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Research Article

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Half-space Gaussian symmetrization: applications to semilinear elliptic problems

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Abstract: We consider a class of semilinear equations with an absorption nonlinear zero order term of power type, where elliptic condition is given in terms of Gauss measure. In the case of the superlinear equation we introduce a suitable definition of solutions in order to prove the existence and uniqueness of a solution in $\mathbb{R}^N$ without growth restrictions at infinity. A comparison result in terms of the half-space Gaussian symmetrized problem is also proved. As an application, we give some estimates in measure of the growth of the solution near the boundary of its support for sublinear equations. Finally we generalize our results to problems with a nonlinear zero order term not necessary of power type.

Keywords: semilinear equations, Gauss measure, Gaussian symmetrization

MSC: 35J61, 35J70, 35B45

1 Introduction

In this paper we focus our attention on a class of semilinear elliptic Dirichlet problems, whose prototype is

$$\begin{cases} -\operatorname{div}(\nabla u(x)\,\varphi(x)) + c_0\,|u(x)|^{p-1}\,u(x)\varphi(x) = f(x)\varphi(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $c_0 > 0$, $p > 0$, Ω is an open subset of $\mathbb{R}^N$ not necessary bounded, $\varphi(x) = \varphi_N(x) := (2\pi)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{2}\right)$ is the density of standard N−dimensional Gauss measure γ and the datum f belongs to a suitable Zygmund space. A more general diffusion operator is in fact considered in all this paper.

Problem (1.1) is related to Ornstein-Uhlenbeck operator $Lu := \triangle u - x \cdot \nabla u$ and our approach allows us to consider an extra semilinear zero order term $c_0\,|u|^{p-1}\,u$, $\Omega = \mathbb{R}^N$ and a weak assumption on the summability of datum. Notice that we can formally write $\operatorname{div}(\nabla u\,\varphi(x)) = \triangle u\varphi(x) + \nabla\varphi(x)\cdot\nabla u$ which justifies the multiplicative role of $\varphi(x)$ in the equation of (1.1). The idea of "symmetrizing the operator" $-\Delta u(x) + x \cdot \nabla u$ in order to solve the drift equation

$$\begin{cases} -\Delta u(x) + x\cdot\nabla u + c_0\,|u(x)|^{p-1}\,u(x) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

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comes back from a pioneering paper [27] by Kolmogorov in 1937 for $c_0 = 0$. For some recent survey in this direction see [32]. It is well-known the above diffusion operator with a drift is related to the stochastic Ornstein–Uhlenbeck process with applications in financial mathematics and in physical sciences (a model for the velocity of a massive Brownian particle under the influence of friction). This is sometimes also written in terms of a Langevin ordinary differential equation with noise (see, *e.g.* [31]).

We remark that if $\varphi \equiv 1$ problem (1.1) was largely considered in the literature (see, *e.g.* [9] when Ω is bounded and [37] and [19] when Ω is unbounded and $p > 1$).

In the weighted case, when $p \leq 1$ and $\gamma(\Omega) < 1$, Lax-Milgram Theorem guarantees the existence of a solution for problem (1.1) $u \in H_0^1(\Omega, \gamma)$ (the weighted Sobolev space) once we assume f belongs to its dual. For example we can require $f \in L^2 \log L^{-\frac{1}{2}}(\Omega, \gamma)$, a functional space which we recalled in Section 2, where other preliminary notions will also be collected.

If we consider $\Omega = \mathbb{R}^N$ one of the difficulties that arises when solving (1.1) is due to lack of a Poincaré inequality. As a consequence we have to consider the Banach space $H^1(\mathbb{R}^N, \gamma)$ equipped with the norm $\|u\|_{L^2(\mathbb{R}^N,\gamma)} + \|\nabla u\|_{L^2(\mathbb{R}^N,\gamma)}$. In the linear case, $p = 1$, the existence and uniqueness of a weak solution $u \in H^1(\mathbb{R}^N, \gamma)$ to (1.1) follows again from Lax-Milgram Theorem and we also get the correct growth condition on $f(x)$ as $|x| \to +\infty$ (see, *e.g.* the exposition made in [30]).

The superlinear case $p > 1$ is different. In Section 3 we shall prove the existence and uniqueness giving a suitable notion of weak solution for the case of $\Omega = \mathbb{R}^N$ and $p > 1$. We point out that in the superlinear case when $\gamma(\Omega) < 1$ the existence and uniqueness of a weak solution can be obtained through an easy adaptation of the results of Brezis and Browder [9]. In order to consider $\Omega = \mathbb{R}^N$ we follow an idea of [19], giving an alternative proof and enlarging the applications of the pioneering result on superlinear problems by Brezis [8]. Thanks to the assumption $p > 1$ we get some a priori estimates on any given half-space allowing us to obtain a general existence and uniqueness result *without any growth condition at infinity* on the datum f (and so with less summability than f in $L^2 \log L^{-\frac{1}{2}}(\mathbb{R}^N, \gamma)$).

In a second part of the paper (Sections 4 and 5) we deal with comparison results in terms of the half-space Gaussian symmetrization of solutions of (1.1) and its generalizations. We point out that when Ω is bounded the usual radially symmetrization method, when applied to an elliptic operator with a drift term as (1.2) modifies drastically the drift term in the symmetrized equation (see, *e.g.*, [34]). In contrast to that, the half-space Gaussian symmetrization method allows to preserve the more important facts of the drift term (see Remark 11) as well as to deal with an unbounded domain. We recall that in [3] the authors compare the solution u to problem (1.1) , when $\gamma(\Omega) < 1$, with the solution v to a simpler problem, called the half-space Gaussian symmetrized problem, without zero order term defined in the half-space $\Omega^\star = \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_1 > \omega\}$ with ω such that $\gamma(\Omega^\star) = \gamma(\Omega)$ and with a datum depending only on the first variable. Here we are interested to prove some comparisons in term of the solution to symmetrized problem keeping also a nonlinear zero order term. As in the unweighted case (see *e.g.* [16], [18]) we are able to obtain some integral estimates, that imply a comparison between Lebesgue norms. Other comparison results related to Gauss measure are contained in [20, 21], [12] for the parabolic case and [25] in the non-local case. As an application, we give some estimates in measure of the growth of the solution near the boundary of its support for sublinear equations $p \in (0, 1)$ when the datum f possibly vanishes on a positively measured subset of Ω.

Finally in Section 5 we generalize our results to problems with more general zero order terms $b(u)$, not necessary of power type as in previous Sections.

2 Preliminaries

Let γ be the N-dimensional Gauss measure on $\mathbb{R}^N$ defined by

$$d\gamma := \varphi_N(x)\, dx := (2\pi)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{2}\right)\, dx, \qquad x \in \mathbb{R}^N$$

normalized by $\gamma\left(\mathbb{R}^N\right) = 1$. In what follows we will set $\varphi_N(x) = \varphi(x)$ for simplicity.

It is well-known that an isoperimetric inequality for Gauss measure holds (see *e.g.* [11]): among all measurable sets of $\mathbb{R}^N$ with prescribed Gauss measure, *the half-spaces* take the smallest *perimeter* (in the sense of this measure). In particular, *the perimeter of a* $(N-1)$*–rectifiable set E of* R^N *with respect to the Gauss measure* is defined as

$$P(E) = (2\pi)^{-\frac{N}{2}} \int_{\partial E} \exp\left(-\frac{|x|^2}{2}\right) \mathcal{H}_{N-1}(dx),$$

where $\mathcal{H}_{N-1}$ denotes the $(N-1)$–dimensional Hausdorff measure. The following isoperimetric estimate (see *e.g.* [11])

$$P(E) \geq \varphi_1(\Phi^{-1}(\gamma(E))) \tag{2.1}$$

holds for all subsets $E \subset \mathbb{R}^n$, where $\varphi_1(s) = (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{|s|^2}{2}\right)$ for $s \in \mathbb{R}$ and

$$\Phi(\lambda) = \int_{\lambda}^{+\infty} \varphi_1(s)ds \text{ for } \lambda \in \mathbb{R} \cup \{-\infty, +\infty\}, \tag{2.2}$$

the 1– dimensional Gauss measure of line (λ, ∞).

Now we introduce the notion of *rearrangement with respect to Gauss measure* (see *e.g.* [22]). Here the balls of Schwartz symmetrization is replaced by half-spaces

$$H_\omega := \left\{x = (x_1, \ldots, x_N) \in \mathbb{R}^N : x_1 > \omega\right\} \text{ for some } \omega \in \mathbb{R} \cup \{-\infty, +\infty\}. \tag{2.3}$$

If u is a measurable function in Ω, we denote by $u^{\circledast}$ the *decreasing rearrangement of* u *with respect to Gauss measure*, *i.e.*

$$u^{\circledast}(s) = \inf\{t \geq 0 : \gamma_u(t) \leq s\} \qquad s \in]0, 1], \tag{2.4}$$

where $\gamma_u(t) = \gamma(\{x \in \Omega : |u| > t\})$ is the distribution function of u with respect to the Gauss measure. Moreover the rearrangement with respect to Gauss measure of u is defined as

$$u^{\star}(x) = u^{\circledast}(\Phi(x_1)) \qquad x \in \Omega^{\star}, \tag{2.5}$$

where

$$\Omega^{\star} := H_\omega \text{ with } \omega \text{ such that } \gamma\left(\Omega^{\star}\right) = \gamma(\Omega) \tag{2.6}$$

and Φ is defined in (2.2).

By definition $u^{\star}$ is a function which depend only on the first variable and its level sets are half-spaces. Moreover u, $u^{\circledast}$ and $u^{\star}$ have the same distribution function.

If $u(x)$, $v(x)$ are measurable functions an Hardy-Littlewood inequality hods:

$$\int_{\Omega} |u(x)v(x)| \, d\gamma \leq \int_{\Omega^{\star}} u^{\star}(x)v^{\star}(x) \, d\gamma = \int_0^{\gamma(\Omega)} u^{\circledast}(s)v^{\circledast}(s) \, ds. \tag{2.7}$$

For general results about the properties of rearrangement with respect to a positive measure see, for example, [13].

We recall that for every open set $\Omega \subseteq \mathbb{R}^N$ the weighted Lebesgue space $L^p(\Omega, \gamma)$ with $p \geq 1$ is the space of measurable functions u such that

$$\|u\|_{L^p(\Omega,\gamma)} := \left(\int_{\Omega} |u|^p \, d\gamma\right)^{\frac{1}{p}}.$$

Moreover, as usual, $H^1(\Omega, \gamma)$ states for the weighted Sobolev space of functions u such that $u, |\nabla u| \in L^2(\Omega, \gamma)$ equipped with the norm $\|u\|_{L^2(\Omega,\gamma)} + \|\nabla u\|_{L^2(\Omega,\gamma)}$. Finally we denote by $H_0^1(\Omega, \gamma)$ the closure of $C_0^\infty(\Omega)$ under the norm $\|\nabla u\|_{L^2(\Omega,\gamma)}$. We remark that a Poincaré inequality holds only when $\gamma(\Omega) < 1$:

$$\int_{\Omega} |\nabla u|^2 \, d\gamma \geq C_P \int_{\Omega} u^2 \, d\gamma \tag{2.8}$$

for every $u \in H_0^1(\Omega, \gamma)$, where C_P is a positive constant depending on Ω.

The Sobolev space $H_0^1(\Omega, \gamma)$ is continuously embedded in the Zygmund space $L^2(\log L)^{\frac{1}{2}}(\Omega, \gamma)$ (see [26], [24], [23] and references therein). We recall that given $1 \le p < \infty$ and $-\infty < \alpha < +\infty$, a measurable function u belongs to the Zygmund space $L^p(\log L)^\alpha(\Omega, \gamma)$ if

$$||u||_{L^p(\log L)^\alpha(\Omega,\gamma)} := \left(\int_0^{\gamma(\Omega)} \left[(1 - \log t)^\alpha u^\circledast(t) \right]^q dt \right)^{\frac{1}{q}} < \infty. \tag{2.9}$$

Then, it is well-known that there exists a constant C_S depending on Ω such that

$$\|u\|_{L^2(\log L)^{\frac{1}{2}}(\Omega,\gamma)} \le C_S \|\nabla u\|_{L^2(\Omega,\gamma)} \tag{2.10}$$

for all $u \in H_0^1(\Omega, \gamma)$. This explain why Zygmund spaces are the natural spaces for the data of problems as (1.1). We observe that these spaces give a refinement of the usual Lebesgue spaces. Indeed by definition (2.9) the space $L^p(\log L)^0(\Omega, \gamma) = L^p(\Omega, \gamma)$. For the definition and properties of the classical Zygmund space we refer to [5].

When $\Omega = \mathbb{R}^N$ we explicitly underline that inequality (2.10) holds by replacing the norm of the gradient by the norm of $H^1(\mathbb{R}^N, \gamma)$.

3 Existence and uniqueness of solutions for the superlinear problem in $\mathbb{R}^N$ without growing conditions

In the present section we focus our attention to existence and uniqueness of solutions to problem (1.1) when $\Omega = \mathbb{R}^N$. Precisely we consider the more general second order elliptic problem

$$-\sum_{i,j=1}^N \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial u}{\partial x_i}(x) \right) + c(x) \left|u(x)\right|^{p-1} u(x)\varphi(x) = f(x)\varphi(x) \text{ in } \mathbb{R}^N. \tag{3.1}$$

As recalled in the introduction, to deal with $\Omega = \mathbb{R}^N$ we will need a suitable definition of weak solution. We refer to [8, 19] for unweighted case $\varphi(x) \equiv 1$.

We introduce the natural energy space for the linear problem (for the case $p = 1$)

$$V(\mathbb{R}^N) = \{w : w \in H^1(H_\omega, \gamma) \text{ for any } \omega \in \mathbb{R}\},$$

where H_ω is defined in (2.3). We stress that for a given function f such that $f \in L^2(\log L)^{-\frac{1}{2}}(H_\omega, \gamma)$ *for any* H_ω the integrals

$$\int_{H_\omega} f\psi \, d\gamma$$

are well-defined for any $\psi \in V(\mathbb{R}^N)$ (even if f is not necessarily in the dual space of $H^1(\mathbb{R}^N, \gamma)$).

In this section we will assume the following structural conditions:

(A1) $p > 1$

(A2) $\frac{a_{ij}}{\varphi} \in L^\infty(H_\omega) \quad \forall \omega \in \mathbb{R}$ and $\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \ge \alpha\varphi(x)|\xi|^2$ for *a.e.* $x \in \mathbb{R}^N$, $\forall \xi \in \mathbb{R}^N$ with $\alpha > 0$

(A3) $c \in L^1(H_\omega, \gamma)$ and $c(x) \ge c_0 > 0$ for $x \in H_\omega$ for any $\omega \in \mathbb{R}$

(A4) $f \in L^2(\log L)^{-\frac{1}{2}}(H_\omega, \gamma)$ for any $\omega \in \mathbb{R}$.

Definition 1. *A function $u \in V(\mathbb{R}^N)$ is a weak solution to problem* (3.1) *if $c\,|u|^{p-1} u \in L^1(H_\omega, \gamma)$ for any $\omega \in \mathbb{R}$ and*

$$\sum_{i,j=1}^N \int_{\mathbb{R}^N} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \psi}{\partial x_j} dx + \int_{\mathbb{R}^N} c\,|u|^{p-1} u\psi \, d\gamma = \int_{\mathbb{R}^N} f\psi \, d\gamma \tag{3.2}$$

for every $\psi \in H^1(H_\omega, \gamma) \cap L^\infty(H_\omega)$ with support contained in H_ω for any $\omega \in \mathbb{R}$.

We stress that under our assumptions all terms in (3.2) are well-defined. Obviously in Definition (1) we can consider any half-space

$$\{x \in \mathbb{R}^N : x \cdot \xi > \omega\} \quad \forall \xi \in \mathbb{R}^N \text{ with } \|\xi\| = 1 \text{ and } \forall \omega \in \mathbb{R},$$

not only the ones with boundary perpendicular to e_1.

Let us fix $\omega_0 > 0$ and let us introduce for any $\omega \in \mathbb{R}$ the auxiliary function

$$\Theta_\omega(x) = \theta_\omega^2(x_1) \text{ for any } x_1 \in \mathbb{R},$$

where $\theta_\omega \in C^\infty(\mathbb{R})$ is such that $\theta_\omega(x_1) = 1$ for $x_1 \geq \omega + \omega_0$ and $\theta_\omega(x_1) = 0$ for $x_1 \leq \omega$.

Theorem 2. *Let us suppose that (A1)-(A4) hold. Then, there exists a unique weak solution in the sense of Definition 1 to problem* (3.1) *such that $c\,|u|^{p+1} \in L^1(H_\omega, \gamma)$ for any $\omega \in \mathbb{R}$ and* (3.2) *holds for $\psi = u\Theta_\omega$ for any $\omega \in \mathbb{R}$.*

Proof. **Step 1. Existence.** For a given $M \in \mathbb{N}$ let us consider the following localized problem

$$\begin{cases} -\sum_{i,j=1}^{N} \partial_{x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\right) + c(x)\,|u(x)|^{p-1}\,u(x)\varphi(x) = f(x)\varphi(x) & \text{in } H_{-M} \\ u = 0, & \text{on } \partial H_{-M}. \end{cases} \tag{P_{-M}}$$

We will adapt Brezis-Browder's proof (see [9]) to prove the existence and uniqueness of a weak solution $u_M \in H_0^1(H_{-M}, \gamma)$, *i.e.* $c|u_M|^p \in L^1(H_{-M}, \gamma)$ and

$$\sum_{i,j=1}^{N} \int_{H_{-M}} a_{ij}\frac{\partial u_M}{\partial x_i}\frac{\partial \psi}{\partial x_j}\,dx + \int_{H_{-M}} c\,|u_M|^{p-1}\,u_M\psi\,d\gamma = \int_{H_{-M}} f\psi\,d\gamma \tag{3.3}$$

for $\psi \in H_0^1(H_{-M}) \cap L^\infty(H_{-M})$. Moreover we get $c|u_M|^{p+1} \in L^1(H_{-M}, \gamma)$ and for $\omega > -M$

$$\sum_{i,j=1}^{N} \int_{H_\omega} a_{ij}\frac{\partial u_M}{\partial x_i}\frac{\partial}{\partial x_j}(u_M\Theta_\omega)dx + \int_{H_\omega} c\,|u_M|^{p+1}\,\Theta_\omega\,d\gamma = \int_{H_\omega} fu_M\Theta_\omega\,d\gamma. \tag{3.4}$$

Indeed take $\psi = T_m(u_M)\Theta_\omega$ in (3.3), where

$$T_m(r) := \min\{m, |r|\}\,\mathrm{sign}(r) \quad \forall r \in \mathbb{R} \text{ and } \forall m \in \mathbb{N}. \tag{3.5}$$

When m goes to ∞ we get (3.4).

Finally we extend u_M by zero over $\mathbb{R}^N \setminus H_{-M}$ and we denote again this extension by u_M.

Now we prove an estimate of u_M, which is independent of M thanks to the crucial assumption $p > 1$.

Lemma 3. *Let us assume that $u \in H^1(H_{-M}, \gamma)$, $c|u|^{p+1} \in L^1(H_{-M}, \gamma)$, $\omega_0 > 0$ and for any $\omega > -M$*

$$\sum_{i,j=1}^{N} \int_{H_\omega} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_j}(u\Theta_\omega)dx + \int_{H_\omega} c\,|u|^{p+1}\,\Theta_\omega\,d\gamma \leq \int_{H_\omega} fu\Theta_\omega\,d\gamma \tag{3.6}$$

holds when $f \in L^2\log L^{-\frac{1}{2}}(H_\omega, \gamma)$ for any $\omega > -M$. Then

$$\begin{aligned} \int_{H_{\omega+\omega_0}} |\nabla u|^2\,d\gamma \leq & K_1\left[\int_0^{+\infty} s^{\frac{2(m-1)(p+1)}{p-1} - \frac{4m}{p-1}}\varphi(\omega+s)\,ds + \int_0^{+\infty} s^{\frac{2(m-1)(p+1)}{p-1}}\varphi(\omega+s)\,ds\right] \\ & + K_2\|f\|^2_{L^2\log L^{-\frac{1}{2}}(H_\omega,\gamma)} \end{aligned} \tag{3.7}$$

and

$$\int_{H_{\omega+\omega_0}} |u|^{p+1}\,d\gamma \le K_1\left[\int_0^{+\infty} s^{\frac{2(m-1)(p+1)}{p-1}-\frac{4m}{p-1}}\varphi(\omega+s)\,ds + \int_0^{+\infty} s^{\frac{2(m-1)(p+1)}{p-1}}\varphi(\omega+s)\,ds\right] + K_2\|f\|^2_{L^2\log L^{-\frac{1}{2}}(H_\omega,\gamma)} \tag{3.8}$$

for some positive constant K_1 and K_2 independent of u, ω, p and f.

Proof. Indeed by (A2) and (A3) we get

$$\begin{aligned}\alpha\int_{H_{\omega+\omega_0}}|\nabla u|^2\,d\gamma &= \alpha\int_{H_{\omega+\omega_0}}|\nabla u|^2\Theta_\omega\,d\gamma \le \alpha\int_{H_\omega}|\nabla u|^2\Theta_\omega\,d\gamma \le \sum_{i,j=1}^N\int_{H_\omega} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\Theta_\omega dx\\ &= \sum_{i,j=1}^N\int_{H_\omega} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_j}(u\Theta_\omega)dx - \sum_{i,j=1}^N\int_{H_\omega} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \Theta_\omega}{\partial x_j}u\,dx\end{aligned} \tag{3.9}$$

and

$$c_0\int_{H_{\omega+\omega_0}}|u|^{p+1}\,d\gamma = c_0\int_{H_{\omega+\omega_0}}|u|^{p+1}\Theta_\omega\,d\gamma \le c_0\int_{H_\omega}|u|^{p+1}\Theta_\omega\,d\gamma \le \int_{H_\omega} c|u|^{p+1}\Theta_\omega\,d\gamma. \tag{3.10}$$

Using (3.6), (3.9) and (3.10) we get

$$\alpha\int_{H_\omega}|\nabla u|^2\Theta_\omega\,d\gamma + \int_{H_\omega}c_0\,|u|^{p+1}\,\Theta_\omega\,d\gamma \le \int_{H_\omega} fu\Theta_\omega\,d\gamma - \sum_{i,j=1}^N\int_{H_\omega}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial\Theta_\omega}{\partial x_j}u\,dx. \tag{3.11}$$

Since $p > 1$, by Young inequality we get that

$$\begin{aligned} -\sum_{i,j=1}^N\int_{H_\omega}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial\Theta_\omega}{\partial x_j}u\,dx &= -\sum_{i,j=1}^N\int_{H_\omega}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_j}\left[\theta^2_\omega(x_1)\right]u\,dx\\ &\le -2\sum_{i,j=1}^N\int_{H_\omega}a_{ij}\frac{\partial u}{\partial x_i}\theta'_\omega(x_1)\theta_\omega(x_1)\,u\,dx \le 2\max_{i,j}\left\|\frac{a_{ij}}{\varphi}\right\|_\infty\int_{H_\omega}|\nabla u|\left|\theta'_\omega(x_1)\right|\theta_\omega(x_1)\,|u|\,d\gamma\\ &\le 2\varepsilon\max_{i,j}\left\|\frac{a_{ij}}{\varphi}\right\|_\infty\int_{H_\omega}|\nabla u|^2\Theta_\omega\,d\gamma + 2C(\varepsilon)\max_{i,j}\left\|\frac{a_{ij}}{\varphi}\right\|_\infty\int_{H_\omega}\left|\theta'_\omega(x_1)\right|^2|u|^2\,d\gamma\\ &\le 2\varepsilon\max_{i,j}\left\|\frac{a_{ij}}{\varphi}\right\|_\infty\int_{H_\omega}|\nabla u|^2\Theta_\omega\,d\gamma + 2\delta C(\varepsilon)\max_{i,j}\left\|\frac{a_{ij}}{\varphi}\right\|_\infty\int_{H_\omega}|u|^{p+1}\Theta_\omega\,d\gamma\\ &\quad + 2C(\delta,\varepsilon)\max_{i,j}\left\|\frac{a_{ij}}{\varphi}\right\|_\infty\int_{H_\omega}\left|\theta'_\omega(x_1)\right|^{\frac{2(p+1)}{p-1}}\Theta_\omega^{-\frac{2}{p-1}}\,d\gamma\end{aligned} \tag{3.12}$$

for some positive constants ε and δ that can be chosen later. Moreover we have

$$\int_{H_\omega} fu\Theta_\omega\,d\gamma \le C(\varepsilon')\|f\|^2_{L^2\log L^{-\frac{1}{2}}(H_\omega,\gamma)} + \varepsilon'\|u\Theta_\omega\|^2_{L^2\log L^{\frac{1}{2}}(H_\omega,\gamma)} \tag{3.13}$$

for some positive constant ε' that can be chosen later. Sobolev inequality (2.10) and Young inequality allow us to obtain

$$\begin{aligned}\|u_M\Theta_\omega\|^2_{L^2\log L^{\frac{1}{2}}(H_\omega)} &\le k_0\left[\int_{H_\omega}|u_M\Theta_\omega|^2\,d\gamma + \int_{H_\omega}\left|\nabla(u\Theta_\omega)\right|^2\,d\gamma\right]\\ &\le k_1\gamma(H_\omega) + k_2\int_{H_\omega}|u|^{p+1}\,\Theta_\omega + k_3\int_{H_\omega}|\nabla u|^2\,\Theta_\omega\,d\gamma + k_4\int_{H_\omega}\left|\theta'_\omega(x_1)\right|^2u^2\Theta_\omega\,d\gamma\end{aligned} \tag{3.14}$$

for some positive constants k_0, k_1, k_2, k_3, k_4. As before we get

$$\int_{H_\omega} \left|\theta_\omega'(x_1)\right|^2 |u|^2 \Theta_\omega \, d\gamma \le \delta' \int_{H_\omega} |u|^{p+1} \Theta_\omega \, d\gamma + C(\delta') \int_{H_\omega} \left|\theta(x_1)'\right|^{\frac{2(p+1)}{p-1}} \Theta_\omega \, d\gamma. \tag{3.15}$$

for some positive constant δ' that can be chosen later. Taking $\theta_\omega'(x_1) = O\left(|\omega - x_1|^{m-1}\right)$ with $m > 0$, then

$$\begin{aligned}\int_{H_\omega} \left|\theta_\omega'(x_1)\right|^{\frac{2(p+1)}{p-1}} \Theta_\omega^{-\frac{2}{p-1}}(x_1) \, d\gamma &\le k_5 \int_{H_\omega} |\omega - x_1|^{\frac{2(m-1)(p+1)}{p-1} - \frac{4m}{p-1}} \, d\gamma \\ &= k_5 \int_0^{+\infty} s^{\frac{2(m-1)(p+1)}{p-1} - \frac{4m}{p-1}} \varphi(\omega + s) \, ds\end{aligned} \tag{3.16}$$

for some positive constant k_5 and the last integral is finite if $m > \frac{p+1}{p-1}$. Moreover

$$\int_{H_\omega} \left|\theta_\omega'(x_1)\right|^{\frac{2(p+1)}{p-1}} \Theta_\omega(x_1) \, d\gamma \le k_6 \int_0^{+\infty} s^{\frac{2(m-1)(p+1)}{p-1}} \varphi(\omega + s) \, ds \tag{3.17}$$

for some positive constant k_6 and the last integral is finite if $m > 1$. Choosing $\varepsilon, \varepsilon', \delta$ and δ' small enough and using (3.16), (3.17), (3.15), (3.14), (3.13), (3.12) in (3.11) we get

$$\begin{aligned}&\int_{H_\omega} |\nabla u|^2 \Theta_\omega \, d\gamma + \int_{H_\omega} |u|^{p+1} \Theta_\omega \, d\gamma \\ &\le k_7 \left[\int_0^{+\infty} s^{\frac{2(m-1)(p+1)}{p-1} - \frac{4m}{p-1}} \varphi(\omega + s) \, ds + \int_0^{+\infty} s^{\frac{2(m-1)(p+1)}{p-1}} \varphi(\omega + s) \, ds \right] + k_8 \|f\|^2_{L^2 \log L^{-\frac{1}{2}}(H_\omega, \gamma)}\end{aligned} \tag{3.18}$$

for some positive constant k_7, k_8. Using (3.9), (3.10) and (3.18) we obtain (3.7) and (3.8). □

Using (3.7) and (3.8) we can conclude that u_M is bounded in $H^1(H_{\omega+\omega_0}, \gamma)$ and it follows that there exists u such that (up a subsequence) $u_M \to u$ weakly in $H^1(H_\omega, \gamma)$ for any $\omega \in \mathbb{R}$, weakly in $L^{p+1}(H_\omega, \gamma)$ for any $\omega \in \mathbb{R}$, strongly in $L^q(H_\omega, \gamma)$ for any $\omega \in \mathbb{R}$ with $q < p + 1$ and *a.e.* in H_ω for any $\omega \in \mathbb{R}$. Using these convergences and the monotonicity of function $G(s) = |s|^{p-1}s$ we can pass to the limit in (3.3) and we conclude.

Step 2. Uniqueness. Let u_1 and u_2 be two different weak solutions to Problem (3.1) such that $c|u_1|^{p+1}, c|u_2|^{p+1} \in L^1(H_\omega, \gamma)$ for any $\omega \in \mathbb{R}$. We stress that they satisfy (3.2) with $\psi = u_1\Theta_\omega$ and $\psi = u_2\Theta_\omega$ for any $\omega \in \mathbb{R}$. Let $v = u_1 - u_2$. Since $v \in H^1(H_\omega, \gamma) \cap L^{p+1}(H_\omega, \gamma)$ for any $\omega \in \mathbb{R}$ we get

$$\sum_{i,j=1}^N \int_{\mathbb{R}^N} a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial(v\Theta_\omega)}{\partial x_j} dx + \int_{\mathbb{R}^N} \Gamma(x) \, |v|^{p+1} \, \Theta_\omega \, d\gamma = 0,$$

where

$$\Gamma(x) = \begin{cases} c(x) \frac{|u_1(x)|^{p-1}u_1(x) - |u_2(x)|^{p-1}u_2(x)}{|u_1(x) - u_2(x)|^{p-1}(u_1(x) - u_2(x))} & \text{if } u_1(x) \ne u_2(x) \\ c_0 & \text{if } u_1(x) = u_2(x). \end{cases}$$

For $p > 1$ we have $\Gamma(x) \ge c_0$ *a.e.* in $\mathbb{R}^N$. Then Lemma 3 can be applied obtaining

$$\int_{H_{\omega+\omega_0}} |\nabla u_M|^2 \, d\gamma \le K_1 \left[\int_0^{+\infty} s^{\frac{2(m-1)(p+1)}{p-1} - \frac{4m}{p-1}} \varphi(\omega + s) \, ds \right] \tag{3.19}$$

and

$$\int_{H_{\omega+\omega_0}} |u_M|^{p+1} \, d\gamma \le K_1 \left[\int_0^{+\infty} s^{\frac{2(m-1)(p+1)}{p-1} - \frac{4m}{p-1}} \varphi(\omega + s) \, ds \right] \tag{3.20}$$

for some positive constant K_1 . Lebesgue's dominated convergence theorem allows

$$\lim_{\omega\to-\infty}\int_0^{+\infty} s^{\frac{2(m-1)(p+1)}{p-1}-\frac{4m}{p-1}}\varphi(\omega+s)\,ds=0.$$

Putting ω goes to $-\infty$ we get $\nabla v = 0$ *a.e.* on $\mathbb{R}^N$ by (3.19) and then $v = 0$ *a.e.* on $\mathbb{R}^N$ using (3.20). This is a contradiction and the uniqueness result follows. □

Remark 4. *Arguing as in the proof of the uniqueness result it is possible to prove that if u_1 is a weak supersolution and u_2 is a weak solution to problem* (3.1)*, then $u_1 \le u_2$ a.e. on $\mathbb{R}^N$.*

Remark 5. *We stress that Definition 1 and Theorem 2 can be easily adapted to problems defined in open subsets of $\mathbb{R}^N$ with $\gamma(\Omega) = 1$.*

Remark 6. *The hypothesis $p > 1$ is crucial to prove existence of a weak solution in the sense of Definition 1 relaxing the standard assumption on the datum, namely f belongs to the dual space of $H^1(\mathbb{R}^N,\gamma)$. Otherwise in the case $p \le 1$ if the datum belongs to the dual space of the energy space, the existence of a solution in $H^1(\mathbb{R}^N,\gamma)$ follows arguing as in Theorem 4.2 of [15], because the space $H^1(\mathbb{R}^N,\gamma)$ coincides with $H^1(\mathbb{R}^N,\gamma)\cap L^{p+1}(\mathbb{R}^N,\gamma)$ and both topologies are equivalent.*

Remark 7. *Since in Theorem 2 the existence (and uniqueness) of solutions is obtained without any decay condition on f it is natural to search about possible decay estimates of the solutions when $|x| \to +\infty$. The estimates obtained in [19] was extended to other different settings by several authors (see, e.g. [29] and its references).*

4 Comparison results in terms of the half-space Gaussian symmetrization

In this section we will give results comparing a solution of problem of type (1.1) with the solution to a simpler problem defined in an half-space having data depending only on one variable.

We need starting with the case $\gamma(\Omega) < 1$ and we consider the following class of Dirichlet problems

$$\begin{cases} -\sum_{i,j=1}^{N}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\right)+c(x)\left|u(x)\right|^{p-1}u(x)\varphi(x)=f(x)\varphi(x) & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

The structural assumptions (instead of $(A1) - (A4)$) are now the following:

- $(A0)$ Ω is an open subset of $\mathbb{R}^N$ ($N \ge 2$) such that $\gamma(\Omega) < 1$,
- $(A1')$ $p > 0$,
- $(A2')$ $\frac{a_{ij}}{\varphi} \in L^\infty(\Omega)$ and $\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge \alpha\varphi(x)\,|\xi|^2$ for *a.e.* $x \in \Omega, \forall \xi \in \mathbb{R}^N$ with $\alpha > 0$,
- $(A3')$ $c \in L^1(\Omega,\gamma)$, $c(x) \ge c_0 > 0$,
- $(A4')$ $f \in L^2\log L^{-1/2}(\Omega,\gamma)$.

We recall that under assumption $(A2')$ a Poincaré inequality holds and that $f \in L^2\log L^{-1/2}(\Omega,\gamma)$ can be identified with an element in the dual space of $H_0^1(\Omega,\gamma)$ (see [4]). Then, under our assumptions, all terms in the corresponding notion of weak formulation are well-defined using (2.10) and (2.8). Since $\gamma(\Omega) < 1$, the existence of a weak solution to (4.1) comes easily by adapting the Brezis-Browder's proof ([9]). Moreover, the uniqueness of solutions is standard.

The first result of this section shows a suitable integral comparison between the solution u to problem (3.1) and the solution v to the following symmetrized problem

$$\begin{cases} -\operatorname{div}(\alpha\nabla v(x)\varphi(x)) + c_0 \left|v(x)\right|^{p-1} v(x)\varphi(x) = \widetilde{f}(x)\varphi(x) & \text{in } \Omega^\bigstar \\ v = 0 & \text{on } \partial\Omega^\bigstar, \end{cases} \tag{4.2}$$

where $\Omega^\bigstar$ is the half-space defined in (2.6) and $\widetilde{f}$ is such that $\widetilde{f} = \widetilde{f}^\bigstar$, the rearrangement with respect to Gauss measure of $\widetilde{f}$ defined in (2.5).

First of all, we prove that the solution to (4.2) coincides with its half-space Gaussian rearrangement.

Proposition 8. *Let assume that $\widetilde{f} \in L^2 \log L^{-\frac{1}{2}}(\Omega^\bigstar, \gamma)$. Then problem (4.2) has a unique nonnegative weak solution such that $v(x) = v^\bigstar(x)$ in $\Omega^\bigstar$.*

Proof. Since $\widetilde{f} = \widetilde{f}^\bigstar$, then $\widetilde{f} \geq 0$. As a consequence the existence of a unique nonnegative weak solution is standard. We only detail the proof of $v(x) = v^\bigstar(x)$.

Let $\widetilde{v}(x_1)$ be the solution to

$$\begin{cases} -\left(\alpha\widetilde{v}'(x_1)\varphi_1(x_1)\right)' + c_0 \left|\widetilde{v}(x_1)\right|^{p-1} \widetilde{v}(x_1)\varphi_1(x_1) = \widetilde{f}(x_1)\varphi_1(x_1) & \text{in } (\omega, +\infty) \\ \widetilde{v}(\omega) = 0, \end{cases}$$

where $\varphi_1(x_1)$ is the density of 1-dimensional Gauss measure and ω is such that $\gamma(\Omega^\star) = \gamma(\Omega)$. By uniqueness $v(x) = \widetilde{v}(x_1)$ is the unique weak solution to Problem (4.2). Thus it remains to be proved the monotonicity. Since

$$\frac{1}{\alpha}\int_{x_1}^{+\infty}\left[\widetilde{f}(s) - c_0\left|\widetilde{v}(s)\right|^{p-1}\widetilde{v}(s)\right]\varphi_1(s)\,ds = \widetilde{v}'(x_1)\varphi_1(x_1) \text{ for } x_1 \geq \omega,$$

it is enough to show

$$\Psi(x_1) := \int_{x_1}^{+\infty}\left[\widetilde{f}(s) - c_0\left|\widetilde{v}(s)\right|^{p-1}\widetilde{v}(s)\right]\varphi_1(s)\,ds \geq 0 \text{ for } x_1 \geq \omega. \tag{4.3}$$

Suppose that $\Psi(x_1) < 0$ for some $x_1 \geq \omega$ and consider $\overline{x_1} \in [\omega, +\infty)$ such that $\Psi(\overline{x_1}) = \min_{[\omega,+\infty)} \Psi(x_1)$. It is obvious that $\Psi(\overline{x_1}) < 0$. We have that $\overline{x_1} > \omega$, otherwise it follows that $\widetilde{v}'(x_1) < 0$ in some neighborhood of ω, in contrast with $\widetilde{v}(x_1) \geq 0$ in $(\omega, +\infty)$ and $\widetilde{v}(\omega) = 0$.

In a similar way we show that there exists $\underline{x_1} \in (\omega, \overline{x_1})$ such that $\Psi(\underline{x_1}) > 0$. Indeed otherwise $\widetilde{v}'(x_1) \leq 0$ in $(\omega, \overline{x_1})$, in contrast with $\widetilde{v}(x_1) \geq 0$ in $(\omega, +\infty)$ and $\widetilde{v}(\omega) = 0$.

Since $\Psi(\underline{x_1}) > 0$ there exists $\widetilde{x_1} \leq \overline{x_1} \leq \widehat{x_1}$ such that $\Psi(\widetilde{x_1}) = 0$ and $\Psi(x_1) < 0$ in $\left(\widetilde{x_1}, \widehat{x_1}\right)$ and $\min_{[\widetilde{x_1},\widehat{x_1}]} \Psi(x_1) = \Psi(\overline{x_1})$. Then $\widetilde{v}'(x_1) \leq 0$ in $\left[\widetilde{x_1}, \widehat{x_1}\right]$. As a consequence $\widetilde{f} - c_0\left|\widetilde{v}\right|^{p-1}\widetilde{v}$ is increasing in $\left[\widetilde{x_1}, \widehat{x_1}\right]$, *i.e.* $\Psi'(x_1)$ is decreasing and $\Psi(x_1)$ is concave in $\left[\widetilde{x_1}, \widehat{x_1}\right]$. Since $\Psi(x_1)$ has a minimum in $\left[\widetilde{x_1}, \widehat{x_1}\right]$), it follows that $\Psi(x_1) \equiv 0$ in $\left[\widetilde{x_1}, \widehat{x_1}\right]$, in contrast with $\Psi(\overline{x_1}) < 0$ and $\Psi(\widetilde{x_1}) = 0$. This proves (4.3). □

Now we are in position to prove the following comparison result.

Theorem 9. *Assume that (A0) and (A1′)-(A4′) hold. Let f be a nonnegative function, $\widetilde{f} \in L^2 \log L^{-1/2}(\Omega^\bigstar, \gamma)$, let u and v be the nonnegative weak solution of* (4.1) *and* (4.2), *respectively. Then*

$$\|(\mathcal{U} - \mathcal{V})_+\|_{L^\infty(0,\gamma(\Omega))} \leq \frac{1}{c_0}\|(\mathcal{F} - \widetilde{\mathcal{F}})_+\|_{L^\infty(0,\gamma(\Omega))},$$

where

$$\mathcal{U}(s)=\int_0^s [u^{\circledast}(t)]^p\,dt \qquad \mathcal{V}(s)=\int_0^s [v^{\circledast}(t)]^p\,dt \tag{4.4}$$

$$\mathcal{F}(s)=\int_0^s f^{\circledast}(t)\,dt \qquad \widetilde{\mathcal{F}}(s)=\int_0^s \widetilde{f}^{\circledast}(t)\,dt,$$

for $s\in(0,\gamma(\Omega)]$ and $g^{\circledast}(t)$, the decreasing rearrangement of g with respect to Gauss measure, is defined in (2.4). *In particular, if we suppose that*

$$\mathcal{F}(s)\le\widetilde{\mathcal{F}}(s) \qquad \textit{for any } s\in[0,\gamma(\Omega)],$$

then

$$\mathcal{U}(s)\le\mathcal{V}(s) \qquad \textit{for any } s\in[0,\gamma(\Omega)].$$

Proof. We argue as in [16], [18]. Let us define the functions $u_{\kappa,t}:\Omega\to\mathbb{R}$ as

$$u_{\kappa,t}(x)=\begin{cases}0 & \text{if } |u(x)|\le t,\\ (|u(x)|-t)\,\mathrm{sign}\,(u(x)) & \text{if } t<|u(x)|\le t+\kappa\\ \kappa\,\mathrm{sign}\,(u(x)) & \text{if } t+\kappa<|u(x)|\end{cases}$$

for any fixed t and $\kappa>0$. Observing that $u_{\kappa,t}$ belongs to $H_0^1(\Omega,\gamma)$, and $\nabla u_{\kappa,t}=\chi_{\{t<|u|\le t+\kappa\}}\nabla u$ *a.e.* in Ω, function $u_{\kappa,t}$ can be chosen as test function in (18) and by $(A3')$ we get

$$\frac{1}{\kappa}\int_{t<|u|\le t+\kappa}|\nabla u|^2\,d\gamma+\frac{1}{\kappa}\int_{t<|u|\le t+\kappa}c\,|u|^{p-1}u\,(|u|-t)\,\mathrm{sign}(u)\,d\gamma+\int_{|u|>t+\kappa}c\,|u|^{p-1}u\,\mathrm{sign}(u)\,d\gamma$$
$$\le\frac{1}{\kappa}\int_{t<|u|\le t+\kappa}f(|u|-t)\,\mathrm{sign}(u)\,d\gamma+\int_{|u|>t+\kappa}f\,\mathrm{sign}(u)\,d\gamma\,.$$

In the standard way by $(A5)$ we have

$$-\frac{d}{dt}\int_{|u|>t}|\nabla u|^2\,d\gamma\le\int_{|u|>t}|f|\,d\gamma-\int_{|u|>t}c_0\,|u|^{p-1}u\,\mathrm{sign}u\,d\gamma \qquad \text{for } t>0.$$

By Hardy-Littlewood inequality (2.7), we obtain

$$-\frac{d}{dt}\int_{|u|>t}|\nabla u|^2\,d\gamma\le\int_0^{\gamma_u(t)}f^{\circledast}(s)\,ds-\int_0^{\gamma_u(t)}c_0[u^{\circledast}(s)]^p\,ds \qquad \text{for } t>0.$$

Using (2.1) by standard arguments (see [33]) it follows that

$$1\le\frac{-\gamma_u'(t)}{\left(\varphi_1\left(\Phi^{-1}\left(\gamma_u(t)\right)\right)\right)^2}\left[\mathcal{F}(\gamma_u(t))-c_0\,\mathcal{U}(\gamma_u(t))\right] \qquad \text{for } t>0.$$

Then

$$\left(-u^{\circledast}(s)\right)'\le\frac{1}{\left(\varphi_1\left(\Phi^{-1}(s)\right)\right)^2}\left[\mathcal{F}(s)-c_0\,\mathcal{U}(s)\right] \qquad \text{for } s\in(0,\gamma(\Omega)). \tag{4.5}$$

By (4.4) the derivative of $\mathcal{U}$ equals

$$\mathcal{U}'(s)=[u^{\circledast}(s)]^p \qquad \text{for } a.e.\ s\in(0,\gamma(\Omega)). \tag{4.6}$$

Relations (4.5), (4.6) and (4.4) yield

$$\begin{cases} (\varphi_1(\Phi^{-1}(s)))^2\left[-\frac{d}{ds}\left(\left(\mathcal{U}'(s)\right)^{\frac{1}{p}}\right)\right] + c_0\,\mathcal{U}(s) \le \mathcal{F}(s) & \text{for } s \in (0,\gamma(\Omega)) \\ \mathcal{U}(0)=0, \quad \mathcal{U}'(\gamma(\Omega))=0\,. \end{cases} \tag{4.7}$$

Now let us consider problem (4.2). The solution v to problem (4.2) is unique and $v(x) = v^{\star}(x)$ (see Proposition 8).

By the properties of v we can repeat arguments used to prove (4.5) replacing all the inequalities by equalities and obtaining

$$\left(-v^{\circledast}(s)\right)' = \frac{1}{(\varphi_1(\Phi^{-1}(s)))^2}\left[\widetilde{\mathcal{F}}(s) - c_0\mathcal{U}(s)\right] \qquad \text{for } s \in (0,\gamma(\Omega))\,, \tag{4.8}$$

and thus

$$\begin{cases} (\varphi_1(\Phi^{-1}(s)))^2\left[-\frac{d}{ds}\left(\left(\mathcal{V}'(s)\right)^{\frac{1}{p}}\right)\right] + c_0\mathcal{V}(s) = \widetilde{\mathcal{F}}(s) & \text{for } s \in (0,\gamma(\Omega)) \\ \mathcal{V}(0)=0, \quad \mathcal{V}'(\gamma(\Omega))=0. \end{cases} \tag{4.9}$$

Putting together (4.7) and (4.9) we get

$$\begin{aligned} \left(\varphi_1\left(\Phi^{-1}(s)\right)\right)^2 &\left[-\frac{d}{ds}\left(\left(\mathcal{U}'(s)\right)^{\frac{1}{p}}\right) + \frac{d}{ds}\left(\left(\mathcal{V}'(s)\right)^{\frac{1}{p}}\right)\right] \\ &\le \mathcal{F}(s) - \widetilde{\mathcal{F}}(s) + c_0\left(\mathcal{V}(s) - \mathcal{U}(s)\right) \end{aligned} \tag{4.10}$$

Since $\mathcal{U}, \mathcal{V} \in \mathcal{C}([0,\gamma(\Omega)])$, there exists $s_0 \in (0,\gamma(\Omega))$ such that

$$\|(\mathcal{U}-\mathcal{V})_+\|_{L^\infty(0,\gamma(\Omega))} = \frac{1}{c_0}(\mathcal{U}-\mathcal{V})(s_0).$$

We argue by absurdum. Suppose that

$$(\mathcal{U}-\mathcal{V})(s_0) > \frac{1}{c_0}\|(\mathcal{F}-\widetilde{\mathcal{F}})_+\|_{L^\infty(0,\gamma(\Omega))}\,.$$

If $s_0 < \gamma(\Omega)$, by (4) it follows that

$$\mathcal{F}(s) - \widetilde{\mathcal{F}}(s) + c_0\left(\mathcal{V}(s)-\mathcal{U}(s)\right) \le \|(\mathcal{F}-\widetilde{\mathcal{F}})_+\|_{L^\infty(0,\gamma(\Omega))} - c_0(\mathcal{U}-\mathcal{V})(s) < 0 \qquad \text{for } s \in (s_0-\varepsilon, s_0+\varepsilon). \tag{4.11}$$

We set

$$Z = \mathcal{U} - \mathcal{V} \in H^{2,\infty}(s_0-\varepsilon, s_0+\varepsilon).$$

Then

$$\left(\mathcal{U}'(s)\right)^{\frac{1}{p}} - \left(\mathcal{V}'(s)\right)^{\frac{1}{p}} = Z'(s)\,\rho(s), \tag{4.12}$$

where

$$\rho(s) = \int_0^1 \frac{1}{p}\left(\tau\mathcal{U}'(s) + (1-\tau)\mathcal{V}'(s)\right)^{\frac{1}{p}-1} d\tau > 0. \tag{4.13}$$

As a consequence of (4.10), (4.11), (4.12) and (4.13) we obtain

$$\begin{aligned} \left(\varphi_1\left(\Phi^{-1}(s)\right)\right)^2 &\left[-\frac{d}{ds}\left(\left(\mathcal{U}'(s)\right)^{\frac{1}{p}}\right) + \frac{d}{ds}\left(\left(\mathcal{V}'(s)\right)^{\frac{1}{p}}\right)\right] \\ &= \left(\varphi_1\left(\Phi^{-1}(s)\right)\right)^2 h(s)\frac{d}{ds}\left(\rho(s)Z'(s)\right) < 0, \end{aligned}$$

where

$$h(s) = \int_0^1 \left\{ \left[-\tau \frac{d}{ds} \left((\mathcal{U}'(s))^{\frac{1}{p}} - (1-\tau) \frac{d}{ds} \left((\mathcal{V}'(s))^{\frac{1}{p}} \right) \right] \right\} d\tau > 0.$$

We can conclude that

$$-\frac{d}{ds}\left(\varrho(s)Z'(s)\right) < 0 \qquad \text{for } s \in (s_0 - \varepsilon, s_0 + \varepsilon), \tag{4.14}$$

which is in contradiction with the assumption that Z has a maximum in s_0.
If $s_0 = \gamma(\Omega)$, (4.14) holds for $(\gamma(\Omega) - \varepsilon, \gamma(\Omega))$, then $Z'(\gamma(\Omega)) > 0$, but we know that $Z'(\gamma(\Omega)) = 0$ and again a contradiction arises. □

Remark 10. *Under the same assumption of Theorem 9, it is well-known that we deduce also that*

$$\|u\|_{L^{rp}(\Omega,\gamma),} \le \|v\|_{L^{rp}(\Omega^\star,\gamma)} \quad \textit{for any } 1 \le r \le \infty.$$

Remark 11. *As mentioned in the Introduction, when Ω is bounded the usual radially symmetrization method, when applied to an elliptic operator with a drift term as (1.2), modifies drastically the drift term in the symmetrized equation. For instance, we can apply Theorem 1 of [34] to equation (1.2), which can be formulated in divergence form as*

$$-\Delta u(x) + \operatorname{div}(xu) - Nu + c_0 \left|u(x)\right|^{p-1} u(x) = f(x),$$

so that in the notation of [34] we must take $b_i(x) = x_i$ and $c(x) = -N$. Then the corresponding symmetrized problem built in [34] is

$$\begin{cases} -\Delta v(y) + B\frac{y}{|y|} \cdot \nabla v(y) + Bv \operatorname{div}\left(\frac{y}{|y|}\right) - Nv + c_0 \left|v(y)\right|^{p-1} v(y) = f^\#(y) & \text{in } \Omega^\#, \\ v = 0 & \text{on } \partial\Omega^\#, \end{cases} \tag{4.15}$$

where $\Omega^\#$ is now a ball with the same volume than Ω, $B = \||x|\|_{L^\infty(\Omega)}$, and $f^\#$ is, for instance, the radially decreasing symmetric rearrangement of f. Notice that, in contrast with the "artificial" first order terms arising in problem (4.15), the half-space Gaussian symmetrization problem (4.2) preserves the same type of drift than the original problem (1.2)

Using the Definition 1 the above comparison result can be extended to the case of $\Omega = \mathbb{R}^N$ and $p > 1$ under the assumptions of the previous Section.

Theorem 12. *Let $\Omega = \mathbb{R}^N$, $p > 1$ and the rest of conditions of Theorem 2. Let f be a nonnegative function and let $\widetilde{f} \in L^2 \log L^{-1/2}(H^\star_\omega, \gamma)$ for any $\omega \in \mathbb{R}$ such that $\widetilde{f} = \widetilde{f}^\star$. Then*

i) problem

$$-\operatorname{div}\left(\alpha \nabla v(x)\varphi(x)\right) + c_0 \left|v(x)\right|^{p-1} v(x)\varphi(x) = \widetilde{f}(x)\varphi(x) \text{ in } \mathbb{R}^N \tag{4.16}$$

admits a unique weak solution v.

ii) let u and v be the nonnegative weak solution of (3.1) *and of* (4.16)*, respectively. Then, for any $\varepsilon \in (0, 1)$*

$$\|(\mathcal{U} - \mathcal{V})_+\|_{L^\infty(0,1-\varepsilon))} \le \frac{1}{c_0} \|(\mathcal{F} - \widetilde{\mathcal{F}})_+\|_{L^\infty(0,1-\varepsilon)},$$

where $\mathcal{U}$, $\mathcal{V}$, $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are defined as in (4.4) *for $s \in (0, 1)$. In particular, if we suppose that*

$$\mathcal{F}(s) \le \widetilde{\mathcal{F}}(s) \qquad \textit{for any } s \in [0, 1),$$

then

$$\mathcal{U}(s) \le \mathcal{V}(s) \qquad \textit{for any } s \in [0, 1).$$

Proof. Part i) is a consequence of Theorem 2 (see also [6]). To prove ii), as in the proof of Theorem 2, we first consider u_M, the solution of the corresponding localized problem (P_{-M}) on H_{-M}, and v_M the solution of the symmetrized problem

$$\begin{cases} -\mathrm{div}(\alpha\nabla v\varphi) + c_0\,|v|^{p-1}\,v\varphi(x) = \widetilde{f}(x)\varphi(x) & \text{in } H_{-M} \\ v = 0 & \text{on } \partial H_{-M}, \end{cases}$$

where now $H_{-M}{}^{\star} = H_{-M}$ and $\widetilde{f}$ is such that $\widetilde{f} = \widetilde{f}^{\star}$. We extend u_M and v_M by zero over $\mathbb{R}^N \setminus H_{-M}$ and we denote again these extensions by u_M and v_M. Then, from the proof of Theorem 2 we know that $\{u_M\}$, $\{v_M\}$ are bounded in $H^1(H_{\omega+\omega_0},\gamma)$ with $\omega \in \mathbb{R}$ and $\omega_0 > 0$. It follows that there exists u and v such that (up a subsequence) $u_M \to u$ and $v_M \to v$ weakly in $H^1(H_\omega)$ for any $\omega \in \mathbb{R}$, weakly in $L^{p+1}(H_\omega,\gamma)$ for any $\omega \in \mathbb{R}$, strongly in $L^q(H_\omega,\gamma)$ for any $\omega \in \mathbb{R}$ with $q < p+1$ and *a.e.* in H_ω for any $\omega \in \mathbb{R}$. Since the rearrangement application $u \to u^\star$ is a contraction in $L^r(H_\omega,\gamma)$ for any $r \geq 1$, we get that $u_M^\star \to u^\star$ in $L^q(H_\omega,\gamma)$ for any $\omega \in \mathbb{R}$ with $q < p+1$ and *a.e.* in H_ω for any $\omega \in \mathbb{R}$ (and weakly in $L^{p+1}(H_\omega,\gamma)$ for any $\omega \in \mathbb{R}$). On the other hand, by the Polya-Sezgo theorem

$$\left\|\nabla u_M^\star\right\|_{L^2(H_{-M},\gamma)} \leq \|\nabla u_M\|_{L^2(H_{-M},\gamma)}$$

which implies (thanks to the assumption $p > 1$: Lemma 3) that $\left\{u_M^\star\right\}$ is bounded in $H^1(H_{\omega+\omega_0},\gamma)$ and thus $u_M^\star \to u^\star$ weakly in $H^1(H_\omega,\gamma)$ for any $\omega \in \mathbb{R}$. In particular, if we define

$$\mathcal{U}_M(s) = \int_0^s [u_M^\circledast(t)]^p\,dt \quad \mathcal{V}_M(s) = \int_0^s [v_M^\circledast(s)]^p\,dt,$$

we get that

$$\|(\mathcal{U}_M - \mathcal{V}_M)_+\|_{L^\infty(0,\gamma(H_{-M}))} \leq \frac{1}{c_0}\|(\mathcal{F} - \widetilde{\mathcal{F}})_+\|_{L^\infty(0,\gamma(H_{-M}))},$$

for any $M \in \mathbb{N}$. Moreover $\mathcal{U}_M \to \mathcal{U}$ and $\mathcal{V}_M \to \mathcal{V}$ strongly on $L^\infty(0,1-\varepsilon)$ for any $\varepsilon \in (0,1)$ and thus we get the desired conclusion. □

Remark 13. *Notice that we have proved the existence and, specially, the uniqueness of a solution of the problem*

$$\begin{cases} \left(\varphi_1\left(\Phi^{-1}(s)\right)\right)^2\left[-\dfrac{d}{ds}\left(\left(\mathcal{V}'(s)\right)^{\frac{1}{p}}\right)\right] + c_0\mathcal{V}(s) = \widetilde{\mathcal{F}}(s) & \textit{for } s \in (0,1) \\ \mathcal{V}(0) = 0, \quad \mathcal{V}'(1) = 0. \end{cases}$$

when we assume only $0 \leq \widetilde{\mathcal{F}}(s)$ *and* $\widetilde{\mathcal{F}} \in L^1_{loc}(0,1)$, *to be more precise* $\widetilde{\mathcal{F}} \in L^1(0,1-\varepsilon)$ *for any* $\varepsilon \in (0,1)$. *For instance we could consider function of the type*

$$\widetilde{\mathcal{F}}(s) = \frac{s}{(1-s)^\alpha}, \textit{ for any } \alpha > 0.$$

This type of questions is related with the study of removable singularities for quasilinear equations (see, e.g. Section 5.2 of Veron [36]). In this theory, usually it is assumed $N \geq 2$.

Remark 14. *The above passing to the limit in* $M \in \mathbb{N}$ *also holds (for very different arguments, see Remark 6) when* $p \leq 1$ *once we assume that* $f \in L^1(\mathbb{R}^N,\gamma)$ *is in the dual space of* $H^1(\mathbb{R}^N,\gamma)$ *(which, essentially, corresponds to the case in which* $\widetilde{\mathcal{F}} \in L^1(0,1)$*).*

We end this Section with a qualitative property for the case $p < 1$ which holds as an application of the above comparison Theorem. First of all we recall that this assumption allows the formation of a free boundary in the sense that if $f = 0$ over some suitable large subset of Ω, with

$$\gamma(\{x \in \Omega : f(x) = 0\}) = s_f \in (0,\gamma(\Omega)),$$

then the solution u of (4.1) have compact support on Ω (*i.e.*, $N_u := \{x \in \Omega : u(x) = 0\}$, contained in $\{x \in \Omega : f(x) = 0\}$, is not empty) (see, *e.g.* [15]). Thus

$$\gamma(\{x \in \Omega : u(x) = 0\}) = \tau, \text{ for some } \tau \in [0, s_f]. \tag{4.17}$$

Notice that in terms of the corresponding function $\mathcal{U}(s)$ it means that $\mathcal{U}$ attaints its maximum on a subinterval $[\gamma(\Omega) - \tau, \gamma(\Omega)]$. Solutions $\mathcal{V}(s)$ of the symmetrized problem (4.9) may have also this property (once the data $\widetilde{\mathcal{F}}(s)$ take its maximum on an interval $[\gamma(\Omega) - s_{\widetilde{\mathcal{F}}}, \gamma(\Omega)]$). This is possible since the diffusion operator of (4.9) becomes degenerate over the sets where $\mathcal{V}'(s) \equiv 0$, because $p < 1$ (see, Theorem 1.14 of [15]). The following result gives some estimates about the decaying (in measure) of $u^{\circledast}(t)$ near the boundary of its support $t = \gamma(\Omega) - \tau$ (notice that τ could be zero). Since our goal is of local nature we shall need some additional condition which holds, for instance, when f is a bounded function:

$$\|u\|_{L^\infty(\Omega)} \le M_\infty, \text{ for some } M_\infty > 0. \tag{4.18}$$

Proposition 15. *Assume that $p < 1$, $(A2') - (A3')$ holds, f belongs to the dual space of $H^1(\Omega, \gamma)$, $f \ge 0$ and*

$$\gamma(\Omega) < 1 \text{ and } \tau \ge 0 \text{ or } \Omega = \mathbb{R}^N \text{ and } \tau > 0. \tag{4.19}$$

Let u be the solution of (4.1) *and assume (4.18). Let $\tau \in [0, s_f]$ given by (4.17). Assume data f and Ω, and u be such that,*

$$\mathcal{U}(\gamma(\Omega) - \tau - \delta) \le M_u - \theta\delta^{\frac{p+1}{1-p}} \tag{4.20}$$

for some $\delta \in (0, \gamma(\Omega) - \tau)$, and

$$\mathcal{F}(s) \le c_0 M_u + \frac{k(\theta)}{K_\delta(\Omega)} \left[\gamma(\Omega) - \tau - s\right]_+^{\frac{p+1}{1-p}} \text{ for } s \in \left(\gamma(\Omega) - \tau - \delta, \min\left(\gamma(\Omega) - \frac{\tau}{2}, \gamma(\Omega)\right)\right), \tag{4.21}$$

where $\theta > 0$ is some constant such that

$$\theta > \left[K_\delta(\Omega)c_0 \frac{(1-p)^{\frac{p+1}{p}}}{2p(p+1)^{\frac{1}{p}}}\right]^{\frac{p}{p-1}}, \tag{4.22}$$

$$K_\delta(\Omega) = \frac{1}{\min_{s \in (\gamma(\Omega) - \tau - \delta, \min(\gamma(\Omega) - \frac{\tau}{2}, \gamma(\Omega))} \left(\varphi_1\left(\Phi^{-1}(s)\right)\right)^2}, \tag{4.23}$$

$$k(\theta) = \theta^{\frac{1}{p}} \frac{2(p+1)^{\frac{1}{p}} p}{(1-p)^{\frac{p+1}{p}}} - \theta K_\delta(\Omega) c_0$$

and

$$M_u = \max_{s \in [0, \gamma(\Omega)]} \mathcal{U}(s) = \mathcal{U}(\gamma(\Omega)) = \int_0^{\gamma(\Omega)} [u^{\circledast}(t)]^p \, dt = \int_0^{\gamma(\Omega) - \tau} [u^{\circledast}(t)]^p \, dt = \|u^p\|_{L^1(\Omega:\gamma)}.$$

Then

$$\int_s^{\gamma(\Omega) - \tau} [u^{\circledast}(t)]^p \, dt \le \theta(\gamma(\Omega) - \tau - s)^{\frac{p+1}{1-p}} \text{ for any } s \in (\gamma(\Omega) - \tau - \delta, \gamma(\Omega) - \tau).$$

Proof. Since $-\frac{d}{ds}\left(\left(\mathcal{U}'(s)\right)^{\frac{1}{p}}\right) \ge 0$ $\left(\left(\mathcal{U}'(s)\right)^{\frac{1}{p}} = u^{\circledast}(s)\right.$ which is a decreasing function), for any $\delta > 0$ in a neighborhood of $\gamma(\Omega) - \tau$ we have

$$\begin{aligned} &\min_{s \in \left(\gamma(\Omega) - \tau - \delta, \min\left(\gamma(\Omega) - \frac{\tau}{2}, \gamma(\Omega)\right)\right)} \left(\varphi_1\left(\Phi^{-1}(s)\right)\right)^2 \left[-\frac{d}{ds}\left(\left(\mathcal{U}'(s)\right)^{\frac{1}{p}}\right)\right] \\ &\quad \le \left(\varphi_1\left(\Phi^{-1}(s)\right)\right)^2 \left[-\frac{d}{ds}\left(\left(\mathcal{U}'(s)\right)^{\frac{1}{p}}\right)\right], \quad \text{for } s \in \left(\gamma(\Omega) - \tau - \delta, \min\left(\gamma(\Omega) - \frac{\tau}{2}, \gamma(\Omega)\right)\right). \end{aligned} \tag{4.24}$$

Notice that due to (4.19) then $\min_{s\in(\gamma(\Omega)-\tau-\delta,\min(\gamma(\Omega)-\frac{\tau}{2},\gamma(\Omega))}\left(\varphi\left(\Phi^{-1}(s)\right)\right)^2 > 0$ since $\min(\gamma(\Omega)-\frac{\tau}{2},\gamma(\Omega)) < 1$.
Moreover, from (4.18)

$$0 \le \mathcal{U}'(s) = [u^{\circledast}(s)]^p \le M_\infty^p$$

which, in particular, implies

$$\mathcal{U}(\gamma(\Omega)-\tau-\delta) \le M_\infty^p(\gamma(\Omega)-\tau-\delta).$$

Then, simplifying the notation $K = K_\delta(\Omega)$ in (4.23), (4.7) and (4.24) we get

$$\begin{cases} \left[-\frac{d}{ds}\left(\left(\mathcal{U}'(s)\right)^{\frac{1}{p}}\right)\right] + Kc_0\mathcal{U}(s) \le K\mathcal{F}(s) & \text{for } s \in \left(\gamma(\Omega)-\tau-\delta, \min\left(\gamma(\Omega)-\frac{\tau}{2},\gamma(\Omega)\right)\right), \\ \mathcal{U}(\gamma(\Omega)-\tau-\delta) \le M_\infty^p(\gamma(\Omega)-\tau-\delta), \ \mathcal{U}'(\min(\gamma(\Omega)-\frac{\tau}{2},\gamma(\Omega))) = 0. \end{cases} \tag{4.25}$$

When $\Omega = \mathbb{R}^N$ we recall that the existence of solutions can be proved by well-known methods since the perturbation is sublinear (see *e.g.* Theorem 4.2 of [15] and Remark 14).

Let us construct now a supersolution of (4.9). We define the function

$$\overline{W}(s) = \begin{cases} M_u - \eta(\gamma(\Omega)-\tau-s) & \text{if } s \in [\gamma(\Omega)-\tau-\delta, \gamma(\Omega)-\tau], \\ M_u & \text{if } s \in [\gamma(\Omega)-\tau, \min(\gamma(\Omega)-\frac{\tau}{2},\gamma(\Omega))], \end{cases}$$

where

$$\eta(r) = \theta r^{\frac{p+1}{1-p}} \quad \text{with } \theta \text{ satisfying (4.22)}.$$

Since $p < 1$, according Lemma 1.3 and Lemma 1.6 of [15] (where its conclusion implies that $k(\theta) > 0$ when (4.22) holds), we get that

$$\begin{cases} \left[-\frac{d}{ds}\left(\left(\overline{W}'(s)\right)^{\frac{1}{p}}\right)\right] + Kc_0\overline{W}(s) = Kc_0M_u + k(\theta)\left[\gamma(\Omega)-\tau-s\right]_+^{\frac{p+1}{p-1}} \text{for } s \in \left(0, \min\left(\gamma(\Omega)-\frac{\tau}{2},\gamma(\Omega)\right)\right), \\ \overline{W}(\gamma(\Omega)-\tau-\delta) = M_u - \theta\left(\delta\right)^{\frac{p+1}{1-p}}, \ \overline{W}'(\min(\gamma(\Omega)-\frac{\tau}{2},\gamma(\Omega))) = 0. \end{cases}$$

Notice that the support $M_u - \overline{W}$ is the interval $[\gamma(\Omega)-\tau-\delta, \gamma(\Omega)-\tau]$ and that $\gamma(\Omega)-\tau \ge \gamma(\Omega)-s_f$.
From the assumption (4.20) $\mathcal{U}(\gamma(\Omega)-\tau-\delta) \le \overline{W}(\gamma(\Omega)-\tau-\delta))$.
Moreover, by (4.21) we have

$$\begin{cases} \left[-\frac{d}{ds}\left(\left(\mathcal{U}'(s)\right)^{\frac{1}{p}}\right)\right] + Kc_0\mathcal{U}(s) \le \left[-\frac{d}{ds}\left(\left(\overline{W}'(s)\right)^{\frac{1}{p}}\right)\right] + Kc_0\overline{W}(s) \ s \in \left(\gamma(\Omega)-\tau-\delta, \min\left(\gamma(\Omega)-\frac{\tau}{2},\gamma(\Omega)\right)\right), \\ \mathcal{U}(\gamma(\Omega)-\tau-\delta) \le \overline{W}(\gamma(\Omega)-\tau-\delta), \ \mathcal{U}'(\min(\gamma(\Omega)-\frac{\tau}{2},\gamma(\Omega))) = \overline{W}'(\min(\gamma(\Omega)-\frac{\tau}{2},\gamma(\Omega))) = 0. \end{cases}$$

Thus, by the comparison principle,

$$\mathcal{U}(s) \le \overline{W}(s) \text{ for any } s \in (\gamma(\Omega)-\tau-\delta, \min(\gamma(\Omega)-\frac{\tau}{2},\gamma(\Omega))),$$

i.e.

$$\int_0^s [u^{\circledast}(t)]^p\,dt \ge \int_0^{\gamma(\Omega)-\tau} [u^{\circledast}(t)]^p\,dt - \eta(\gamma(\Omega)-\tau-s)$$

and then

$$\int_s^{\gamma(\Omega)-\tau} [u^{\circledast}(t)]^p\,dt \le \eta(\gamma(\Omega)-\tau-s) \text{ for any } s \in (\gamma(\Omega)-\tau-\delta, \gamma(\Omega)-\tau),$$

which gives the result. □

Remark 16. *The above result improves Proposition 5 of [17]. We send the reader to [15] and [17] for many other results concerning solutions with compact support and dead cores, when $p < 1$. In particular, it is well known that a suitable balance between the "sizes" of f and the set $\{x \in \Omega : f(x) = 0\}$ is needed for the occurrence of a free boundary: in some sense the last set must big enough. Such a balance appears here written in terms of the assumptions (4.20) and (4.21). Notice the above results says that if condition (4.20) holds for $s = \gamma(\Omega) - \tau - \delta$ then we get the decay inequality for any $s \in (\gamma(\Omega) - \tau - \delta, \gamma(\Omega) - \tau)$. Finally, notice that if in (4.25) there is an equality, instead an inequality, and if $s_f > 0$ then, necessarily $M_u = M_f$, where*

$$M_f = \mathcal{F}(\gamma(\Omega)) = \int_0^{\gamma(\Omega)} [f^{\circledast}(t)]^p \, dt = \int_0^{\gamma(\Omega)-s_f} [f^{\circledast}(t)]^p \, dt = \|f^p\|_{L^1(\Omega:\gamma)} .$$

5 Comparison in mass for problems with a more general non linearity

The results of the previous section can be generalized to a class of elliptic problem with a more general zero order term. Several directions of improvement are possible. We could work with solutions outside the energy space, for instance when $f(x)\varphi(x) \in L^1(\Omega)$, as in the famous paper by Brezis and Strauss [10], but we prefer to continue working with solutions in the energy space and so, to fix ideas, we consider in this Section the following generalization

$$\begin{cases} -\sum_{i,j=1}^N \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}(x)\right)_{x_j} + c(x)b(u(x))\varphi(x) = f(x)\varphi(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

We assume the structural assumptions (A0), ($A2'$)-($A4'$) and replace ($A1'$) by

(B) b is a continuous increasing function such that $b(0) = 0$ and $b(u)u > 0$

Moreover arguing as in Theorem 9 it is possible to prove a comparison result between the concentration of the solution u to problem (5.1) and the solution $v \in H_0^1(\Omega^\star, \gamma)$ to the following problem

$$\begin{cases} -\mathrm{div}(\alpha\nabla v(x)\varphi(x)) + c_0\, b(v(x))\varphi(x) = \widetilde{f}(x)\varphi(x) & \text{in } \Omega^\star \\ v = 0 & \text{on } \partial\Omega^\star, \end{cases} \tag{5.2}$$

where $\Omega^\star$ is defined in (2.6) and $\widetilde{f} = \widetilde{f}^\star$, the Gauss rearrangement of $\widetilde{f}$.

Theorem 17. *Suppose that (A0), ($A2'$)-($A4'$) and (B) hold. Let f be a nonnegative function, $\widetilde{f} \in L^2 \log L^{-1/2}(\Omega^\star, \gamma)$ and let u and v be the nonnegative weak solution of* (5.1) *and* (5.2), *respectively. Then*

$$\|(\mathcal{U} - \mathcal{V})_+\|_{L^\infty(0,\gamma(\Omega))} \le \frac{1}{c_0}\|(\mathcal{F} - \widetilde{\mathcal{F}})_+\|_{L^\infty(0,\gamma(\Omega))},$$

where

$$\mathcal{U}(s) = \int_0^s b(u^{\circledast}(t))\, dt \qquad \mathcal{V}(s) = \int_0^s b(v^{\circledast}(s))\, dt$$

$$\mathcal{F}(s) = \int_0^s f^{\circledast}(t)\, dt \qquad \widetilde{\mathcal{F}}(s) = \int_0^s \widetilde{f}^{\circledast}(t)\, dt,$$

for $s \in (0, \gamma(\Omega)]$.

The proof of Theorem 17 runs as in the case $b(u) = |u|^{p-1}u$, but as a preliminary step we need that the analogue of Proposition 8 is in force. For reader convenience we detail the following result of existence.

Proposition 18. *Suppose that (A0), (A2')-(A4') and (B) hold. If f is nonnegative, then Problem (5.1) has a unique nonnegative weak solution* $u \in H_0^1(\Omega, \gamma)$*, i.e. such that* $c(x)\, b(u) \in L^1(\Omega, \gamma)$ *and*

$$\int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial \psi}{\partial x_j} dx + \int_\Omega c\, b(u)\psi \, d\gamma = \int_\Omega f\psi \, d\gamma$$

for every $\psi \in H_0^1(\Omega, \gamma) \cap L^\infty(\Omega)$ *holds. Moreover* $c(x)\, b(u)u \in L^1(\Omega, \gamma)$.

Proof. We give only some details about existence, because the proof of positivity and uniqueness is standards and runs using the monotonicity of b. We introduce the following class of approximated problems:

$$\begin{cases} -\sum_{i,j=1}^N \partial_{x_j}(a_{ij}(x)\frac{\partial u_k}{\partial x_i}) + T_k\left(c\, b(u_k)\right)\varphi(x) = f(x)\varphi(x) & \text{in } \Omega \\ \\ u_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.3}$$

where $T_k(s)$ is defined as in (3.5). Since $|T_k\left(c(x)\, b(u_k)\right)|\varphi(x) \le k\varphi(x)$ and $T_k\left(c(x)\, b(u_k)\right) u_k \ge 0$, the existence of a variational weak solution $u_k \in H_0^1(\Omega, \gamma)$ is well-known (see, *e.g.* [9]). Taking u_k as test function and using Log-Sobolev inequality (2.10) we obtain

$$\|\nabla u_k\|_{L^2(\Omega,\gamma)} \le C\|f\|_{L^2 \log L^{-1/2}(\Omega,\gamma)}$$

and

$$\int_\Omega T_k\left(c(x)\, b(u_k)\right) u_k \, d\gamma \le C\|f\|_{L^2 \log L^{-1/2}(\Omega,\gamma)}$$

for some positive constant C independent of u_k. Then the sequence u_k is bounded in $H_0^1(\Omega, \gamma)$, then there exists a function $u \in H_0^1(\Omega, \gamma)$ such that (up a subsequence

$$u_k \rightharpoonup u \quad \text{in } H_0^1(\Omega, \gamma) \quad \text{and} \quad u_k \to u \text{ a.e. in } \Omega$$

hold. In particular

$$T_k\left(c(x)\, b(u_k)\right) u_k \to c(x)\, b(u)u \quad \text{a.e. in } \Omega.$$

By Fatou's Lemma and estimate (5) we get

$$\int_\Omega c(x)\, b(u)\, u\, d\gamma \le \liminf \int_\Omega T_k\left(c(x)\, b(u_k)\right) u_k \, d\gamma \le C\|f\|_{L^2 \log L^{-1/2}(\Omega,\gamma)},$$

then $c(x)b(u)u \in L^1(\Omega, \gamma)$. Moreover for some $\delta > 0$ and every $E \subset \Omega$ by (5) we get

$$\begin{aligned} \int_E T_k\left(c(x)\, b(u_k)\right) d\gamma &= \int_{E\cap\{u_k<1/\delta\}} T_k\left(c(x)\, b(u_k)\right) d\gamma + \int_{E\cap\{u_k>1/\delta\}} T_k\left(c(x)\, b(u_k)\right) d\gamma \\ &\le b\left(\frac{1}{\delta}\right) \int_E c(x)\, d\gamma + \delta C\|f\|_{L^2 \log L^{-1/2}(\Omega,\gamma)}. \end{aligned}$$

Choosing $\delta = \int_E c(x)\, d\gamma$ if $\sup b(s) < +\infty$ or otherwise $\delta = \frac{1}{b^{-1}\left(\frac{1}{\int_E c(x)\, d\gamma}\right)}$, we get the equintegrability and Vitali's Theorem allow us to conclude that

$$T_k\left(c(x)\, b(u_k)\right) \to c(x)\, b(u) \quad \text{in } L^1(\Omega, \gamma). \tag{5.4}$$

Now we are able to pass to the limit in (5.3) for every $\psi \in H_0^1(\Omega, \gamma) \cap L^\infty(\Omega)$ and the result holds. □

Remark 19. *Theorem 17 also holds if we assume in (B) that b is merely non-decreasing $b(0) = 0$ and $b(u)u > 0$. The only difficulty arises when dealing with b^{-1} because now is not necessarily a function but a maximal monotone graph of $\mathbb{R}^2$ and some technicalities are needed (see, e.g., [10], [15] and [35]).*

Remark 20. *A different extension concerns the case in which we replace f by a general datum $F = f - divg$ with f that satisfy $(A4')$ and $g \in (L^2(\Omega,\gamma))^N$. To have nonnegative solutions we have to require $< F, \psi >\geq 0$ for every nonnegative test function.*

We can also compare (in the sense of rearrangements) problems with different nonlinearities. Just to give an idea, let us consider problem (5.1) when the domain Ω is H_ω, the half-space $x_1 > \omega$ with $\omega \in \mathbb{R}$. We take into account two smooth strictly increasing functions b and $\widetilde{b}$ having the same domain such that $b(0) = \widetilde{b}(0) = 0$, and two positive increasing functions of x_1 variable f and $\widetilde{f}$ defined in H_ω. Recalling the symmetrized problem (4.5) it is natural to require some conditions on the inverse of the zero order term functions. Indeed let us assume that b and $\widetilde{b}$ are smooth functions such that

$$((\widetilde{b})^{-1})'(s) \leq \left(b^{-1}\right)'(s) \qquad \text{for every } \; s \in \mathbb{R}, \tag{5.5}$$

where b^{-1} and $(\widetilde{b})^{-1}$ the inverse functions of b and $\widetilde{b}$ respectively and that the datum f is "less concentrated" than the datum $\widetilde{f}$, namely

$$\int_{H_v} f(x)\,dx \leq \int_{H_v} \widetilde{f}(x)\,dx \qquad \text{for every } \; v > \omega.$$

Then, we are going to prove that

$$\int_{H_v} b(u^\star(x))\,dx \leq \int_{H_v} \widetilde{b}\left(\widetilde{u}(x)\right)\,dx \qquad \text{for every } \; v > \omega, \tag{5.6}$$

where $u^\star$ is the rearrangement with respect to Gauss measure of the solution u to problem (5.1) and $\widetilde{u}$ is the solution to the following problem

$$\begin{cases} -\text{div}\left(\alpha\nabla\widetilde{u}\right) + \widetilde{b}(\widetilde{u})\varphi(x) = \widetilde{f}(x)\varphi(x) & \text{in } H_\omega \\ \widetilde{u} = 0 & \text{on } \partial H_\omega. \end{cases}$$

We refer to [35] for unweighted case $\varphi(x) \equiv 1$. A more general result, implying conclusion (5.6) can be proved. To be more precise, let b_1, b_2 be two continuous non decreasing functions. We say that b_1 is *weaker* than b_2, and we write

$$b_1 \prec b_2, \tag{5.7}$$

if they have the same domain of definition, and there exists a contraction $\rho : \mathbb{R} \to \mathbb{R}$ (*i.e.* such that $|\rho(a) - \rho(b)| \leq |a - b|$ for $a, b \in \mathbb{R}$) and $b_1 = \rho \circ b_2$ (notice that this implies condition (5.5) when they are differentiable). We are now in position to state a comparison result between the concentration of the solution $u \in H_0^1(\Omega)$ to problem (5.1) with $c(x) \equiv 1$ (for simplicity) and the solution $v \in H_0^1(\Omega^\star,\gamma)$ to the following problem

$$\begin{cases} -\text{div}(\alpha\nabla v(x)\varphi(x)) + \widetilde{b}(v(x))\varphi(x) = \widetilde{f}(x)\varphi(x) & \text{in } \Omega^\star \\ v = 0 & \text{on } \partial\Omega^\star, \end{cases} \tag{5.8}$$

where $\Omega^\star$ is defined in (2.6), $\widetilde{f} = \widetilde{f}^\star$ and $(\widetilde{b})^{-1} \prec b^{-1}$.

Theorem 21. *Suppose $c(x) \equiv 1$, (A0), (A2')-(A4') and (B) hold with $c_0 \equiv 1$ and $(\widetilde{b})^{-1} \prec b^{-1}$. Let f be a nonnegative function, $\widetilde{f} \in L^2 \log L^{-1/2}(\Omega^\star,\gamma)$ and let u and v be the weak nonnegative solution of (5.1) and (5.8), respectively. Then we get*

$$\|(\mathcal{B} - \widetilde{\mathcal{B}})_+\|_{L^\infty(0,\gamma(\Omega))} \leq \|(\mathcal{F} - \widetilde{\mathcal{F}})_+\|_{L^\infty(0,\gamma(\Omega))},$$

where

$$\mathcal{B}(s) = \int_0^s b(u^{\circledast}(t))\,dt \qquad \widetilde{\mathcal{B}}(s) = \int_0^s \widetilde{b}(w^{\circledast}(t))\,dt$$

$$\mathcal{F}(s) = \int_0^s f^{\circledast}(t)\,dt \qquad \widetilde{\mathcal{F}}(s) = \int_0^s \widetilde{f}^{\circledast}(t)\,dt$$

for $s \in (0, \gamma(\Omega)]$.

Proof. By using the Yosida approximation of functions $(\widetilde{b})^{-1}$ and b^{-1} it is enough to prove the conclusion when both functions are differentiable and strictly increasing (see, *e.g.* [10], [35]). As in the proof of Theorem 9 we get

$$\begin{cases} \left(\varphi_1\left(\Phi^{-1}(s)\right)\right)^2\left[-\dfrac{d}{ds}\left(b^{-1}\left(\mathcal{B}'(s)\right)\right)\right] + c_0\,\mathcal{B}(s) \le \mathcal{F}(s) & \text{for } s \in (0, \gamma(\Omega)) \\ \mathcal{B}(0) = 0, \quad \mathcal{B}'(\gamma(\Omega)) = 0\,, \end{cases}$$

and

$$\begin{cases} \left(\varphi_1\left(\Phi^{-1}(s)\right)\right)^2\left[-\dfrac{d}{ds}\left((\widetilde{b})^{-1}\left(\widetilde{\mathcal{B}}'(s)\right)\right)\right] + c_0\widetilde{\mathcal{B}}(s) = \widetilde{\mathcal{F}}(s) & \text{for } s \in (0, \gamma(\Omega)) \\ \widetilde{\mathcal{B}}(0) = 0, \quad \widetilde{\mathcal{B}}'(\gamma(\Omega)) = 0. \end{cases}$$

Assume that $\|(\mathcal{B} - \widetilde{\mathcal{B}})_+\|_{L^\infty(0,\gamma(\Omega))} > 0$ (otherwise the conclusion is trivial). Since $\mathcal{B}, \widetilde{\mathcal{B}} \in \mathcal{C}^0[0, \gamma(\Omega)]$ the above norm is attainted in some point $s_0 \in (0, \gamma(\Omega)]$ (is clear that $s_0 > 0$), so that

$$(\mathcal{B} - \widetilde{\mathcal{B}})(s_0) = \frac{1}{c_0}\|(\mathcal{B} - \widetilde{\mathcal{B}})_+\|_{L^\infty(0,\gamma(\Omega))}.$$

On the other hand, since $(\widetilde{b})^{-1}$ and b^{-1} are differentiable and strictly increasing we get that $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ are convex functions on $(0, \gamma(\Omega)]$, from the assumption (5.7) we get that

$$\left(\varphi\left(\Phi^{-1}(s)\right)\right)^2\left[-\frac{d}{ds}\left((b)^{-1}\left(\widetilde{\mathcal{B}}'(s)\right)\right)\right] + c_0\widetilde{\mathcal{B}}(s) \ge \widetilde{\mathcal{F}}(s).$$

Then we can reproduce the final arguments of the proof of Theorem 9 and the result holds. □

Example 22. *Condition (5.7) holds, for instance, if* $b(u) = e^{\alpha u} - 1$ *and* $\widetilde{b}(u) = e^{\beta u} - 1$ *with* $\beta \ge \alpha > 0$. *It also holds for* $b(u) = u^\alpha$ *when* $\widetilde{b}(u) = \begin{cases} u^\alpha & \text{if } 0 \le u < 1 \\ u^\beta & \text{if } u \ge 1, \end{cases}$ *with* $\beta \ge \alpha > 0$.

Remark 23. *The pointwise comparison between* $b(u^\star(x))$ *and* $\widetilde{b}\left(\widetilde{u}(x)\right)$ *usually fails. This type of pointwise comparison was studied in [2] for the case of some evolution problems which are related with problem (5.1) through its implicit time discretization.*

Remark 24. *Proposition 15 can be extended to the case of a function* $b(u)$ *more general than* $|u|^{p-1}u$ *with* $p < 1$. *The condition* $p < 1$ *is now replaced by a condition stated in terms of an improper integral*

$$\int_0^\tau \frac{ds}{\sigma^{-1}(\frac{c_0}{2}s^2)} < +\infty, \text{ for any } \tau \in (0, 1),$$

where $\sigma(r) = \int_0^r (b^{-1})'(s)s\,ds$ *(see Lemma 1.3 of [15]).*

Remark 25. *Theorem 2 can be also extended to the structural assumptions of this Section when* $\Omega = \mathbb{R}^N$ *by replacing the expression* $|u|^{p+1}$ *by* $b(u)u$ *and by working in the Orlicz space* $L^A_{loc}(\mathbb{R}^N, \gamma)$*, where* $A(t) = b(t)t$ *and by asking some additional conditions. We recall that b verifies a* Δ_2 *condition, i.e. there exist a constant K and* s_0 *such that*

$$b(2s) \le K\, b(s) \quad \forall s > s_0.$$

We stress that all estimates in Lemma 3 holds replacing $|u|^{p-1}u$ *with* $b(u)$*, when*

$$b(ku)ku \ge |u|^{p+1} \quad \text{for } u > u_0$$

for some $k, u_0 > 0$ *and for some* $p > 1$ *and the only crucial step is the weak convergence in the Orlicz space* $L^A_{loc}(\mathbb{R}^N, \gamma)$*. Clearly we have to require more on b in order to have that A is an N-function (see [1]). For example b has to be an odd function. Moreover when b verifies a* Δ_2 *condition, it is easy to check that A does. Then the Orlicz space* $L^A_{loc}(\mathbb{R}^N, \gamma)$ *is reflexive and the boundedness of* $\|u_M\|_{L^A_{loc}(\mathbb{R}^N,\gamma)}$ *allows as to pass to the limit in the sequence of approximate problems.*

For some results in this framework but with the Lebesgue measure see [28] and its references. For other generalizations of Brezis result [8] see for instance [14], [7], [6], [28].

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References

[1] R. Adams and J. Fournier, *Sobolev Spaces*, Academic Press, New York, 2003.
[2] Ph. Bénilan and J.I. Díaz, *Comparison of solutions of nonlinear evolutions equations with different nonlinear terms*, Isr. J. Math. 42 (1982), 241-257.
[3] M.F. Betta, F. Brock, A. Mercaldo and M. R. Posteraro, *A comparison result related to Gauss measure*, C. R. Math. Acad. Sci. Paris, 334 (2002), 451-456.
[4] M.F. Betta, F. Feo and M. R. Posteraro, *Uniqueness results for strongly monotone operators related to Gauss measure*, Isr. J. Math. 233 (2019), 297-310.
[5] C. Bennett and K. Rudnick, *On Lorentz-Zygmund spaces*, Dissertationes Math. (Rozprawy Mat.), 175 (1980).
[6] F. Bernis, *Elliptic and parabolic semilinear problems without conditions at infinity*, Arch. Rational Mech. Anal. 106 (1989) n.3, 217-241.
[7] L. Boccardo, T. Gallouet and J. L. Vázquez, *Nonlinear elliptic equations in* R^N *without growth restrictions on the data*, J. Differential Equations 105 (1993),n.2, 334-363.
[8] H. Brezis, *Semilinear equations in* R^N *without condition at infinity*, Applied Mathematics and Optimization 12 (1984), 271-282.
[9] H. Brezis and F. E. Browder, *Some properties of higher order Sobolev spaces,* J. Math. Pures Appl. 61 (1982), 245-259.
[10] H. Brezis and W. Strauss, *Semilinear second order elliptic equations in* L^1, J. Math. Soc. Japan 25 (1974), 831-844.
[11] C. Borell, *The Brunn-Minkowski inequality in Gauss space*, Invent. Math. 30 (1975), n. 2, 207-216.
[12] F. Chiacchio, *Comparison results for linear parabolic equations in unbounded domains via Gaussian symmetrization*, Differential Integral Equations 17 (2004), n.3-4, 241-258.
[13] K. M. Chong and N. M. Rice, *Equimeasurable rearrangements of functions*. Queen's Papers in Pure and Applied Mathematics, 28. Queen's University, Kingston, Ont., 1971.
[14] G. Díaz, *Uniqueness and interior* L^∞*-estimates for fully nonlinear equations in* $\mathbb{R}^N$ *without conditions at infinity*. In "Contributions to nonlinear partial differential equations, Vol. II", ed. by J. I. Díaz& P. L. Lions. Pitman Research Notes in Mathematics 155 (1987), 91-102.
[15] J. I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries*, Pitman, London 1985.

[16] J. I. Díaz, *Inequalities of isoperimetric type for the Plateau problem and the capillarity problem*, Rev. Acad. Canaria Cienc. **3** (1991), 127-166.

[17] J. I. Díaz, *Simetrización de problemas parabólicos no lineales: Aplicación a ecuaciones de reacción-difusión*, Memorias de la Real Acad. de Ciencias Exactas, Físicas y Naturales, Tomo XXVII. Madrid, 1991.

[18] J. I. Díaz, *Symmetrization of nonlinear elliptic and parabolic problems and applications: a particular overview*. In *Progress in partial differential equations.elliptic and parabolic problems*, Proc. First European Conf. on Elliptic and Parabolic Problems, Progress in partial differential equations: elliptic and parabolic problems (Pont-à-Mousson, 1991), Pitman Res. Notes Math. Ser. 266, Longman Sci.Tech., Harlow,1992, 1-16.

[19] J. I. Díaz and O. A. Oleinik, *Nonlinear elliptic boundary value problems in unbounded domains and the asymptotic behaviour of its solutions*, Comptes Rendus Acad. Sci. Paris,t. 315 (1992), Série I, 787-792.

[20] G. di Blasio, Linear elliptic equations and Gauss measure, Journal of Inequalities in Pure and Applied Mathematics 4 (2003).

[21] G. di Blasio, F. Feo and M. R. Posteraro, *Regularity results for degenerate elliptic equations related to Gauss measure*, Mathematical Inequalities and Applications 10 (2007), n. 4, 771-797.

[22] A. Ehrhard. *Inégalités isopérimétriques et intégrales de Dirichlet gaussiennes*, Ann. Sci. Ecole Norm. Sup. (4),17 (1984), n. 2, 317-332.

[23] F. Feo and G. Paderni, *Anisotropic Logarithmic Sobolev Inequality with a Gauss weight and its applications*, Electronic Journal of Differential Equations (2019), n. 89, 1-15.

[24] F. Feo and M. R. Posteraro, *Logarithmic Sobolev trace inequalities.*, Asian J. Math. 17 (2013), n. 3, 569-582.

[25] F. Feo, P. R. Stinga and B. Volzone, *The fractional nonlocal Ornstein-Uhlenbeck equation, Gaussian symmetrization and regularity*, Discrete & Continuous Dynamical Systems - A 38 (2018), n.7, 3269-3298.

[26] L. Gross, *Logarithmic Sobolev inequalities,* Amer. J. Math., 97 (1976), 1061-1083.

[27] A.N. Kolmogorov,*Zur Umkehrbarkeit der statistischen Naturgesetze*, Math. Ann. 113 (1937), 766-772.

[28] L.M. Kozhevnikova and A. A. Khadzhi, *Existence of solutions of anisotropic elliptic equations with nonpolynomial nonlinearities in unbounded domains*, Sb. Math. 206 (2015), 1123-1149.

[29] L. M. Kozhevnikova and A. A. Nikitina, *Qualitative properties of solutions to elliptic equations with nonpower nonlinearities in* $\mathbb{R}^n$, Journal of Mathematical Sciences, 228 4 (2018), 395-408.

[30] O. A. Oleinik, *Some Asymptotic Problems in the Theory of Partial Differential Equations*. Cambridge University Press, Cambridge, 1996.

[31] H. Risken, *The Fokker–Planck Equation: Method of Solution and Applications*. Springer-Verlag. New York, 1989.

[32] M. Röckner, L^p*-analysis of finite and infinite dimensional diffusion operators*, in the book, *Stochastic PDE's and Kolmogorov Equations in Infinite Dimensions*(G. Da Prato ed.), Springer, Berlin (1999), 65-116.

[33] G. Talenti, *Elliptic equations and rearrangements*, Ann. Sc. Norm. Sup. Pisa IV 3 (1976), 697-718.

[34] G. Trombetti and J. L. Vázquez, *A symmetrization result for elliptic equations with lower-order terms*, Annales de la Faculté des Sciences de Toulouse 7, Université Paul Sabatier, 5^e série, tome 7(1985), n. 2, 137-150

[35] J. L. Vazquez, *Symmetrization and Mass Comparison for Degenerate Nonlinear Parabolic and related Elliptic Equations*, Advanced Nonlinear Studies 5 (2005), 87-131.

[36] L. Veron, *Singularities of solutions of second order quasilinear equations*, Longman, Harlow, Essex, 1996.

[37] D. V. Widder, *Positive temperatures in an infinite rod*, Trans. Amer. Math. Soc. 55 (1944), 85-95.