g})).$$

The map $G_k$ is continuous in $B_k(2R_1)$. The self-adjointness of $A(\underline{z})$ implies that $|F_A^{-1}(\underline{z})| \le |\underline{z}|$ hence for $\bar{t} \le t_1$ one has that

$$\sup_{t\in[t_k,t_k+\bar{t}\,]} |G_k(\underline{g})(t)| \le \sup_{t\in[t_k,t_k+\bar{t}\,]} |f_k(t) + I_k(\underline{g})(t)| \le 2R_1,$$

which means that $G_k$ maps $B_k(2R_1)$ into itself.

By Assumption 3.2, we have that the maps $F_A^{-1}(\underline{z})$ and $A(\underline{z})\underline{z} = i2c(\underline{z} - F_A(\underline{z}))$ are locally Lipschitz. More precisely, for any $\underline{z}_1$ and $\underline{z}_2$ such that $|\underline{z}_1|, |\underline{z}_2| \le R$, there exist two constants $\kappa_F(R)$ and $\kappa_A(R)$ such that

$$|F_A^{-1}(\underline{z}_1) - F_A^{-1}(\underline{z}_2)| \le \kappa_F(R)|\underline{z}_1 - \underline{z}_2| \quad \text{and} \quad |A(\underline{z}_1)\underline{z}_1 - A(\underline{z}_2)\underline{z}_2| \le \kappa_A(R)|\underline{z}_1 - \underline{z}_2|.$$

Take $\bar{t} \le t_1$. For any $\underline{g}_1, \underline{g}_2 \in B_k(2R_1)$, one has that $\sup_{t\in[t_k,t_k+\bar{t}\,]} |f_k(t) + I_k(\underline{g}_j)(t)| \le 2R_1$ and

$$\begin{aligned}\sup_{t\in[t_k,t_k+\bar{t}\,]} |G_k(\underline{g}_1(t)) - G_k(\underline{g}_2(t))| &\le \kappa_F(2R_1) \sup_{t\in[t_k,t_k+\bar{t}\,]} |I_k(\underline{g}_1)(t) - I_k(\underline{g}_2)(t)| \\ &\le \kappa_F(2R_1)\kappa_A(2R_1)C\bar{t} \sup_{t\in[t_k,t_k+\bar{t}\,]} |\underline{g}_1(t) - \underline{g}_2(t)|.\end{aligned}$$

Set $t_2 = (2\kappa_F(2R_1)\kappa_A(2R_1)C)^{-1}$ independent on $k$, and $\bar{t} = \min\{t_1, t_2\}$; then the map $G_k$ is a contraction in $B_k(2R_1)$. By the Banach–Caccioppoli fixed point theorem, this implies that there exists a unique solution $\underline{q}^*(t) \in B_k(2R_1)$ of (3.6).

By construction, the function $\underline{q}^{k+1}(t)$ which is equal to $\underline{q}^k(t)$ for $t \in [0, t_k]$, and to $\underline{q}^*(t)$ for $t \in [t_k, t_k + \bar{t}\,]$, is indeed in $C[0, t_k + \bar{t}\,] \otimes \mathbb{C}^2$ and solves (3.4) for $t \in [0, t_k + \bar{t}\,]$.

By

$$\underline{q}(t) = F_A^{-1}\left(\Psi^f(y,t) - I(t)\right), \quad I(t) := i\int_0^t ds\, K(0, t-s)(A(\underline{q})\underline{q})(s),$$

since $\Psi^f(y,\cdot) \in H^1(0,T) \otimes \mathbb{C}^2$ (see Proposition A.1 in Appendix A), $I \in C^1[0,T] \otimes \mathbb{C}^2$, and $F_A^{-1}$ is Lipschitz continuous. In conclusion, $\underline{q} \in H^1(0,T) \otimes \mathbb{C}^2$. ∎

Following the previous results we can prove the global well-posedness of the Cauchy problem (3.2).

THEOREM 3.5. *Let $A(\underline{z})$ be such that Assumption* 3.2 *is satisfied. Then, for any* $\Psi_\circ \in D(H_A^{nl})$, *the formulae* (3.3) *and* (3.4) *provide the unique, global-in-time solution* $\Psi(t)$ *of the Cauchy problem* (3.2); *more precisely,* $\Psi \in C^1(\mathbb{R}_+, L^2(\mathbb{R}) \otimes \mathbb{C}^2)$, $\Psi(t) \in D(H_A^{nl})$, *and* (3.2) *holds for any* $t \geq 0$.

*Proof.* By (3.1) and by the same reasonings as in the linear case provided in section 2 (replacing $A\underline{q}$ with $A(\underline{q})\underline{q}$), one has that $\Psi(t)$ given in formula (3.3) solves the distributional Cauchy problem

$$\begin{cases} i\hbar \dfrac{d}{dt}\Psi(t) = D_m\Psi(t) + \hbar(A(\underline{q})\underline{q})(t)\,\delta_y, \\ \Psi(0) = \Psi_\circ, \end{cases}$$

and, since $\underline{q}(t)$ solves (3.4), one gets $\Psi(t) \in D(H_A^{nl})$ for any $t \geq 0$. Therefore, to conclude the proof we need to show that the map $t \mapsto \Psi(t)$ belongs to $C^1(\mathbb{R}_+, L^2(\mathbb{R}) \otimes \mathbb{C}^2)$. Since $\Psi(t) \in D(H_A^{nl}) \subset H^1(\mathbb{R}\backslash\{y\})$, we have the decomposition (in the following we suppose $m > 0$; similar considerations hold in the $m = 0$ case)

$$\Psi(t) = \Phi(t) + G\underline{\xi}(t), \quad \Phi(t) \in H^1(\mathbb{R}) \otimes \mathbb{C}^2, \quad \underline{\xi}(t) = i\hbar c\,\sigma_1[\Psi(t)]_y. \tag{3.7}$$

Let us notice that, since

$$\underline{\xi}(t) = \hbar(A(\underline{q})\underline{q})(t) \tag{3.8}$$

and $\underline{z} \mapsto A(\underline{z})\underline{z}$ is Lipschitz continuous, $t \mapsto \underline{\xi}(t)$ belongs to $H^1(0,T) \otimes \mathbb{C}^2$ for any $T > 0$ by Lemma 3.4. Moreover, since

$$H_A^{nl}\Psi = D_m(\Phi + G\underline{\xi}(t)) + \underline{\xi}(t)\,\delta_y = H\Phi, \tag{3.9}$$

one has that $\Phi(t)$ solves the Cauchy problem

$$\begin{cases} i\hbar \dfrac{d}{dt}\Phi(t) = H\Phi(t) - i\hbar\, G\dot{\underline{\xi}}(t), \\ \Phi(0) = \Phi_\circ \end{cases} \tag{3.10}$$

with $\Phi_\circ := \Psi_\circ - G\underline{\xi}(0) \in H^1(\mathbb{R}) \otimes \mathbb{C}^2$. Therefore,

$$\Psi(t) = e^{-\frac{i}{\hbar}tH}\Phi_\circ - \int_0^t ds\, e^{-\frac{i}{\hbar}(t-s)H} G\dot{\underline{\xi}}(s) + G\underline{\xi}(t).$$

Since $t \mapsto e^{-\frac{i}{\hbar}tH}\Phi_\circ$ belongs to $C^1(\mathbb{R}_+, L^2(\mathbb{R}) \otimes \mathbb{C}^2)$ and $-HG\dot{\underline{\xi}} = \dot{\underline{\xi}}\delta_y$, to conclude we need to show that the map

$$t \mapsto \Upsilon(t) := \frac{d}{dt}\left(\int_0^t ds\, e^{-\frac{i}{\hbar}(t-s)H} G\dot{\underline{\xi}}(s) - G\underline{\xi}(t)\right) = \frac{i}{\hbar}\int_0^t ds\, e^{-\frac{i}{\hbar}(t-s)H}\dot{\underline{\xi}}(s)\delta_y$$

belongs to $C(\mathbb{R}_+, L^2(\mathbb{R}) \otimes \mathbb{C}^2)$. By the same calculations that led to (2.10), one gets

$$\Upsilon(t) = \frac{i}{\hbar}\left(\frac{1}{2c}\,\Upsilon_1(t) + \Upsilon_2(t)\right),$$

where

$$\Upsilon_1(x,t) = \theta\left(t - \frac{|x-y|}{c}\right)\begin{bmatrix} 1 & \operatorname{sgn}(x-y) \\ \operatorname{sgn}(x-y) & 1 \end{bmatrix} \underline{\dot{\xi}}\left(t - \frac{|x-y|}{c}\right),$$
$$\Upsilon_2(x,t) = \theta\left(t - \frac{|x-y|}{c}\right)\int_0^{t-\frac{|x-y|}{c}} d\tau\, K(x-y,t-\tau)\underline{\dot{\xi}}(\tau).$$

Let $\kappa$ denote the bound for the kernel $K$:

$$\kappa := \max_{i,j=1,2}\ \sup_{0<t<t_\circ, |x|<ct} |K_{i,j}(x,t)|.$$

One has (supposing $0 \le s < t \le t_\circ$, the same kind of reasonings hold in the case $0 \le t < s \le t_\circ$)

$$\begin{aligned}\|\Upsilon_1(t) - \Upsilon_1(s)\|^2 &\le C\left(\int_0^{cs} dx\, \left|\underline{\dot{\xi}}\left(t - \frac{x}{c}\right) - \underline{\dot{\xi}}\left(s - \frac{x}{c}\right)\right|^2 + \int_{cs}^{ct} dx\, \left|\underline{\dot{\xi}}\left(t - \frac{x}{c}\right)\right|^2\right) \\ &\le C\left(\int_0^{t_\circ} dx\, \left|\underline{\dot{\xi}}\,(t-s+x) - \underline{\dot{\xi}}\,(x)\right|^2 + \int_0^{t-s} dx\, \left|\underline{\dot{\xi}}(x)\right|^2\right)\end{aligned}$$

and

$$\begin{aligned}&\|\Upsilon_2(t) - \Upsilon_2(s)\|^2 \\ &\quad\le 2\int_{|x|\le cs} dx \left|\int_0^{t-|x|/c} d\tau\, K(x,t-\tau)\underline{\dot{\xi}}(\tau) - \int_0^{s-|x|/c} d\tau\, K(x,s-\tau)\underline{\dot{\xi}}(\tau)\right|^2 \\ &\qquad + 2\int_{cs\le|x|\le ct} dx \left|\int_0^{t-|x|/c} d\tau\, K(x,t-\tau)\underline{\dot{\xi}}(\tau)\right|^2 \\ &\quad\le 4\int_{|x|\le cs} dx \left(\int_{|x|/c}^{s} d\tau\, \left|K(x,\tau)\left(\underline{\dot{\xi}}(t-\tau) - \underline{\dot{\xi}}(s-\tau)\right)\right|\right)^2 \\ &\qquad + 4\int_{|x|\le cs} dx \left(\int_s^t d\tau\, \left|K(x,\tau)\underline{\dot{\xi}}(t-\tau)\right|\right)^2 \\ &\qquad + 2\int_{cs\le|x|\le ct} dx \left(\int_0^{t-|x|/c} d\tau\, \left|K(x,t-\tau)\underline{\dot{\xi}}(\tau)\right|\right)^2 \\ &\quad\le 16\kappa^2 \int_{|x|\le cs} dx \left(\int_{|x|/c}^{s} d\tau \left|\underline{\dot{\xi}}(t-\tau) - \underline{\dot{\xi}}(s-\tau)\right|\right)^2 \\ &\qquad + 16\kappa^2\int_{|x|\le cs} dx\left(\int_s^t d\tau\left|\underline{\dot{\xi}}(t-\tau)\right|\right)^2 + 8\kappa^2\int_{cs\le|x|\le ct} dx\left(\int_0^{t-|x|/c} d\tau\, \left|\underline{\dot{\xi}}(\tau)\right|\right)^2 \\ &\quad\le C\int_0^s d\tau\, \left|\underline{\dot{\xi}}(t-\tau) - \underline{\dot{\xi}}(s-\tau)\right|^2 + C(t-s)\|\underline{\dot{\xi}}\|^2_{H^1(0,t_\circ)} \\ &\quad\le C\int_0^{t_\circ} d\tau\left|\underline{\dot{\xi}}(t-s+\tau) - \underline{\dot{\xi}}(\tau)\right|^2 + C(t-s)\|\underline{\dot{\xi}}\|^2_{H^1(0,t_\circ)}.\end{aligned}$$

Since

$$\lim_{s\to t}\int_0^{t_\circ} dx\,\left|\dot{\underline{\xi}}\,(t-s+x)-\dot{\underline{\xi}}\,(x)\right|^2=0$$

(by $\underline{\xi}\in H^1(0,T)$ and by the continuity of the shift operator; see. e.g., [37, page 11]) and

$$\|\Upsilon(t)-\Upsilon(s)\|\le C(\|\Upsilon_1(t)-\Upsilon_1(s)\|+\|\Upsilon_2(t)-\Upsilon_2(s)\|),$$

we conclude

$$\lim_{s\to t}\|\Upsilon(t)-\Upsilon(s)\|=0. \qquad \square$$

*Remark* 3.6. Let us define, as in the proof of Theorem 3.5, the map $t\to\Phi(t)=\Psi(t)-G\underline{\xi}(t)$, which, according to (3.7), gives the time evolution of the $H^1(\mathbb{R})\otimes\mathbb{C}^2$ component of the solution. Since, by (3.10), $i\hbar\frac{d}{dt}\Psi(t)=H\Phi(t)$, and we proved that $\Psi\in C^1(\mathbb{R}_+,L^2(\mathbb{R})\otimes\mathbb{C}^2)$, one has that $H\Phi\in C^0(\mathbb{R}_+,L^2(\mathbb{R})\otimes\mathbb{C}^2)$. Therefore, $\Phi\in C^0(\mathbb{R}_+,H^1(\mathbb{R})\otimes\mathbb{C}^2)$.

*Remark* 3.7. The Dirac differential operator $D_m$ has many different equivalent representations: given any unitary map $U:\mathbb{C}^2\to\mathbb{C}^2$, one defines an equivalent Dirac operator by $\tilde{D}_m:=(\mathbb{1}\otimes U^*)D_m(\mathbb{1}\otimes U)$, i.e.,

$$\tilde{D}_m\Psi=-i\hbar\,c\,\tilde{\sigma}_1\frac{d\Psi}{dx}+m\,c^2\tilde{\sigma}_3\Psi,\qquad \tilde{\sigma}_k:=U^*\sigma_kU.$$

The relation between the corresponding nonlinear operators is

$$(\mathbb{1}\otimes U^*)H_A^{nl}(\mathbb{1}\otimes U)\Psi=\tilde{H}_{\tilde{A}}^{nl}\Psi:=\tilde{D}_m\Psi+\hbar\tilde{A}(\underline{q})\underline{q}\,\delta_y,\qquad \tilde{A}(\underline{z}):=U^*A(U\underline{z})U.$$

Since

$$F_{\tilde{A}}(\underline{z})=U^*\left(\mathbb{1}+\frac{i}{2c}\,A(U\underline{z})\right)U\underline{z}=U^*F_A(U\underline{z}),$$

$F_A$ satisfies Assumption 3.2 if and only if $F_{\tilde{A}}$ does. This shows that our global well-posedness result holds in any representation and Assumption 3.2 is an invariant one.

THEOREM 3.8 (mass conservation). *Let $\Psi_\circ\in D(H_A^{nl})$; then the $L^2$-norm is conserved along the flow associated to the Cauchy problem* (3.2).

*Proof.* We take the derivative

$$\frac{d}{dt}\|\Psi(t)\|^2=2\,\mathrm{Re}\left\langle\dot{\Psi}(t),\Psi(t)\right\rangle.$$

We write $\Psi(t)$ as in (3.7) and use (3.10) to get

$$\left\langle\dot{\Psi}(t),\Psi(t)\right\rangle=\frac{i}{\hbar}\left\langle H\Phi(t),\Phi(t)\right\rangle+\frac{i}{\hbar}\left\langle H\Phi(t),G\underline{\xi}(t)\right\rangle. \tag{3.11}$$

Since $-D_mG=\delta_y\otimes\mathbb{1}$, we have that

$$\begin{aligned}\frac{i}{\hbar}\left\langle H\Phi(t),G\underline{\xi}(t)\right\rangle&=-\frac{i}{\hbar}\left\langle\Phi(y,t),\underline{\xi}(t)\right\rangle_{\mathbb{C}^2}\\&=-i\left\langle\underline{q}(t),\left(\left(A(\underline{q})+\frac{A(\underline{q})\sigma_3A(\underline{q})}{2c}\right)\underline{q}\right)(t)\right\rangle_{\mathbb{C}^2},\end{aligned} \tag{3.12}$$

where we used (3.8), the boundary condition in (2.4), and the fact that $A$ is self-adjoint. Using the latter identity in (3.11), and noticing that $\operatorname{Im}\langle H\Phi, \Phi\rangle = 0$ and $\operatorname{Im}\left\langle \underline{q}, \left(A + \frac{A\sigma_3 A}{2c}\right)\underline{q}\right\rangle_{\mathbb{C}^2} = 0$, we conclude that $\operatorname{Re}\langle\dot{\Psi}(t), \Psi(t)\rangle = 0$, which in turn implies that the $L^2$-norm is conserved. ∎

To state the conservation of the energy, we look at $H_A^{nl}$ as a Hamiltonian vector field with respect to the pair of canonical coordinates $\Psi$ and $\overline{\Psi}$. For this reason, in the next theorem we use the notation $A(\underline{q}) = \mathcal{A}(\underline{\bar{q}}, \underline{q})$.

THEOREM 3.9 (energy conservation). *Assume that* $\mathcal{A}(\underline{\bar{q}}, \underline{q}) = \mathcal{A}(\underline{q}, \underline{\bar{q}})$, *and let* $\Psi_\circ \in D(H_{\mathcal{A}}^{nl})$. *Then the energy*

$$E(\Psi) = \left\langle \Psi, H_{\mathcal{A}}^{nl}\Psi\right\rangle - \hbar\left\langle \underline{q}, \mathcal{A}(\underline{\bar{q}}, \underline{q})\underline{q}\right\rangle_{\mathbb{C}^2} + \hbar W(\underline{\bar{q}}, \underline{q}), \tag{3.13}$$

*where* $W : \mathbb{C}^4 \to \mathbb{R}$ *is such that* $W(\underline{\bar{q}}, \underline{q}) = W(\underline{q}, \underline{\bar{q}})$, *and*

$$\nabla_{\underline{\bar{q}}} W(\underline{\bar{q}}, \underline{q}) = \mathcal{A}(\underline{\bar{q}}, \underline{q})\underline{q} \tag{3.14}$$

*is conserved along the flow associated to the Cauchy problem* (3.2).

*Proof.* As a first step, we rewrite the energy functional in a different form. Recall that $\Psi \in D(H_{\mathcal{A}}^{nl})$ can be decomposed as in (3.7). By (3.9) it follows that

$$\left\langle \Psi, H_{\mathcal{A}}^{nl}\Psi\right\rangle = \langle\Phi, H\Phi\rangle + \left\langle G\underline{\xi}, H\Phi\right\rangle.$$

Repeating the calculations in (3.12), one obtains

$$\left\langle G\underline{\xi}, H\Phi\right\rangle = -\left\langle \Phi(y), \underline{\xi}\right\rangle_{\mathbb{C}^2} = -\hbar\left\langle \underline{q}, \left(\mathcal{A}(\underline{\bar{q}}, \underline{q}) + \frac{\mathcal{A}(\underline{\bar{q}}, \underline{q})\sigma_3\mathcal{A}(\underline{\bar{q}}, \underline{q})}{2c}\right)\underline{q}\right\rangle_{\mathbb{C}^2}.$$

Hence, for any state $\Psi \in D(H_{\mathcal{A}}^{nl})$, the energy functional can be written as

$$E(\Psi) = \langle\Phi, H\Phi\rangle - 2\hbar\left\langle \underline{q}, \left(\mathcal{A}(\underline{\bar{q}}, \underline{q}) + \frac{\mathcal{A}(\underline{\bar{q}}, \underline{q})\sigma_3\mathcal{A}(\underline{\bar{q}}, \underline{q})}{4c}\right)\underline{q}\right\rangle_{\mathbb{C}^2} + \hbar W(\underline{\bar{q}}, \underline{q}).$$

Next, we compute the time derivative of the $E(\Psi(t))$ when $\Psi(t)$ is the solution of the Cauchy problem (3.2). By using again the decomposition in (3.7), we have that

$$\begin{aligned} \frac{d}{dt}\langle\Phi(t), H\Phi(t)\rangle &= \lim_{s\to 0}\frac{1}{s}\left(\langle\Phi(t+s), H\Phi(t+s)\rangle - \langle\Phi(t), H\Phi(t)\rangle\right) \\ &= \lim_{s\to 0}\frac{1}{s}\left(\langle\Phi(t+s), H\Phi(t+s)\rangle - \langle\Phi(t), H\Phi(t+s)\rangle\right. \\ &\qquad \left. + \langle\Phi(t), H\Phi(t+s)\rangle - \langle\Phi(t), H\Phi(t)\rangle\right) \\ &= 2\operatorname{Re}\langle\dot{\Phi}(t), H\Phi(t)\rangle, \end{aligned} \tag{3.15}$$

where we used the fact that $\Phi(t)$ is in $D(H)$ for all $t \geq 0$ and is a continuous function of $t$, and that $H$ is self-adjoint.

In what follows, to shorten the notation, we do not make explicit the dependence of functions on $t$. We have that

$$2\,\mathrm{Re}\langle\dot\Phi, H\Phi\rangle = 2\,\mathrm{Re}\left\langle \dot{\underline{\xi}}, \Phi(y)\right\rangle_{\mathbb{C}^2},$$

where we used the fact that $\dot\Phi$ satisfies the equation in (3.10), and that $-D_m G = \delta_y \otimes \mathbb{1}$. From the relations (3.8) and (2.3), we have that

$$\Phi(y) = \underline{q} + \frac{\sigma_3 \xi}{2\hbar c} = \left(\mathbb{1} + \frac{\sigma_3 \mathcal{A}(\underline{\bar q}, \underline{q})}{2c}\right)\underline{q}.$$

Hence,

$$\begin{aligned}\frac{d}{dt}\langle\Phi, H\Phi\rangle &= 2\hbar\,\mathrm{Re}\left\langle \frac{d}{dt}\left(\mathcal{A}(\underline{\bar q}, \underline{q})\underline{q}\right), \left(\mathbb{1} + \frac{\sigma_3\mathcal{A}(\underline{\bar q}, \underline{q})}{2c}\right)\underline{q}\right\rangle_{\mathbb{C}^2}\\ &= 2\hbar\left\langle \underline{q}, \dot{\mathcal{A}}(\underline{\bar q}, \underline{q})\underline{q}\right\rangle_{\mathbb{C}^2} + 2\hbar\,\mathrm{Re}\left\langle \dot{\underline{q}}, \mathcal{A}(\underline{\bar q}, \underline{q})\underline{q}\right\rangle_{\mathbb{C}^2}\\ &\quad + \frac{\hbar}{c}\mathrm{Re}\left\langle \underline{q}, \dot{\mathcal{A}}(\underline{\bar q}, \underline{q})\sigma_3\mathcal{A}(\underline{\bar q}, \underline{q})\underline{q}\right\rangle_{\mathbb{C}^2}\\ &\quad + \frac{\hbar}{c}\mathrm{Re}\left\langle \dot{\underline{q}}, \mathcal{A}(\underline{\bar q}, \underline{q})\sigma_3\mathcal{A}(\underline{\bar q}, \underline{q})\underline{q}\right\rangle_{\mathbb{C}^2},\end{aligned}$$

where we used the fact that $\mathcal{A}(\underline{\bar q}, \underline{q})$ and $\dot{\mathcal{A}}(\underline{\bar q}, \underline{q})$ are self-adjoint. We note that

$$\begin{aligned}\left\langle \underline{q}, \dot{\mathcal{A}}(\underline{\bar q}, \underline{q})\underline{q}\right\rangle_{\mathbb{C}^2} &= \left\langle \underline{q}, \left(\dot{\underline{\bar q}}\cdot\nabla_{\underline{\bar q}}\mathcal{A}(\underline{\bar q}, \underline{q})\right)\underline{q}\right\rangle_{\mathbb{C}^2} + \left\langle \underline{q}, \left(\dot{\underline{q}}\cdot\nabla_{\underline{q}}\mathcal{A}(\underline{\bar q}, \underline{q})\right)\underline{q}\right\rangle_{\mathbb{C}^2}\\ &= 2\,\mathrm{Re}\left\langle \underline{q}, \left(\dot{\underline{\bar q}}\cdot\nabla_{\underline{\bar q}}\mathcal{A}(\underline{\bar q}, \underline{q})\right)\underline{q}\right\rangle_{\mathbb{C}^2},\end{aligned}$$

where we used $\overline{\nabla_q \mathcal{A}_{i,j}(\underline{\bar q}, \underline{q})} = \nabla_{\underline{\bar q}}\mathcal{A}_{j,i}(\underline{\bar q}, \underline{q})$, which is a consequence of the assumption $\mathcal{A}(\underline{\bar q}, \underline{q}) = \mathcal{A}(\underline{q}, \underline{\bar q})$ and of the fact that $\mathcal{A}$ is self-adjoint. From the latter identity, it follows that

$$\left\langle \underline{q}, \dot{\mathcal{A}}(\underline{\bar q}, \underline{q})\underline{q}\right\rangle_{\mathbb{C}^2} = 2\,\mathrm{Re}\left(\left\langle \dot{\underline{q}}, \nabla_{\underline{\bar q}}\langle \underline{q}, \mathcal{A}(\underline{\bar q}, \underline{q})\underline{q}\rangle_{\mathbb{C}^2}\right\rangle_{\mathbb{C}^2} - \left\langle \dot{\underline{q}}, \mathcal{A}(\underline{\bar q}, \underline{q})\underline{q}\right\rangle_{\mathbb{C}^2}\right).$$

In a similar way, by using the identity $\overline{\langle \underline{q}, (\nabla_{\underline{q}}\mathcal{A})\sigma_3\mathcal{A}q\rangle_{\mathbb{C}^2}} = \langle \underline{q}, \mathcal{A}\sigma_3(\nabla_{\underline{\bar q}}\mathcal{A})q\rangle_{\mathbb{C}^2}$, one can prove that

$$\begin{aligned}\mathrm{Re}\left\langle \underline{q}, \dot{\mathcal{A}}(\underline{\bar q}, \underline{q})\sigma_3\mathcal{A}(\underline{\bar q}, \underline{q})\underline{q}\right\rangle_{\mathbb{C}^2} &= \mathrm{Re}\left(\left\langle \dot{\underline{q}}, \nabla_{\underline{\bar q}}\langle \underline{q}, \mathcal{A}(\underline{\bar q}, \underline{q})\sigma_3\mathcal{A}(\underline{\bar q}, \underline{q})\underline{q}\rangle_{\mathbb{C}^2}\right\rangle_{\mathbb{C}^2}\right.\\ &\quad \left. - \langle \dot{\underline{q}}, \mathcal{A}(\underline{\bar q}, \underline{q})\sigma_3\mathcal{A}(\underline{\bar q}, \underline{q})\underline{q}\rangle_{\mathbb{C}^2}\right).\end{aligned} \tag{3.16}$$

Hence,

$$\begin{aligned}\frac{d}{dt}\langle\Phi, H\Phi\rangle &= 4\hbar\,\mathrm{Re}\left\langle \dot{\underline{q}}, \nabla_{\underline{\bar q}}\left\langle \underline{q}, \left(\mathcal{A}(\underline{\bar q}, \underline{q}) + \frac{\mathcal{A}(\underline{\bar q}, \underline{q})\sigma_3\mathcal{A}(\underline{\bar q}, \underline{q})}{4c}\right)\underline{q}\right\rangle_{\mathbb{C}^2}\right\rangle_{\mathbb{C}^2}\\ &\quad - 2\hbar\,\mathrm{Re}\left\langle \dot{\underline{q}}, \mathcal{A}(\underline{\bar q}, \underline{q})\underline{q}\right\rangle_{\mathbb{C}^2}.\end{aligned} \tag{3.17}$$

Taking into account the fact that

$$\begin{aligned}&2\hbar\frac{d}{dt}\left\langle \underline{q}, \left(\mathcal{A}(\underline{\bar q}, \underline{q}) + \frac{\mathcal{A}(\underline{\bar q}, \underline{q})\sigma_3\mathcal{A}(\underline{\bar q}, \underline{q})}{4c}\right)\underline{q}\right\rangle_{\mathbb{C}^2}\\ &\quad = 4\hbar\,\mathrm{Re}\left\langle \dot{\underline{q}}, \nabla_{\underline{\bar q}}\left\langle \underline{q}, \left(\mathcal{A}(\underline{\bar q}, \underline{q}) + \frac{\mathcal{A}(\underline{\bar q}, \underline{q})\sigma_3\mathcal{A}(\underline{\bar q}, \underline{q})}{4c}\right)\underline{q}\right\rangle_{\mathbb{C}^2}\right\rangle_{\mathbb{C}^2},\end{aligned}$$

and that

$$\frac{d}{dt}W(\underline{\bar{q}},\underline{q}) = 2\,\mathrm{Re}\langle \underline{\dot{q}}, \nabla_{\bar{q}} W(\underline{\bar{q}},\underline{q})\rangle_{\mathbb{C}^2},$$

together with (3.13) and (3.14), it follows that $\frac{d}{dt}E[\Psi]=0$. ∎

**4. Examples.** Given $M:\mathbb{C}^2\to\mathbb{C}^2$ self-adjoint, let $\phi(\underline{z}) := \langle \underline{z}, M\underline{z}\rangle_{\mathbb{C}^2}$ be the corresponding quadratic form. We suppose that

$$A(\underline{z}) = A_\circ(\phi(\underline{z}))$$

with $A_\circ$ such that

$$A_\circ(\varrho) M A_\circ(\varrho) = a(\varrho) M, \quad a:\mathbb{R}\to\mathbb{R}. \tag{4.1}$$

Therefore, $\phi(A(\underline{z})\underline{z}) = a(\phi(\underline{z}))\phi(\underline{z})$ and

$$\phi(F_A(\underline{z})) = \phi(\underline{z}) + \frac{1}{4c^2}\,\phi(A(\underline{z})\underline{z}) = f_a(\phi(\underline{z})), \qquad f_a(\varrho) := \left(1+\frac{1}{4c^2}\,a(\varrho)\right)\varrho.$$

So, if $x\mapsto a(\varrho)\varrho$ is $C^1$ and

$$\lim_{|x|\to+\infty}\left|1+\frac{1}{4c^2}\,a(\varrho)\right|\,|\varrho| = +\infty, \tag{4.2}$$

$$1+\frac{1}{4c^2}\,\frac{d}{d\varrho}\,(a(\varrho)\varrho)\neq 0, \tag{4.3}$$

then $f_a:\mathbb{R}\to\mathbb{R}$ is a $C^1$-diffeomorphism by Hadamard's theorem and $F_A$ has a global inverse given by

$$F_A^{-1}(\underline{z}) = \left(\mathbb{1}+\frac{i}{2c}\,A_\circ\big(f_a^{-1}(\phi(\underline{z}))\big)\right)^{-1}\underline{z}.$$

Therefore, $F_A^{-1}$ is a $C^1$-diffeomorphism (and hence Assumption 3.2 holds) whenever $\varrho\mapsto A_\circ(\varrho)$ is a $C^1$ map such that (4.2) and (4.3) hold. Let us notice that, as the next example shows, Assumption 3.2 can hold true under weaker conditions.

**4.1. Nonlinear Gesztesy–Šeba models.** The two simplest models are the ones in which $M = M_\pm := \frac{1}{2}(\mathbb{1}\pm\sigma_3)$, i.e., either $\phi(\underline{z}) = \phi_+(\underline{z}) := |z_1|^2$ or $\phi(\underline{z}) = \phi_-(\underline{z}) := |z_2|^2$. These give nonlinear versions of the models introduced in [19]. One has that (4.1) holds if and only if

$$A_\circ(\varrho) = \alpha(\varrho)\,M_\pm, \qquad \alpha:\mathbb{R}\to\mathbb{R}, \qquad a=\alpha^2.$$

By straightforward computation, one gets that the Jacobian determinant of $F_A$ never vanishes if and only if

$$1+\frac{1}{4c^2}\,\frac{d}{d\varrho}(\alpha^2(\varrho)\varrho)\neq 0.$$

So, for example, Assumption 3.2 holds true whenever $\alpha(\varrho) = \kappa\,\varrho^{2\sigma}$, $\kappa\in\mathbb{R}$ and $\sigma\in\mathbb{R}_+$; the case $\sigma\in(0,1/2)$ shows that Assumption 3.2 can be true even if $x\mapsto A_\circ(\varrho)$ is not a $C^1$ map.

Let us notice that, in the linear case, the nonrelativistic limit of the "+" case gives a Schrödinger operator with a delta interaction of strength $\alpha$, whereas the nonrelativistic limit of the "−" case gives a Schrödinger operator with a delta prime interaction of strength $-1/\alpha$ (see [7]).

**4.2. Bragg resonance.** If $M = \mathbb{1}$, i.e., $\phi(\underline{z}) = |\underline{z}|^2$, then (4.1) holds if and only if either

$$A_\circ(\varrho) = \alpha(\varrho)\,\mathbb{1}, \qquad \alpha : \mathbb{R} \to \mathbb{R}, \qquad a = \alpha^2,$$

or

$$A_\circ(\varrho) = \begin{bmatrix} \alpha(\varrho) & \gamma(\varrho) \\ \bar{\gamma}(\varrho) & -\alpha(\varrho) \end{bmatrix}, \quad \alpha : \mathbb{R} \to \mathbb{R}, \quad \gamma : \mathbb{R} \to \mathbb{C}, \quad a = \alpha^2 + |\gamma|^2.$$

As a special case, taking $A(\underline{z}) = 3\beta|\underline{z}|^2\mathbb{1}$, $\beta \in \mathbb{R}$, i.e., $W(\underline{z}) = \frac{3}{2}\,\beta\,|\underline{z}|^4$, one obtains the "concentrated" version of the Bragg resonance model; see [34]. There a different representation of the Dirac operator is used; with reference to Remark 3.7, it corresponds to the choice

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

so that

$$\tilde{D}_m = -i\hbar c\,\sigma_3 \frac{d}{dx} - mc^2\sigma_1.$$

In such a representation, the corresponding potential is given by

$$\tilde{W}(\underline{z}) = \beta\big(|\underline{z}|^4 + 2\,|z_1|^2|z_2|^2\big).$$

**4.3. Example.** If $M = \sigma_1$, i.e., $\phi(\underline{z}) = z_1\bar{z}_2 + \bar{z}_1 z_2$, then (4.1) holds if and only if either

$$A_\circ(\varrho) = \gamma(\varrho)\,\sigma_1, \quad \gamma : \mathbb{R} \to \mathbb{R}, \qquad a = \gamma^2,$$

or

$$A_\circ(\varrho) = \begin{bmatrix} \alpha_1(\varrho) & \gamma(\varrho) \\ \bar{\gamma}(\varrho) & \alpha_2(\varrho) \end{bmatrix}, \quad \alpha_1 : \mathbb{R} \to \mathbb{R},\ \alpha_2 : \mathbb{R} \to \mathbb{R},\ \gamma : \mathbb{R} \to i\mathbb{R}, \quad a = \alpha_1\alpha_2 - |\gamma|^2.$$

**4.4. Example.** If $M = \sigma_2$, i.e., $\phi(\underline{z}) = i(z_1\bar{z}_2 - \bar{z}_1 z_2)$, then (4.1) holds if and only if either

$$A_\circ(\varrho) = \gamma(\varrho)\,\sigma_2, \qquad \gamma : \mathbb{R} \to \mathbb{R}, \qquad a = \gamma^2,$$

or

$$A_\circ(\varrho) = \begin{bmatrix} \alpha_1(\varrho) & \gamma(\varrho) \\ \gamma(\varrho) & \alpha_2(\varrho) \end{bmatrix}, \quad \alpha_1 : \mathbb{R} \to \mathbb{R},\ \alpha_2 : \mathbb{R} \to \mathbb{R},\ \gamma : \mathbb{R} \to \mathbb{R}, \quad a = \alpha_1\alpha_2 - \gamma^2.$$

**4.5. Soler-type models.** If $M = \sigma_3$, i.e., $\phi(\underline{z}) = |z_1|^2 - |z_2|^2$, then (4.1) holds if and only if either

$$A_\circ(\varrho) = \alpha(\varrho)\,\sigma_3, \qquad \alpha : \mathbb{R} \to \mathbb{R}, \qquad a = \alpha^2,$$

or

$$A_\circ(\varrho) = \begin{bmatrix} \alpha(\varrho) & \gamma(\varrho) \\ \bar{\gamma}(\varrho) & \alpha(\varrho) \end{bmatrix}, \quad \alpha : \mathbb{R} \to \mathbb{R}, \quad \gamma : \mathbb{R} \to \mathbb{C}, \quad a = \alpha^2 - |\gamma|^2.$$

As a special case, taking $A(\underline{z}) = 4\left(|z_1|^2 - |z_2|^2\right)\sigma_3$, i.e., $W(\underline{z}) = 2\left(|z_1|^2 - |z_2|^2\right)^2$, one obtains the "concentrated" version of the massive Gross–Neveu model (see [23, 34]) which corresponds to the 1-D Soler model (see [38]). Notice that, with respect to the representation $\tilde{D}_m$ of the Dirac operator used in [34], the potential is given by $\tilde{W}(\underline{z}) = 2(\overline{z}_1 z_2 + z_1 \overline{z}_2)^2$.

**Appendix A. A regularity result for the free evolution.** Here we recall the basic results for the Cauchy problem for the 1-D free Dirac operator:

$$\begin{cases} i\hbar \dfrac{d}{dt}\Psi^f(t) = D_m \Psi^f(t), \\ \Psi^f(0) = \Psi_\circ. \end{cases} \tag{A.1}$$

By

$$\left(-\frac{i}{\hbar}\, D_m\right)^2 = K_m := c^2 \frac{d^2}{dx^2} - \frac{m^2 c^4}{\hbar^2},$$

such a Cauchy problem is equivalent to

$$\begin{cases} \dfrac{d^2}{dt^2}\Psi^f(t) = K_m \Psi^f(t), \\ \Psi^f(0) = \Psi_\circ, \\ \dfrac{d}{dt}\Psi^f(0) = -\frac{i}{\hbar} D_m \Psi_\circ. \end{cases}$$

The solution of the Cauchy problem for the Klein–Gordon equation is known (see, e.g., [40, section II.5.4] and [35, section 4.1.3-3]):

$$\begin{aligned} \Psi^f(x,t) = {} & \frac{1}{2}\left(\Psi_\circ(x-ct) + \Psi_\circ(x+ct)\right) \\ & - \frac{mc^2 t}{2\hbar} \int_{x-ct}^{x+ct} d\xi\, \frac{J_1\left(\frac{mc}{\hbar}\sqrt{(ct)^2-(x-\xi)^2}\right)}{\sqrt{(ct)^2-(x-\xi)^2}}\, \Psi_\circ(\xi) \\ & + \frac{1}{2c}\int_{x-ct}^{x+ct} d\xi\, J_0\left(\frac{mc}{\hbar}\sqrt{(ct)^2-(x-\xi)^2}\right)\left(-\frac{i}{\hbar} D_m \Psi_\circ\right). \end{aligned}$$

Posing

$$\Psi^f(t) = \begin{pmatrix} \psi_1^f(t) \\ \psi_2^f(t) \end{pmatrix}, \quad \Psi_\circ = \begin{pmatrix} \psi_1^\circ \\ \psi_2^\circ \end{pmatrix},$$

integrating by parts and by $\frac{d}{dx}J_0(x) = -J_1(x)$, the solution can be rewritten in an equivalent way as

$$\begin{aligned} \psi_k^f(x,t) = {} & \frac{1}{2}\left((\psi_k^\circ + \psi_j^\circ)(x-ct) + (\psi_k^\circ - \psi_j^\circ)(x+ct)\right) \\ & - \frac{mc}{2\hbar}\int_{x-ct}^{x+ct}\Bigg(ct\, \frac{J_1\left(\frac{mc}{\hbar}\sqrt{(ct)^2-(x-\xi)^2}\right)}{\sqrt{(ct)^2-(x-\xi)^2}} \\ & \qquad - i(-1)^k J_0\left(\frac{mc}{\hbar}\sqrt{(ct)^2-(x-\xi)^2}\right)\Bigg)\, \psi_k^\circ(\xi) d\xi \\ & - \frac{mc}{2\hbar}\int_{x-ct}^{x+ct} \frac{J_1\left(\frac{mc}{\hbar}\sqrt{(ct)^2-(x-\xi)^2}\right)}{\sqrt{(ct)^2-(x-\xi)^2}}(x-\xi)\psi_j^\circ(\xi)\, d\xi, \quad j,k=1,2; \quad j \neq k. \end{aligned} \tag{A.2}$$

Therefore, defining the matrix-valued kernel function

$$K(x,t) = -\frac{mc}{2\hbar}\left( i\sigma_3 J_0\left(\frac{m\,c}{\hbar}\sqrt{(ct)^2-x^2}\right) + (ct\mathbb{1} + x\sigma_1)\,\frac{J_1\left(\frac{m\,c}{\hbar}\sqrt{(ct)^2-x^2}\right)}{\sqrt{(ct)^2-x^2}}\right),$$

the solution of the Cauchy problem (A.1) can be written as

$$\begin{aligned}\Psi^f(x,t) &= \left(e^{-\frac{i}{\hbar}tH}\Psi_\circ\right)(x) \\ &= \frac{1}{2}\left((\mathbb{1}+\sigma_1)\Psi_\circ(x-ct) + (\mathbb{1}-\sigma_1)\Psi_\circ(x+ct)\right) \\ &\quad + \int_{x-ct}^{x+ct} d\xi\, K(x-\xi,t)\Psi_\circ(\xi).\end{aligned} \tag{A.3}$$

In the following proposition we establish the regularity properties of the map $t \mapsto \Psi^f(y,t)$.

PROPOSITION A.1. *For any* $y \in \mathbb{R}$, $\Psi_\circ \in H^1(\mathbb{R}\backslash\{y\}) \otimes \mathbb{C}$, *and* $T > 0$, $\Psi^f(y,\cdot) \in H^1(0,T)$.

*Proof.* We use identity (A.2), which we rewrite as

$$\psi_k^f(y,t) = u_{1,k}(t) + u_{2,k}(t) + u_{3,k}(t) + u_{4,k}(t)$$

with

$$\begin{aligned} u_{1,k}(t) &= \frac{1}{2}\left((\psi_k^\circ + \psi_j^\circ)(y-ct) + (\psi_k^\circ - \psi_j^\circ)(y+ct)\right), \\ u_{2,k}(t) &= -\frac{m\,c}{2\,\hbar}\int_{y-ct}^{y+ct} ct\,\frac{J_1\left(\frac{m\,c}{\hbar}\sqrt{(ct)^2-(y-\xi)^2}\right)}{\sqrt{(ct)^2-(y-\xi)^2}}\psi_k^\circ(\xi)\,d\xi, \\ u_{3,k}(t) &= i(-1)^k\frac{m\,c}{2\,\hbar}\int_{y-ct}^{y+ct} J_0\left(\frac{m\,c}{\hbar}\sqrt{(ct)^2-(y-\xi)^2}\right)\psi_k^\circ(\xi)\,d\xi, \\ u_{4,k}(t) &= -\frac{mc}{2\,\hbar}\int_{y-ct}^{y+ct}\frac{J_1\left(\frac{m\,c}{\hbar}\sqrt{(ct)^2-(y-\xi)^2}\right)}{\sqrt{(ct)^2-(y-\xi)^2}}\,(y-\xi)\psi_j^\circ(\xi)\,d\xi \end{aligned}$$

with $k,j = 1,2$ and $k \neq j$.

We start by noting that, for $k = 1,2$,

$$\int_0^T |\psi_k^{\circ\prime}(y+ct)|^2\,dt = c\int_y^{y+cT} |\psi_k^{\circ\prime}(s)|^2\,ds \le c\|\psi_k^{\circ\prime}\|^2_{L^2(y,+\infty)}.$$

Similarly,

$$\int_0^T |\psi_k^{\circ\prime}(y-ct)|^2\,dt \le c\|\psi_k^{\circ\prime}\|^2_{L^2(-\infty,y)},$$

and an equivalent bound holds true for $\int_0^T |\psi_k^\circ(y \pm ct)|^2\,dt$. Hence,

$$\|u_{k,1}\|_{H^1(0,T)} \le C\sum_{j=1}^2 \|\psi_j^\circ\|_{H^1(\mathbb{R}\backslash\{y\})}.$$

Next we analyze the integral terms in (A.2). Recall that

$$
\begin{aligned}
\|\psi_k^\circ\|^2_{L^\infty(y,+\infty)} &\le 2\|\psi_k^\circ\|_{L^2(y,+\infty)}\|\psi_k^{\circ\prime}\|_{L^2(y,+\infty)},\\
\|\psi_k^\circ\|^2_{L^\infty(-\infty,y)} &\le 2\|\psi_k^\circ\|_{L^2(-\infty,y)}\|\psi_k^{\circ\prime}\|_{L^2(-\infty,y)}.
\end{aligned}
\tag{A.4}
$$

We shall prove that, for $l = 2,3,4$, $u'_{l,k}(t)$ is bounded for all $t\in[0,T]$; this in turn implies that $u_{l,k}\in H^1(0,T)$.

We start with $u_{2,k}$. We split the integral on the intervals $(y-ct,y)$ and $(y,y+ct)$ and consider first the integration for $\xi\in(y,y+ct)$; we have that

$$
\begin{aligned}
&\int_y^{y+ct} ct\,\frac{J_1\left(\frac{m\,c}{\hbar}\sqrt{(ct)^2-(y-\xi)^2}\right)}{\sqrt{(ct)^2-(y-\xi)^2}}\psi_k^\circ(\xi)\,d\xi\\
&\qquad = ct\int_0^{ct}\frac{J_1\left(\frac{m\,c}{\hbar}\sqrt{(2ct-\eta)\eta}\right)}{\sqrt{(2ct-\eta)\eta}}\psi_k^\circ(ct+y-\eta)\,d\eta.
\end{aligned}
\tag{A.5}
$$

Taking the derivative with respect to $t$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_y^{y+ct} ct\,\frac{J_1\left(\frac{m\,c}{\hbar}\sqrt{(ct)^2-(y-\xi)^2}\right)}{\sqrt{(ct)^2-(y-\xi)^2}}\psi_k^\circ(\xi)\,d\xi\\
&\quad = c\int_0^{ct}\frac{J_1\left(\frac{m\,c}{\hbar}\sqrt{(2ct-\eta)\eta}\right)}{\sqrt{(2ct-\eta)\eta}}\psi_k^\circ(ct+y-\eta)\,d\eta\\
&\qquad + c^2t\frac{J_1\left(\frac{m\,c}{\hbar}\sqrt{(ct)^2}\right)}{\sqrt{(ct)^2}}\psi_k^\circ(y^+)\\
&\qquad + ct\int_0^{ct}\frac{d}{dt}\left(\frac{J_1\left(\frac{m\,c}{\hbar}\sqrt{(2ct-\eta)\eta}\right)}{\sqrt{(2ct-\eta)\eta}}\right)\psi_k^\circ(ct+y-\eta)\,d\eta\\
&\qquad + c^2t\int_0^{ct}\frac{J_1\left(\frac{m\,c}{\hbar}\sqrt{(2ct-\eta)\eta}\right)}{\sqrt{(2ct-\eta)\eta}}\psi_k^{\circ\prime}(ct+y-\eta)\,d\eta.
\end{aligned}
\tag{A.6}
$$

For the first, second, and fourth terms on the r.h.s. we use the bounds (A.4) and the fact that, for any $a\ge 0$, there exists a constant $C$ such that $|J_1(\sqrt{a})/\sqrt{a}|\le C$. So for all $t\in[0,T]$, each of those terms is bounded by $C_T\|\psi_k^\circ\|_{H^1(y,\infty)}$, where $C_T$ is a constant which depends on $T$.

For the third term on the r.h.s. of (A.6) we use the fact that, for any $a>0$ and $b\ge 0$, there exists a constant $C$ such that $|\frac{d}{da}J_1(\sqrt{ab})/\sqrt{ab}|\le C/a$. We have that, for all $\eta\in[0,ct]$,

$$
t\left|\frac{d}{dt}\frac{J_1\left(\frac{m\,c}{\hbar}\sqrt{(2ct-\eta)\eta}\right)}{\sqrt{(2ct-\eta)\eta}}\right|\le\frac{Ct}{2ct-\eta}\le C.
$$

Hence, the third term on the r.h.s. of (A.6) is also bounded by $C_T\|\psi_k^\circ\|_{H^1(y,\infty)}$. The integral of the form (A.5) on the interval $(y-ct,y)$ is bounded in a similar way and we omit the details. We have proved that, for all $t\in[0,T]$, $|u_{2,k}(t)|\le C_T\|\psi_k^\circ\|_{H^1(\mathbb{R}\setminus\{y\})}$.

The analysis of $u_{3,k}$ is straightforward. Splitting again the integration interval and taking the derivative, we obtain

$$\frac{d}{dt}\int_y^{y+ct} J_0\left(\frac{m\,c}{\hbar}\sqrt{(ct)^2-(y-\xi)^2}\right)\psi_k^\circ(\xi)\,d\xi$$
$$= c\psi_k^\circ(y+ct) - \frac{m\,c}{\hbar}c^2 t\int_y^{y+ct}\frac{J_1\left(\frac{m\,c}{\hbar}\sqrt{(ct)^2-(y-\xi)^2}\right)}{\sqrt{(ct)^2-(y-\xi)^2}}\psi_k^\circ(\xi)\,d\xi.$$

The integral on the interval $(y-ct,y)$ can be treated in a similar way. By inequalities (A.4), and since $J_1(a)/a$ is bounded for all $a\geq 0$, we conclude that, for all $t\in[0,T]$, $|u_{3,k}(t)|\leq C_T\|\psi_k^\circ\|_{H^1(-\infty,y)\oplus H^1(y,\infty)}$.

We are left to analyze $u_{4,k}$. Also, in this case, we split the integral on the intervals $(y-ct,y)$ and $(y,y+ct)$, and take the derivative. We obtain

(A.7)
$$\frac{d}{dt}\int_y^{y+ct}\frac{J_1\left(\frac{m\,c}{\hbar}\sqrt{(ct)^2-(y-\xi)^2}\right)}{\sqrt{(ct)^2-(y-\xi)^2}}\,(y-\xi)\psi_j^\circ(\xi)\,d\xi$$
$$= -\frac{m\,c^2}{2\hbar}(ct)\psi_j^\circ(y+ct) + c^2 t\int_y^{y+ct}\frac{d}{d\xi}\left(\frac{J_1\left(\frac{m\,c}{\hbar}\sqrt{(ct)^2-(y-\xi)^2}\right)}{\sqrt{(ct)^2-(y-\xi)^2}}\right)\psi_j^\circ(\xi)\,d\xi,$$

where we used the identity

$$(y-\xi)\frac{d}{dt}\left(\frac{J_1\left(\frac{m\,c}{\hbar}\sqrt{(ct)^2-(y-\xi)^2}\right)}{\sqrt{(ct)^2-(y-\xi)^2}}\right) = c^2 t\frac{d}{d\xi}\left(\frac{J_1\left(\frac{m\,c}{\hbar}\sqrt{(ct)^2-(y-\xi)^2}\right)}{\sqrt{(ct)^2-(y-\xi)^2}}\right).$$

In (A.7), we integrate by parts and obtain

$$\frac{d}{dt}\int_y^{y+ct}\frac{J_1\left(\frac{m\,c}{\hbar}\sqrt{(ct)^2-(y-\xi)^2}\right)}{\sqrt{(ct)^2-(y-\xi)^2}}\,(y-\xi)\psi_j^\circ(\xi)\,d\xi$$
$$= -cJ_1\left(\frac{m\,c^2}{\hbar}t\right)\,\psi_j^\circ(y^+) - c^2 t\int_y^{y+ct}\frac{J_1\left(\frac{m\,c}{\hbar}\sqrt{(ct)^2-(y-\xi)^2}\right)}{\sqrt{(ct)^2-(y-\xi)^2}}\,\psi_j^{\circ\prime}(\xi)\,d\xi.$$

A similar result holds true for the integral on the interval $(y-ct,y)$. By using again the bounds (A.4) and the boundedness of $J_1(a)/a$, we obtain $|u_{3,k}(t)|\leq C_T\|\psi_k^\circ\|_{H^1(\mathbb{R}\setminus\{y\})}$, and this concludes the proof of the proposition. ∎

REFERENCES

[1] M. J. Ablowitz and Y. Zhu, *Nonlinear waves in shallow honeycomb lattices*, SIAM J. Appl. Math., 72 (2012), pp. 240–260.

[2] R. Adami, G. Dell'Antonio, R. Figari, and A. Teta, *The Cauchy problem for the Schrödinger equation in dimension three with concentrated nonlinearity*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 20 (2003), pp. 477–500.

[3] R. Adami, G. Dell'Antonio, R. Figari, and A. Teta, *Blow-up solutions for the Schrödinger equation in dimension three with a concentrated nonlinearity*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 21 (2004), pp. 121–137.

[4] R. Adami, D. Noja, and C. Ortoleva, *Orbital and asymptotic stability for standing waves of a nonlinear Schrödinger equation with concentrated nonlinearity in dimension three*, J. Math. Phys., 54 (2013), 013501.

[5] R. Adami and A. Teta, *A class of nonlinear Schrödinger equations with concentrated nonlinearity*, J. Funct. Anal., 180 (2001), pp. 148–175.

[6] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, 2nd ed., AMS Chelsea Publishing, Providence, RI, 2005.

[7] S. Benvegnù and L. Dąbrowski, *Relativistic point interaction*, Lett. Math. Phys., 30 (1994), pp. 159–167.

[8] N. Bournaveas, *Local well-posedness for a nonlinear Dirac equation in spaces of almost critical dimension*, Discrete Contin. Dyn. Syst., 20 (2008), pp. 605–616.

[9] O. Bulashenko, V. Kochelap, and L. Bonilla, *Coherent patterns and self-induced diffraction of electrons on a thin nonlinear layer*, Phys. Rev. B, 54 (1996), pp. 1537–1540.

[10] C. Cacciapuoti, *On the derivation of the Schrödinger equation with point-like nonlinearity*, Nanosystems: Phys. Chem. Math., 6 (2015), pp. 79–94.

[11] C. Cacciapuoti, D. Finco, D. Noja, and A. Teta, *The NLS equation in dimension one with spatially concentrated nonlinearities: The pointlike limit*, Lett. Math. Phys., 104 (2014), pp. 1557–1570.

[12] C. Cacciapuoti, D. Finco, D. Noja, and A. Teta, *The point-like limit for a NLS equation with concentrated nonlinearity in dimension three*, J. Funct. Anal. (2017), https://doi.org/10.1016/j.jfa.2017.04.011.

[13] T. Candy, *Global existence for an $L^2$ critical nonlinear Dirac equation in one dimension*, Adv. Differential Equations, 16 (2011), pp. 643–666.

[14] R. Carlone, M. Malamud, and A. Posilicano, *On the spectral theory of Gesztesy–Šeba realizations of 1-D Dirac operators with point interactions on a discrete set*, J. Differential Equations, 254 (2013), pp. 3835–3902.

[15] C. M. De Sterke and J. E. Sipe, *Gap solitons*, Prog. Opt., 33 (2008), pp. 203–260.

[16] V. Delgado, *Global solutions of the Cauchy problem for the (classical) coupled Maxwell-Dirac and other nonlinear Dirac equations in one space dimension*, Proc. Amer. Math. Soc., 69 (1978), pp. 289–296.

[17] N. Dror and B. A. Malomed, *Solitons supported by localized nonlinearities in periodic media*, Phys. Rev. A, 83 (2011), 033828.

[18] C. Fefferman and M. Weinstein, *Waves in honeycomb structures*, Journées Équations aux Dérivées Partielles, 2012 (2012), 12.

[19] F. Gesztesy and P. Šeba, *New analytically solvable models of relativistic point interactions*, Lett. Math. Phys., 13 (1987), pp. 345–358.

[20] R. H. Goodman, M. I. Weinstein, and P. J. Holmes, *Nonlinear propagation of light in one-dimensional periodic structures*, J. Nonlinear Sci., 11 (2001), pp. 123–168.

[21] W. B. Gordon, *On the diffeomorphisms of Euclidean space*, Amer. Math. Monthly, 79 (1972), pp. 755–759.

[22] W. B. Gordon, *Addendum to: "On the diffeomorphisms of Euclidean space"*, Amer. Math. Monthly, 80 (1973), pp. 674–675.

[23] D. J. Gross and A. Neveu, *Dynamical symmetry breaking in asymptotically free field theories*, Phys. Rev. D, 10 (1974), pp. 3235–3253.

[24] J. Holmer and C. Liu, *Blow-up for the 1D Nonlinear Schrödinger equation with Point Nonlinearity I: Basic Theory*, preprint, arXiv:1510.03491 [math.AP], 2015.

[25] H. Huh, *Global strong solution to the Thirring model in critical space*, J. Math. Anal. Appl., 381 (2011), pp. 513–520.

[26] H. Huh, *Global solutions to Gross–Neveu equation*, Lett. Math. Phys., 103 (2013), pp. 927–931.

[27] A. I. Komech and A. A. Komech, *Global well-posedness for the Schrödinger equation coupled to a nonlinear oscillator*, Russ. J. Math. Phys., 14 (2007), pp. 164–173.

[28] A. I. Komech and A. A. Komech, *Global attraction to solitary waves for a nonlinear Dirac equation with mean field interaction*, SIAM J. Math. Anal., 42 (2010), pp. 2944–2964.

[29] E. Kopylova, *On global well-posedness for Klein–Gordon equation with concentrated nonlinearity*, J. Math. Anal. Appl., 443 (2016), pp. 1142–1157.

[30] S. Machihara, K. Nakanishi, and K. Tsugawa, *Well-posedness for nonlinear Dirac equations in one dimension*, Kyoto J. Math., 50 (2010), pp. 403–451.

[31] B. A. Malomed and M. Y. Azbel, *Modulational instability of a wave scattered by a nonlinear center*, Phys. Rev. B, 47 (1993), pp. 10402–10406.

[32] D. Noja and A. Posilicano, *Wave equations with concentrated nonlinearities*, J. Phys. A, 38 (2005), pp. 5011–5022.

[33] D. Pelinovsky, *Localization in Periodic Potentials*, London Mathematical Society Lecture Note Series 390, Cambridge University Press, Cambridge, 2011.

[34] D. Pelinovsky, *Survey on global existence in the nonlinear Dirac equations in one spatial dimension*, RIMS Kokyuroku Bessatsu, B26 (2011), pp. 37–50.

[35] A. D. Polyanin and V. E. Nazaikinskii, *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, CRC Press, Boca Raton, FL, 2015.
[36] S. Selberg and A. Tesfahun, *Low regularity well-posedness for some nonlinear Dirac equations in one space dimension*, Differential Integral Equations, 23 (2010), pp. 265–278.
[37] S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics,* 3rd ed., American Mathematical Society, Providence, RI, 1991.
[38] M. Soler, *Classical, stable, nonlinear spinor field with positive rest energy*, Phys. Rev. D, 1 (1970), pp. 2766–2769.
[39] W. E. Thirring, *A soluble relativistic field theory*, Ann. Physics, 3 (1958), pp. 91–112.
[40] A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics*, Oxford, Pergamon Press, 1963.