Lyapunov stability for piecewise affine systems via cone-copositivity

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Abstract

Cone-copositive piecewise quadratic Lyapunov functions (PWQ-LFs) for the stability analysis of continuous-time piecewise affine (PWA) systems are proposed. The state space is assumed to be partitioned into a finite number of convex, possibly unbounded, polyhedra. Preliminary conditions on PWQ functions for their sign in the polyhedra and continuity over the common boundaries are provided. The sign of each quadratic function is studied by means of cone-constrained matrix inequalities which are translated into linear matrix inequalities (LMIs) via cone-copositivity. The continuity is guaranteed by adding equality constraints over the polyhedra intersections. An asymptotic stability result for PWA systems is then obtained by finding a continuous PWQ-LF through the solution of a set of constrained LMIs. The effectiveness of the proposed approach is shown by analyzing an opinion dynamics model and two saturating control systems.

Key words: Piecewise affine systems; piecewise quadratic Lyapunov functions; cone-copositivity; asymptotic stability; Multi-agent system; opinion dynamics; saturated systems.

1 Introduction

Piecewise affine (PWA) systems are characterised by a set of state-dependent switching affine subsystems defined over a state space partitioned into convex polyhedra [22]. There exist numerous important applications which involve PWA systems. At least they can be employed to approximate nonlinear systems and are shown to be equivalent to several classes of hybrid systems [10]. The stability analysis of PWA systems is a difficult issue due to their hybrid nature. A classical sufficient condition is the quadratic stability [2,5] which is however known to be conservative. Different approaches have been investigated in the last years with the aim of obtaining less conservative results. Among others, the multiple Lyapunov function approach, i.e., to combine Lyapunov functions defined over different regions of the state space, has been proposed, see [18]. In particular, piecewise quadratic Lyapunov functions (PWQ-LFs) obtained by patching together *quadratic forms* (for the regions containing the origin) and *quadratic functions* (for the regions which do not contain the origin), have been widely investigated starting from the seminal work [15]. In this framework the stability conditions are typically formulated in terms of constrained inequalities which can be solved by means of a set of linear matrix inequalities (LMIs) by applying the $\mathcal{S}$-procedure [18]. Unfortunately the $\mathcal{S}$-procedure is lossy in general. Several variants of this technique have been proposed in the more recent literature including sliding modes [21], attraction domain estimation [17], and relaxed LMIs for discrete-time PWA systems [11].
In [12] a PWQ-LF approach suitable for Lur'e systems with slab partitions is proposed, however the results therein cannot be directly extended to PWA systems with more general polyhedral partitions of the state space. In [13] conewise linear systems were considered, which excluded the presence of bounded polyhedra in the state space partition. In this paper we propose a new PWQ approach for continuous-time PWA systems where the PWQ-LF, differently from the other approaches, is obtained by suitably combining *quadratic functions* for *all* regions of the state space partition. The stability conditions are expressed in terms of cone-

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constrained inequalities which are translated into LMIs by formulating a cone-copositive problem. The copositive programming for a given matrix analyzed in [3,23] is here exploited with a more challenging perspective. Indeed our problem consists in finding a set of cone-copositive matrices that define a PWQ-LF and whose entries are degrees of freedom for the stability problem. The approach is shown to be effective for the stability analysis of opinion dynamics [26] and saturated control systems [19,4].
The paper is organized as follows. In Sec. 2 some preliminary definitions and concepts are recalled. The sign analysis for a PWQ function is considered in Sec. 3 and its continuity is investigated. The stability problem for PWA systems is presented in Sec. 4. The numerical examples illustrated in Sec. 5 confirm the effectiveness of the approach. Sec. 6 concludes the paper.

2 Preliminaries

Let us recall some useful definitions and concepts.

Definition 1 *Given a finite number ρ of points $\{r_\ell\}_{\ell=1}^{\rho}$, $r_\ell \in \mathbb{R}^n$, $\rho \in \mathbb{N}$, a conical hull $\mathcal{C} = \text{cone}\{r_\ell\}_{\ell=1}^{\rho}$ is the set of points $v \in \mathbb{R}^n$ such that $v = \sum_{\ell=1}^{\rho} \theta_\ell r_\ell$, with $\theta_\ell \in \mathbb{R}_+$, $\mathbb{R}_+$ being the set of nonnegative real numbers. The set $\mathcal{C}$ is also called (polyhedral) cone and the points $\{r_\ell\}_{\ell=1}^{\rho}$ are called rays of the cone. The matrix $R \in \mathbb{R}^{n\times\rho}$ whose columns are the points $\{r_\ell\}_{\ell=1}^{\rho}$ in an arbitrary order is called ray matrix. Any $v \in \mathcal{C}$ can be written as $v = R\theta$ where $\theta \in \mathbb{R}_+^{\rho}$.*

Definition 2 *Given a finite number λ of points $\{v_\ell\}_{\ell=1}^{\lambda}$, $v_\ell \in \mathbb{R}^n$, $\lambda \in \mathbb{N}$, a convex hull, say $\text{conv}\{v_\ell\}_{\ell=1}^{\lambda}$, is a conical hull with $\sum_{\ell=1}^{\lambda}\theta_\ell = 1$.*

Definition 3 *Given a finite number λ of vertices $\{v_\ell\}_{\ell=1}^{\lambda}$ and a finite number ρ of rays $\{r_\ell\}_{\ell=1}^{\rho}$, $v_\ell, r_\ell \in \mathbb{R}^n$, $\lambda, \rho \in \mathbb{N}$, the (convex) set*

$$X = \text{conv}\{v_\ell\}_{\ell=1}^{\lambda} + \text{cone}\{r_\ell\}_{\ell=1}^{\rho} \tag{1}$$

is a polyhedron in $\mathbb{R}^n$. The expression (1) identifies the so-called $\mathcal{V}$-representation *of the polyhedron.*

In the following we assume that in the polyhedron representation (1) all possible redundancies of the set of vertices and rays have been eliminated.
Any non-empty polyhedron can be equivalently represented by using the $\mathcal{H}$-representation or the $\mathcal{V}$-representation [1]. Given an $\mathcal{H}$-representation of a polyhedron there exist numerical tools for obtaining a corresponding $\mathcal{V}$-representation, e.g., [7].

Definition 4 *Denote by int(X) the interior of a full-dimensional set $X \subseteq \mathbb{R}^n$ and S a finite positive integer. A* partition *of X is the family of full-dimensional sets $\{X_s\}_{s=1}^{S}$ satisfying $X = \cup_{s=1}^{S} X_s$ and int$(X_s) \cap$ int$(X_m) = \emptyset$ for $s \neq m$.*

In this paper we are interested in polyhedral partitions of X, i.e., to the case where $\{X_s\}_{s=1}^{S}$ are polyhedra, such that the intersection of two polyhedra is either empty or a common face. If such property does not hold, regions can be subdivided such that the property is fulfilled. An $(n-1)$-dimensional face of a polyhedron is called *facet*. Given a polyhedron one can define two corresponding cones of interest. The conical hull of a polyhedron X represented as in (1) is the cone $\mathcal{C}_X \subseteq \mathbb{R}^n$ defined as

$$\mathcal{C}_X = cone\{\{v_\ell\}_{\ell=1}^{\lambda}, \{r_\ell\}_{\ell=1}^{\rho}\}. \tag{2}$$

In the following we assume that (2) is a minimal representation for $\mathcal{C}_X$, in the sense that in (2) all possible redundancies of the set of generators have been eliminated and the numbers λ and ρ redefined accordingly. The matrix $R = (\, v_1 \;\dots\; v_\lambda \; r_1 \;\dots\; r_\rho \,)$, with $R \in \mathbb{R}^{n\times(\lambda+\rho)}$, is the ray matrix of $\mathcal{C}_X$. Note that if $0 \in \text{int}(X)$ then $\mathcal{C}_X$ is equal to $\mathbb{R}^n$. For the analysis of interest in the sequel of the paper it is assumed without loss of generality that if $0 \in X$ then the origin belongs to the boundary of X.
Another cone related to a polyhedron $X \subset \mathbb{R}^n$, denoted by $\hat{\mathcal{C}}_X \subset \mathbb{R}^{n+1}$, is obtained by means of the homogenization procedure defined below.

Definition 5 *Consider a polyhedron $X \subset \mathbb{R}^n$ with the representation (1). For each vertex $v_\ell \in \mathbb{R}^n$, its vertex-homogenization $\bar{v}_\ell \in \mathbb{R}^{n+1}$ is defined as $\bar{v}_\ell = \text{col}(v_\ell, 1) \in \mathbb{R}^{n+1}$, where $\text{col}(\cdot)$ indicates a vector obtained by stacking in a unique column the column vectors in its argument. For each ray $r_\ell \in \mathbb{R}^n$ its direction-homogenization $\bar{r}_\ell \in \mathbb{R}^{n+1}$ is defined as $\bar{r}_\ell = \text{col}(r_\ell, 0) \in \mathbb{R}^{n+1}$.*

Given a polyhedron $X \subset \mathbb{R}^n$ it is possible to define a corresponding cone $\hat{\mathcal{C}}_X \subset \mathbb{R}^{n+1}$ by moving X to the hyperplane $\mathcal{H} = \{\bar{x} \in \mathbb{R}^{n+1} : \bar{x} = \text{col}(x,1), x \in \mathbb{R}^n\}$ and drawing all the halflines from the origin of $\mathbb{R}^{n+1}$ to any point of X, as stated in the following proposition.

Proposition 6 *Given a polyhedron $X \subset \mathbb{R}^n$, consider the points $\{\bar{v}_\ell\}_{\ell=1}^{\lambda}$ and $\{\bar{r}_\ell\}_{\ell=1}^{\rho}$ in $\mathbb{R}^{n+1}$ obtained by applying the homogenization in Def. 5. Then the cone in $\mathbb{R}^{n+1}$*

$$\hat{\mathcal{C}}_X = \text{cone}\{\{\bar{v}_\ell\}_{\ell=1}^{\lambda}, \{\bar{r}_\ell\}_{\ell=1}^{\rho}\} \tag{3}$$

is such that $\hat{\mathcal{C}}_X \cap \mathcal{H} = \bar{X}$ where $\mathcal{H}$ is the hyperplane defined above and $\bar{X} = \{\bar{x} \in \mathbb{R}^{n+1} : \bar{x} = \text{col}(x,1), x \in X\}$.

For any cone $\hat{\mathcal{C}}_X \subset \mathbb{R}^{n+1}$ defined by Proposition 6, one can obtain a corresponding ray matrix $\hat{R} \in \mathbb{R}^{(n+1)\times(\lambda+\rho)}$ which has the form

$$\hat{R} = \begin{pmatrix} v_1 & \dots & v_\lambda & r_1 & \dots & r_\rho \\ 1 & \dots & 1 & 0 & \dots & 0 \end{pmatrix}. \tag{4}$$

A nonempty intersection of two polyhedra is a polyhedron. In the stability analysis we will need to formulate the continuity condition of a candidate Lyapunov function over the polyhedra intersections. To this aim we will exploit the following result.

Lemma 7 *Given two polyhedra $X_1, X_2 \subset \mathbb{R}^n$ such that $X_1 \cap X_2 \neq \emptyset$, then $\hat{\mathcal{C}}_{X_1 \cap X_2} = \hat{\mathcal{C}}_{X_1} \cap \hat{\mathcal{C}}_{X_2}$.*

PROOF. *The proof easily follows by applying the homogenization procedure and then the definitions of polyhedron and cone.*

We can now present some definitions and results on copositivity and cone-copositivity.

Definition 8 *A symmetric matrix $P \in \mathbb{R}^{n\times n}$ is cone-copositive with respect to a cone $\mathcal{C} \subseteq \mathbb{R}^n$ if it is positive semidefinite with respect to that cone, i.e., if $x^\top P x \geq 0$ for any $x \in \mathcal{C}$. A cone-copositive matrix will be denoted by $P \succcurlyeq^{\mathcal{C}} 0$. If the equality only holds for $x = 0$, then P is strictly cone-copositive and the notation is $P \succ^{\mathcal{C}} 0$. In the particular case $\mathcal{C} = \mathbb{R}^n_+$, a (strictly) cone-copositive matrix is called (strictly) copositive.*

The notation $P \succcurlyeq 0$, i.e., without any superscript on the inequality, indicates that P is positive semidefinite, i.e., $x^\top P x \geq 0$ for any $x \in \mathbb{R}^n$. The cone-copositivity evaluation of a known symmetric matrix P on a cone can be always transformed into an equivalent copositive problem and then to an LMI, as stated by the following result.

Lemma 9 *Let $P \in \mathbb{R}^{n\times n}$ be a symmetric matrix, $\mathcal{C} \subseteq \mathbb{R}^n$ be a cone, $R \in \mathbb{R}^{n\times\rho}$ be the ray matrix of the cone $\mathcal{C}$, N be a symmetric (entrywise) positive matrix. Consider the following constrained inequalities*

$$P \succ^{\mathcal{C}} 0, \tag{5a}$$

$$R^\top P R \succ^{\mathbb{R}^\rho_+} 0, \tag{5b}$$

$$R^\top P R - N \succcurlyeq 0. \tag{5c}$$

Then the following conditions hold

i) (5a) $\Longleftrightarrow$ (5b)
ii) (5c) $\Longrightarrow$ (5a).

PROOF. *i) The equivalence is directly obtained by using Def. 1.*
ii) From (5c) *it is $R^\top P R - N = Q$ with $Q \succcurlyeq 0$ and hence $\theta^\top R^\top P R \theta = \theta^\top (Q + N)\theta$ is strictly positive for $\theta \in \mathbb{R}^\rho_+ - \{0\}$ because $\theta^\top N \theta$ is strictly positive for positive θ. Then* (5b) *holds and from i) the proof is complete.*

3 Continuous PWQ functions

In this section we consider the sign analysis and the continuity problem for a PWQ function defined over a polyhedral partition $\{X_s\}_{s=1}^S$ of $\mathbb{R}^n$. With reference to this partition, denote by Σ_0 the subset of indices s such that $0 \in X_s$ and Σ_1 its complement, i.e., $\Sigma_0 \cup \Sigma_1 = \{1, \ldots, S\}$. Let

$$V(x) = x^\top P_s x + 2\nu_s^\top x + \omega_s, \quad x \in X_s,\ s = 1, \ldots, S \tag{6}$$

be a PWQ function, where $\{P_s\}_{s=1}^S$ are symmetric matrices with $P_s \in \mathbb{R}^{n\times n}$, $\{\nu_s\}_{s=1}^S$ are vectors with $\nu_s \in \mathbb{R}^n$, $\{\omega_s\}_{s=1}^S$ are real scalars with $\omega_s = 0$ for $s \in \Sigma_0$, $\{X_s\}_{s=1}^S$ are polyhedra providing a partition of $\mathbb{R}^n$ and expressed as

$$X_s = conv\{v_{s,\ell}\}_{\ell=1}^{\lambda_s} + cone\{r_{s,\ell}\}_{\ell=1}^{\rho_s} \tag{7}$$

with $s = 1, \ldots, S$. Let us define the matrices

$$R_s = \begin{pmatrix} v_{s,1} & \ldots & v_{s,\lambda_s} & r_{s,1} & \ldots & r_{s,\rho_s} \end{pmatrix}, \tag{8a}$$

$$\hat{R}_s = \begin{pmatrix} v_{s,1} & \ldots & v_{s,\lambda_s} & r_{s,1} & \ldots & r_{s,\rho_s} \\ 1 & \ldots & 1 & 0 & \cdots & 0 \end{pmatrix}, \tag{8b}$$

$$\hat{P}_s = \begin{pmatrix} P_s & \nu_s \\ \nu_s^\top & \omega_s \end{pmatrix}, \tag{8c}$$

with $R_s \in \mathbb{R}^{n\times(\lambda_s+\rho_s)}$, $\hat{R}_s \in \mathbb{R}^{(n+1)\times(\lambda_s+\rho_s)}$, $\hat{P}_s \in \mathbb{R}^{(n+1)\times(n+1)}$. The matrices (8) will be used below for the sign and continuity analysis of the PWQ function (6).

3.1 Sign of quadratic functions on polyhedra

For the sign analysis of (6) we need the following result.

Lemma 10 *Let $P_s \in \mathbb{R}^{n\times n}$ be a symmetric matrix, $\nu_s \in \mathbb{R}^n$ be a vector, ω_s be a real scalar, $\hat{P}_s \in \mathbb{R}^{(n+1)\times(n+1)}$ be given by* (8c)*, $X_s \subset \mathbb{R}^n$ be a polyhedron represented as in* (7)*, and $\hat{\mathcal{C}}_{X_s} \subset \mathbb{R}^{n+1}$ the corresponding cone defined by the homogenization procedure. The constrained inequality $x^\top P_s x + 2\nu_s^\top x + \omega_s \geq 0$ for $x \in X_s$, is equivalent to $\hat{P}_s \succcurlyeq^{\hat{\mathcal{C}}_{X_s}} 0$, with $\omega_s = 0$ if $0 \in X_s$.*

PROOF. *The proof follows from Proposition 2 in [24].*

Remark 11 *If $0 \notin X_s$, Lemma 10 is valid also for strict inequalities.*

In order to analyze the sign of the PWQ function (6) we first consider its sign on a single polyhedron X_s by distinguishing the cases $0 \notin X_s$ and $0 \in X_s$. The corollary at the end of the subsection will unify the results obtained for each single polyhedron to $s \in \Sigma_0 \cup \Sigma_1$ and it will express a condition for the sign of the PWQ function on the whole state space in terms of a set of LMIs. First we analyze the easier problem of the sign of a quadratic function over a polyhedron X_s with $0 \notin X_s$.

Lemma 12 *Let $P_s \in \mathbb{R}^{n\times n}$ be a symmetric matrix, $\nu_s \in \mathbb{R}^n$ be a vector and ω_s be a real scalar, $X_s \subset \mathbb{R}^n$ be a polyhedron represented as in (7) with $0 \notin X_s$ and $\hat{\mathcal{C}}_{X_s} \subset \mathbb{R}^{n+1}$ the corresponding cone obtained by applying the homogenization procedure. Consider the following constrained inequalities*

$$x^\top P_s x + 2\nu_s^\top x + \omega_s > 0, \quad x \in X_s, \tag{9a}$$
$$\hat{P}_s \succ^{\hat{\mathcal{C}}_{X_s}} 0 \tag{9b}$$
$$\hat{R}_s^\top \hat{P}_s \hat{R}_s \succ^{\mathbb{R}_+^{\lambda_s+\rho_s}} 0 \tag{9c}$$
$$\hat{R}_s^\top \hat{P}_s \hat{R}_s - N_s \succcurlyeq 0, \tag{9d}$$

with $\hat{P}_s \in \mathbb{R}^{(n+1)\times(n+1)}$ given by (8c), $\hat{R}_s \in \mathbb{R}^{(n+1)\times(\lambda_s+\rho_s)}$ ray matrix of the cone $\hat{\mathcal{C}}_{X_s}$ given by (8b) N_s a symmetric (entrywise) positive matrix. Then the following conditions hold

i) (9a) $\Longleftrightarrow$ (9b)
ii) (9b) $\Longleftrightarrow$ (9c)
iii) (9d) $\Longrightarrow$ (9a).

PROOF. *The proof of i) follows from Remark 11. The application of Lemma 9 with the cone $\hat{\mathcal{C}}_{X_s}$ furnishes a direct proof of ii) and iii).*

Lemma 12 cannot be used when $0 \in X_s$ because in this case the quadratic function is assumed to be zero in the origin, i.e., $\omega_s = 0$, which implies that (9a) is no more equivalent to (9b). In the following lemma we consider the case when the origin belongs to X_s.

Lemma 13 *Let $P_s \in \mathbb{R}^{n\times n}$ be a symmetric matrix, $\nu_s \in \mathbb{R}^n$ be a vector, $X_s \subset \mathbb{R}^n$ be a polyhedron represented as in (7) with $0 \in X_s$, and $\mathcal{C}_{X_s} \subset \mathbb{R}^n$ the corresponding cone defined as the conical hull of X_s. Consider the following constrained inequalities*

$$x^\top P_s x + 2\nu_s^\top x > 0, \quad x \in X_s - \{0\}, \tag{10a}$$
$$x^\top P_s x + 2\nu_s^\top x > 0, \quad x \in \mathcal{C}_{X_s} - \{0\}, \tag{10b}$$
$$x^\top P_s x \geq 0, \quad 2\nu_s^\top x \geq 0, \quad x \in \mathcal{C}_{X_s} - \{0\}, \tag{10c}$$
$$\nexists \tilde{x} \in \mathcal{C}_{X_s} - \{0\} : \tilde{x}^\top P_s \tilde{x} = 0, 2\nu_s^\top \tilde{x} = 0, \tag{10d}$$
$$\theta^\top R_s^\top P_s R_s \theta + 2\nu_s^\top R_s \theta > 0, \ \theta \in \mathbb{R}_+^{\lambda_s+\rho_s} - \{0\}, \tag{10e}$$
$$R_s^\top P_s R_s \succ^{\mathbb{R}_+^{\lambda_s+\rho_s}} 0, 2\nu_s^\top R_s e_i \geq 0, \tag{10f}$$
$$R_s^\top P_s R_s - N_s \succcurlyeq 0, 2\nu_s^\top R_s e_i \geq 0. \tag{10g}$$

with $\{e_i\}_{i=1}^{\lambda_s+\rho_s}$ being the vectors of the standard basis, i.e., all entries equal to zero except for the i-th element equal to 1, $R_s \in \mathbb{R}^{n\times(\lambda_s+\rho_s)}$ ray matrix of the cone $\mathcal{C}_{X_s}$ given by (8a), and N_s a symmetric (entrywise) positive matrix. Then the following conditions hold

i) (10b) $\Longrightarrow$ (10a)
ii) (10b) $\Longleftrightarrow$ (10c) + (10d)
iii) (10b) $\Longleftrightarrow$ (10e)
iv) (10f) $\Longrightarrow$ (10e)
v) (10g) $\Longrightarrow$ (10a).

PROOF. *i) It follows from the fact that $X_s \subseteq \mathcal{C}_{X_s}$.*
ii) implication is proved by contradiction. Assume there exists a $\tilde{x} \in \mathcal{C}_{X_s}$ such that $\tilde{x}^\top P_s \tilde{x} < 0$ ($2\nu_s^\top \tilde{x} < 0$, respectively) with $\tilde{x}^\top P_s \tilde{x} + 2\nu_s^\top \tilde{x} > 0$. Since $\tilde{x} \in \mathcal{C}_{X_s}$ then also $\tau\tilde{x} \in \mathcal{C}_{X_s}$ for any positive real number τ. Therefore from (10b) it must be $\tau^2\tilde{x}^\top P_s \tilde{x} + 2\tau\nu_s^\top \tilde{x} > 0$, for any $\tau > 0$. We can always choose a sufficiently large τ (sufficiently small τ, respectively) such that the sign of this inequality is determined by the dominant quadratic (linear, respectively) term, provided that this term is not zero in $\tilde{x}$, which contradicts the corresponding initial assumption. Then (10c) holds. Moreover the two terms cannot be zero simultaneously because their sum is strictly positive by assumption, hence (10d).
iii) Using Def. 1 for the cone $\mathcal{C}_{X_s}$ we can write $x = R_s\theta$ where $\theta \in \mathbb{R}_+^{\lambda_s+\rho_s}$ which proves the equivalence.
iv) The first of (10f) implies that $\theta^\top R_s^\top P_s R_s \theta > 0$ with $\theta \in \mathbb{R}_+^{\lambda_s+\rho_s} - \{0\}$. Since any positive θ can be written as a linear combination of the standard basis with nonnegative coefficients, (10f) implies (10e).
v) First notice that the first of (10g) implies the first of (10f) from Lemma 9. Then by using iv) of the present Lemma, we get (10g) $\Longrightarrow$ (10e). Finally from iii), ii) and i) we get the implication (10g) $\Longrightarrow$ (10a).

Lemma 12 and Lemma 13 can be applied to the polyhedra of the partition of $\mathbb{R}^n$ in order to get a set of LMIs which provides a sufficient condition for the sign determination of the PWQ function (6). This is synthesized by the following corollary.

Corollary 14 *Consider the PWQ function (6), the matrices $\{\hat{P}_s\}_{s=1}^S$ with $\hat{P}_s \in \mathbb{R}^{(n+1)\times(n+1)}$ given by (8c), the matrices $\{R_s\}_{s\in\Sigma_0}$ with $R_s \in \mathbb{R}^{n\times(\lambda_s+\rho_s)}$ given by (8a) and the matrices $\{\hat{R}_s\}_{s\in\Sigma_1}$ with $\hat{R}_s \in \mathbb{R}^{(n+1)\times(\lambda_s+\rho_s)}$ given by (8b). Assume that $2\nu_s^\top R_s e_i \geq 0$ hold for $i = 1, \ldots, \lambda_s + \rho_s$, $s \in \Sigma_0$. If there exist entrywise positive matrices $\{N_s\}_{s=1}^S$ with $N_s \in \mathbb{R}_+^{(\lambda_s+\rho_s)\times(\lambda_s+\rho_s)}$ such that the set of LMIs*

$$R_s^\top P_s R_s - N_s \succcurlyeq 0, \ s \in \Sigma_0, \tag{11a}$$
$$\hat{R}_s^\top \hat{P}_s \hat{R}_s - N_s \succcurlyeq 0, \ s \in \Sigma_1, \tag{11b}$$

is satisfied, then the PWQ function (6) is strictly positive.

PROOF. *The proof derives by applying v) in Lemma 13 and iii) in Lemma 12 to the polyhedra of the polyhedral partition of* $\mathbb{R}^n$.

Note that the LMIs (11) are not independent because of possible common vertices and/or rays of different polyhedra of the partition. Each matrix of the set $\{\hat{R}_s\}_{s=1}^S$ has at least one common column with another matrix of the set.

3.2 Continuity of the PWQ function

We now formulate the continuity (6) over the intersections of the polyhedra $\{X_s\}_{s=1}^S$ in terms of equality conditions. Note that any common boundary of the polyhedra is itself a polyhedron. For the continuity we do not need to distinguish the two cases whether the origin belongs to X_s or not.

Lemma 15 *Consider the PWQ function* (6) *and the matrices* $\{\hat{P}_s\}_{s=1}^S$ *with* $\hat{P}_s \in \mathbb{R}^{(n+1)\times(n+1)}$ *given by* (8c). *Let* X_i *and* X_j *be two elements of* $\{X_s\}_{s=1}^S$ *such that* $X_i \cap X_j \neq \emptyset$ *and denote by* $\Gamma_{ij} \in \mathbb{R}^{(n+1)\times m_{ij}}$, $m_{ij} < n+1$ *the matrix of the common rays of the corresponding cones* $\hat{\mathcal{C}}_{X_i}$ *and* $\hat{\mathcal{C}}_{X_j}$ *obtained by applying the homogenization procedure. Then* (6) *is continuous on the common boundary between* X_i *and* X_j *if*

$$\Gamma_{ij}^\top (\hat{P}_i - \hat{P}_j)\Gamma_{ij} = 0 \tag{12}$$

for all $i, j \in \{1, \ldots, S\}$, *such that* $X_i \cap X_j \neq \emptyset$.

PROOF. *Clearly* $X_i \cap X_j \neq \emptyset$ *implies* $\hat{\mathcal{C}}_{X_i} \cap \hat{\mathcal{C}}_{X_j} \neq \emptyset$. *The continuity of* (6) *on the polyhedra boundaries can be expressed as*

$$x^\top P_i x + 2\nu_i^\top x + \omega_i = x^\top P_j x + 2\nu_j^\top x + \omega_j, \tag{13}$$

for $x \in X_i \cap X_j$, i *and* $j \in \{1, \ldots, S\}$, *and* $X_i \cap X_j \neq \emptyset$. *Remind that* $\omega_i = \omega_j = 0$ *if* $0 \in X_i \cap X_j$. *The condition* (13) *can be rewritten as*

$$\bar{x}^\top (\hat{P}_i - \hat{P}_j)\bar{x} = 0 \tag{14}$$

where $\hat{P}_i$ *and* $\hat{P}_j$ *are defined according to* (8c), $\bar{x} = \mathrm{col}(x, 1)$ *with* $x \in X_i \cap X_j$. *By considering that any nonzero* $\hat{x} \in \hat{\mathcal{C}}_{X_i \cap X_j}$ *can be written as* $\hat{x} = t\bar{x}$ *with* $t > 0$, *the equality* (14) *is equivalent to*

$$\hat{x}^\top (\hat{P}_i - \hat{P}_j)\hat{x} = 0, \quad \hat{x} \in \hat{\mathcal{C}}_{X_i \cap X_j} \tag{15}$$

Moreover, by Lemma 7, (15) *is equivalent to*

$$\hat{x}^\top \hat{P}_i \hat{x} = \hat{x}^\top \hat{P}_j \hat{x}, \quad \hat{x} \in \hat{\mathcal{C}}_{X_i} \cap \hat{\mathcal{C}}_{X_j} \tag{16}$$

for i *and* $j \in \{1, \ldots, S\}$. *Since* Γ_{ij} *contains the common rays of* $\hat{\mathcal{C}}_{X_i}$ *and* $\hat{\mathcal{C}}_{X_j}$, *for any* $\hat{x} \in \hat{\mathcal{C}}_{X_i} \cap \hat{\mathcal{C}}_{X_j}$ *one can write* $\hat{x} = \Gamma_{ij}\theta$ *with* $\theta \in \mathbb{R}_+^{m_{ij}}$. *Therefore the continuity conditions* (16) *are guaranteed by* (12) *for all* $i, j \in \{1, \ldots, S\}$, *such that* $X_i \cap X_j \neq \emptyset$.

Remark 16 *For those* X_i *and* X_j *such that* $X_i \cap X_j = \{0\}$ *conditions* (12) *are trivial and do not correspond to any constraint on the corresponding matrices* $\hat{P}_i$ *and* $\hat{P}_j$.

The proof of Lemma 15 allows to argue that the continuity of the PWQ function (6) in $\mathbb{R}^n$ across the common boundaries of the polyhedra $\{X_s\}_{s=1}^S$ is equivalent to the continuity of the PWQ form $\hat{x}^\top \hat{P}_s \hat{x}$ with $\hat{P}_s$ given by (8c), across the common boundaries of the cones $\{\hat{\mathcal{C}}_{X_s}\}_{s=1}^S$ in $\mathbb{R}^{n+1}$.

4 Asymptotic stability of PWA systems

We consider the stability problem for the PWA system

$$\dot{x} = A_s x + b_s, \quad x \in X_s, \quad s = 1, \ldots, S \tag{17}$$

where $A_s \in \mathbb{R}^{n\times n}$, $b_s \in \mathbb{R}^n$, $X_s \subset \mathbb{R}^n$ is a polyhedron represented as in (7), $\{X_s\}_{s=1}^S$ is a polyhedral partition of $\mathbb{R}^n$. It is assumed $b_s = 0$ for all $s \in \Sigma_0$ and that the vector field on the polyhedra intersections takes any value among the vector fields defined by the polyhedra sharing that boundary.

4.1 Solution concept

The following solution concept for (17) is adopted.

Definition 17 *Suppose that there are no left accumulation of switches. Given an initial state* $x(0) = x_0$, *a function* $x(t) : [0, \infty) \to \mathbb{R}^n$ *is a solution of the discontinuous system* (17) *in the sense of Caratheodory, if it is absolutely continuous on each compact subinterval of* $[0, \infty)$ *and satisfies* (17) *almost everywhere.*

A result on the existence and the uniqueness of Caratheodory solutions of system (17) can be found in [8]. Definition 17 assumes that the considered system dynamics is not affected by sliding modes or Zeno behavior. A sufficient condition for these exclusions is the continuity of the vector fields over the boundaries [25]. Some preliminary results which consider sliding modes in the simpler case of conewise linear systems are reported in [14]. A sufficient condition for guaranteeing the absence of regular and higher order sliding modes on a common facet between two polyhedra can be expressed in our framework as shown below. Note that, for the partitions of our interest, two polyhedra cannot have more than one common facet.

Lemma 18 *Consider the polyhedra pairs* $\{X_i, X_j\}$ *with* $i \neq j, i, j \in \{1, \ldots, S\}$ *and* $X_i \cap X_j$ *being a common facet. Denote by* λ_{ij} *and* ρ_{ij} *the number of vertices and rays of the polyhedron* $X_i \cap X_j$*, respectively,* $R_{ij} \in \mathbb{R}^{n\times(\lambda_{ij}+\rho_{ij})}$ *the ray matrix of* $\mathcal{C}_{X_i \cap X_j}$*,* $\hat{R}_{ij} \in \mathbb{R}^{(n+1)\times(\lambda_{ij}+\rho_{ij})}$ *the ray matrix of* $\hat{\mathcal{C}}_{X_i \cap X_j}$*. Let* $\{x :\ h_{ij}^\top x + g_{ij} = 0\}$ *with* $h_{ij} \in \mathbb{R}^n$*,* $g_{ij} \in \mathbb{R}$ *be the hyperplane containing* $X_i \cap X_j$*. Define*

$$\hat{Q}_{ij}^{(k)} = \begin{pmatrix} Q_{ij}^{(k)} & \mu_{ij}^{(k)} \\ \mu_{ij}^{(k)\top} & \zeta_{ij}^{(k)} \end{pmatrix}, \tag{18}$$

with

$$Q_{ij}^{(k)} = A_i^{k^\top} h_{ij} h_{ij}^\top A_j^k \tag{19a}$$

$$\mu_{ij}^{(k)} = \frac{1}{2}(A_j^{k^\top} h_{ij} h_{ij}^\top A_i^{k-1} b_i + A_i^{k^\top} h_{ij} h_{ij}^\top A_j^{k-1} b_j) \tag{19b}$$

$$\zeta_{ij}^{(k)} = b_i^\top A_i^{k-1^\top} h_{ij} h_{ij}^\top A_j^{k-1} b_j. \tag{19c}$$

where k *is a positive integer. Assume that the inequalities* $2\mu_{ij}^{(k)^\top} R_{ij} e_\ell \geq 0$ *hold for those* i, j *such that* $0 \in X_i \cap X_j$*,* $\ell = 1, \ldots, \lambda_{ij} + \rho_{ij}$*, and that, if* $k \geq 2$*,*

$$h_{ij}^\top A_i^{\chi-1}(A_i x + b_i) = h_{ij}^\top A_j^{\chi-1}(A_j x + b_j) = 0, \tag{20}$$

for some $x \in X_i \cap X_j$*,* $\chi = 1, \ldots, k-1$*.*
If there exist symmetric (entrywise) positive matrices N_{ij} *such that the following LMI holds*

$$R_{ij}^\top Q_{ij}^{(k)} R_{ij} - N_{ij} \succcurlyeq 0,\ 0 \in X_i \cap X_j \tag{21a}$$

$$\hat{R}_{ij}^\top \hat{Q}_{ij}^{(k)} \hat{R}_{ij} - N_{ij} \succcurlyeq 0,\ 0 \notin X_i \cap X_j, \tag{21b}$$

then the system (17) *does not present sliding of order* k *on the facet* $X_i \cap X_j$*.*

PROOF. A sufficient condition for no regular sliding mode ($k = 1$) to occur on $X_i \cap X_j$ is that the trajectories cross the common facet, which can be expressed as

$$(x^\top A_i^\top + b_i^\top) h_{ij} \cdot h_{ij}^\top (A_j x + b_j) > 0 \tag{22}$$

for all $x \in X_i \cap X_j$. In the case that (20) hold, analogous sufficient conditions can be written for the absence of higher order sliding modes ($k \geq 2$) by taking higher order time derivatives of $h_{ij}^\top x$:

$$(x^\top A_i^\top + b_i^\top) A_i^{k-1^\top} h_{ij} \cdot h_{ij}^\top A_j^{k-1}(A_j x + b_j) > 0 \tag{23}$$

for all $x \in X_i \cap X_j$. Clearly, the expression (23) reduces to (22) in the case $k = 1$. With simple algebraic manipulations and by using (19), the inequality (23) can be rewritten as

$$x^\top Q_{ij}^{(k)} x + 2\mu_{ij}^{(k)^\top} x + \zeta_{ij}^{(k)} > 0, \tag{24}$$

for all $x \in X_i \cap X_j$. Then by applying Corollary 14, one obtains that (21) imply (24) and hence (23).

In general, to find operative conditions for a PWA system to be free from attracting sliding modes along a face of dimension less than $n-1$ and from Zeno behavior is a nontrivial task, see [8,15]. In the following we assume that for each initial condition the system (17) has an absolutely continuous solution in the sense of Def. 17.

4.2 Stability analysis

By exploiting the results obtained in the previous section we can derive a sufficient condition for the asymptotic stability of (17) given by the feasibility of a suitable set of constrained LMIs. Any solution of the proposed set of LMIs directly provides the matrices of a PWQ-LF.

Theorem 19 *Consider the system* (17) *with the polyhedra* $\{X_s\}_{s=1}^S$ *expressed as* (7)*, the matrices* $\{R_s\}_{s\in\Sigma_0}$ *with* $R_s \in \mathbb{R}^{n\times(\lambda_s+\rho_s)}$ *given by* (8a)*, the matrices* $\{\hat{R}_s\}_{s\in\Sigma_1}$ *with* $\hat{R}_s \in \mathbb{R}^{(n+1)\times(\lambda_s+\rho_s)}$ *given by* (8b)*, and define the matrices*

$$\hat{A}_s = \begin{pmatrix} A_s & b_s \\ 0 & 0 \end{pmatrix} \tag{25}$$

with $s \in \Sigma_1$*. Consider the set of LMIs*

$$R_s^\top P_s R_s - N_s \succcurlyeq 0 \tag{26a}$$

$$-R_s^\top (A_s^\top P_s + P_s A_s) R_s - M_s \succcurlyeq 0 \tag{26b}$$

for all $s \in \Sigma_0$*, and*

$$\hat{R}_s^\top \hat{P}_s \hat{R}_s - N_s \succcurlyeq 0 \tag{27a}$$

$$-\hat{R}_s^\top (\hat{A}_s^\top \hat{P}_s + \hat{P}_s \hat{A}_s) \hat{R}_s - M_s \succcurlyeq 0 \tag{27b}$$

for all $s \in \Sigma_1$*, where* $\hat{P}_s \in \mathbb{R}^{(n+1)\times(n+1)}$ *are symmetric matrices in the form* (8c)*,* N_s*,* M_s *are symmetric (entrywise) positive matrices of appropriate dimensions, together with the set of inequalities*

$$2\nu_s^\top R_s e_i \geq 0, \quad -2\nu_s^\top A_s R_s e_i \geq 0 \tag{28}$$

for $i = 1, \ldots, \lambda_s + \rho_s$*,* $s \in \Sigma_0$*. If the set of LMIs* (26)–(27) *subject to the equality constraints* (12) *and to the inequality constraints* (28) *has a solution* $\{P_s, \nu_s, \omega_s, N_s, M_s\}_{s=1}^S$ *with* $\omega_s = 0$ *for* $s \in \Sigma_0$*, then the system* (17) *is globally asymptotically stable.*

PROOF. *Choose the PWQ function* (6) *as a candidate Lyapunov function. By following arguments similar to [16], it can be shown that if the function* (6) *is continuous, strictly positive, radially unbounded and strictly decreasing along any solution $x(t)$ of* (17)*, then the system is globally asymptotically stable.*
From (12) *through Lemma 15 it follows that the function* (6) *is continuous across the polyhedra boundaries and then it is continuous in the whole state space.*
From (26a), (27a)*, the first of* (28) *and by using Corollary 14, the PWQ function* (6) *is strictly positive in* $\mathbb{R}^n - \{0\}$.
In order to show that (6) *is radially unbounded, consider X_s unbounded in the two cases $s \in \Sigma_0$ and $s \in \Sigma_1$. In the former case clearly if* (6) *is radially unbounded in $\mathcal{C}_{X_s}$ then it is also radially unbounded along any ray of $\mathcal{C}_{X_s}$ contained in X_s. Consider a $\tilde{x} \in \mathcal{C}_{X_s}$ then also $\tau\tilde{x} \in \mathcal{C}_{X_s}$ for any positive real number τ. Therefore for all $x = \tau\tilde{x}$ it is*

$$\lim_{\|x\|\to+\infty} V(x) = \lim_{\tau\to+\infty} (\tau^2 \tilde{x}^\top P_s \tilde{x} + 2\tau \nu_s^\top \tilde{x}) = +\infty \quad (29)$$

where we used (10g) $\implies$ (10c)+(10d) *from Lemma 13. In the case $s \in \Sigma_1$ consider*

$$\lim_{\|x\|\to+\infty} V(x) = \lim_{\|\bar{x}\|\to+\infty} \bar{x}^\top \hat{P}_s \bar{x} \quad (30)$$

where $\bar{x} = \mathrm{col}(x, 1)$ and $x \in X_s$. Since $\bar{x} \in \hat{\mathcal{C}}_{X_s}$, by using (9b)*, which is implied by* (27a)*, we can conclude that the limit* (30) *is infinity. Then* (6) *is radially unbounded.*
Definition 17 assumes that $x(t)$ does not remain on the polyhedra boundaries for any time interval. Therefore (12) *allow to say that* (6) *is continuous and piecewise differentiable when it is evaluated along any solution $x(t)$ of* (17)*. Define the "upper derivative" of V along the solutions of* (17) *as*

$$\dot{V}^*(x) = \max\{\dot{V}_s(x)\}_{s\in\Sigma(x)}, \quad x \in \bigcap_{s\in\Sigma(x)} X_s \quad (31)$$

where $\Sigma(x)$ is the set of all indices $s \in \{1, \ldots, S\}$ such that $x \in X_s$, and

$$\begin{aligned} \dot{V}_s(x) =& x^\top (A_s^\top P_s + P_s A_s) x \\ &+ 2(\nu_s^\top A_s + b_s^\top P_s) x + 2\nu_s^\top b_s, \end{aligned} \quad (32)$$

$s = 1, \ldots, S$. Note that for $x \in int(X_{\bar{s}})$ it is $\Sigma(x) = \bar{s}$ and $\dot{V}^*(x) = \dot{V}_{\bar{s}}(x)$. For $s \in \Sigma_0$ it is $b_s = 0$; from (26b) *together with the second of* (28) *and by using Corollary 14 with $-(A_s^\top P_s + P_s A_s) \leftarrow P_s$ it follows that* (32) *is strictly negative for all $s \in \Sigma_0$. For $s \in \Sigma_1$ from direct substitution of* (25) *and* (8c) *one has*

$$\hat{A}_s^\top \hat{P}_s + \hat{P}_s \hat{A}_s = \begin{pmatrix} A_s^\top P_s + P_s A_s & A_s^\top \nu_s + P_s b_s \\ \nu_s^\top A_s + b_s^\top P_s & 2\nu_s^\top b_s \end{pmatrix}. \quad (33)$$

The matrix (33) *is related to* (32) *similarly to how the matrix* (8c) *is related to* (6)*. From* (27b) *and by using Corollary 14 with $-(\hat{A}_s^\top \hat{P}_s + \hat{P}_s \hat{A}_s) \leftarrow \hat{P}_s$ and* (32) *instead of* (6)*, it follows that* (32) *is strictly negative for all $s \in \Sigma_1$. Then* (31) *is strictly negative. For any $t > 0$ denote by $\{t_i\}_{i=1}^N$ the strictly increasing sequence of time instants in $(0, t)$ such that, for all i, $x(t_i)$ lies on the boundary of X_s for some s. With some abuse of notation let $t_0 = 0$ and $t_{N+1} = t$. Therefore, since* (31) *is strictly negative and* (6) *is continuous one can write*

$$\begin{aligned} V(x(t)) &= V(x(0)) + \sum_{i=0}^{N} \int_{t_i}^{t_{i+1}} \dot{V}^*(x(\tau)) d\tau \\ &\le V(x(0)) - \sum_{i=0}^{N} \gamma_i (t_{i+1} - t_i) \\ &\le V(x(0)) - \gamma t, \end{aligned} \quad (34)$$

where $-\gamma_i = \max_{t\in[t_i, t_{i+1}]} \dot{V}^*(x(t))$ is strictly negative and $-\gamma = \max\{-\gamma_i\}_{i=0}^N$ is strictly negative. Therefore (6) *is strictly decreasing along any solution $x(t)$ of* (17) *and goes asymptotically to zero, which completes the proof.*

The conditions of the stability theorem above can be simplified, for instance, by dropping the continuity constraints (12) when for any given i the state flows exclusively from the region X_i to some region X_j (and not from X_j to X_i).

Remark 20 *The proposed approach and Theorem 19 can be used for quadratic forms too by setting $\nu_s^\top = 0$ and $\omega_s = 0$ in* (6)*. On the other hand, the linear terms allow to increase the number of variables which is useful for the verification of the continuity constraints* (12)*. In particular, the use of the linear terms ν_s allows to increase by nS the number of decision variables.*

Remark 21 *Theorem 19 can be reformulated for studying local stability in a bounded (polyhedral) region containing the origin and proved by using similar arguments, provided that the region is an invariant set. In the case of bounded region which is not an invariant set for the system* (17)*, the local stability result is relative to the largest level set of the Lyapunov function that is fully contained in the analysis region [15]. However, finding such a region can be a complex problem especially in higher dimensions.*

Remark 22 *An analogous of Theorem 19 can be proved by considering a PWQ-LF defined on a new polyhedral partition of the state space obtained by refining the partition indicated by the PWA system structure and by using the same system matrices for the refined polyhedra.Although there exist algorithms for automated partition refinements based on the simplex partition, e.g., the* *bisection along the longest edge technique, see [13,15], the sub-partitioning procedures are not ensured to provide a positive stability answer in finite-time.*

Remark 23 *The approach of this paper can be applied also to discrete-time PWA systems, once the transitions from one polyhedron to another, which can be determined by a reachability analysis, see [6], are known. In this case the continuity conditions can be dropped and the decreasing of the candidate Lyapunov function can be obtained and expressed in terms of LMIs by using Lemma 12 and Lemma 13.*

5 Examples

The problem (26)–(28) with (12) has a number of decision variables which depends on the number of vertices and rays of the polyhedra. For each $s \in \Sigma_0$ there are $0.5n^2 + n + (\lambda_s + \rho_s)^2 + \lambda_s + \rho_s$ decision variables corresponding to P_s, ν_s, N_s and M_s, while for each $s \in \Sigma_1$ there are $0.5n^2 + 1.5n + (\lambda_s + \rho_s)^2 + \lambda_s + \rho_s + 1$ decision variables corresponding to $\hat{P}_s$, N_s and M_s. The inequalities and equalities to be satisfied are: the $2S$ LMIs (26)–(27), the $2\sum_{s\in\Sigma_0}(\lambda_s + \rho_s)$ linear inequalities (28) and the linear matrix equalities (12) whose number is given by the polyhedra pairs sharing a common boundary. In this section the proposed PWQ-LF approach is applied for the stability analysis of a Hegselmann-Krause consensus model and of two saturated systems. The numerical results are obtained by using `CDD` [7] and `CVX` [9] with a PC Intel Dual Core processor at 2.4 GHz.

5.1 Multiagent consensus model

The Hegselmann-Krause model is widely analyzed in the literature dealing with the consensus in multiagent systems, see among others [20,26]. The model consists of a set of N autonomous agents, with a state variable $\xi_i \in \mathbb{R}$ for each agent whose dynamics is described by

$$\dot{\xi}_i = \sum_{j=1}^{N} \phi(\xi_i, \xi_j)(\xi_j - \xi_i) \tag{35}$$

with $\xi_i \in [0,1]$, $i = 1,\dots,N$, $\phi : [0,1]^2 \to \{0,1\}$ a weight function defined as $\phi(\xi_i, \xi_j) = 1$ if $|\xi_i - \xi_j| \le d$ and $\phi(\xi_i, \xi_j) = 0$ if $|\xi_i - \xi_j| \ge d$, $d \in \mathbb{R}_+ - \{0\}$. The dynamics (35) preserves the average $\alpha = \frac{1}{N}\sum_{i=1}^{N} \xi_i(0)$. Depending on the initial conditions, the convergence to an equilibrium or a clustering with different steady state values can occur [20].
By introducing the state translation $x_i = \xi_i - \alpha$, for $i = 1,\dots,N-1$ and by using $\xi_N = \alpha - \sum_{i=1}^{N-1} x_i$, the transformed model becomes

$$\begin{aligned} \dot{x}_i = &\sum_{j=1}^{N-1} \phi\left(x_i, x_j\right)\left(x_j - x_i\right) \\ &+ \phi(x_i + \sum_{j=1}^{N-1} x_j, 0)\,(x_i + \sum_{j=1}^{N-1} x_j) \end{aligned} \tag{36}$$

for $i = 1,\dots,N-1$. The feasibility region in $\mathbb{R}^{N-1}$ can be obtained by considering that $\xi_i \in [0,1]$ implies $x_i \in [-\alpha, 1-\alpha]$, $i = 1,\dots,N-1$, and $\xi_N \in [0,1]$ implies $\sum_{i=1}^{N-1} x_i \in [\alpha - 1, \alpha]$. By assuming $N = 3$, $\alpha = 0.5$ and $d = 0.5$, the feasibility region of (36) is represented by the larger hexagon with dashed sides shown in Fig. 1. By referring to the partition in Fig. 1, the model (36) can be written in the form (17) with

$$A_s = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} \text{ for s=1... 6},$$

$$A_7 = A_8 = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix},\ A_9 = A_{10} = \begin{pmatrix} -1 & 1 \\ 0 & -3 \end{pmatrix},$$

$$A_{11} = A_{12} = \begin{pmatrix} -3 & 0 \\ 1 & -1 \end{pmatrix},\ A_{13} = A_{14} = \begin{pmatrix} -2 & -1 \\ 0 & 0 \end{pmatrix},$$

$$A_{15} = A_{16} = \begin{pmatrix} 0 & 0 \\ -1 & -2 \end{pmatrix},\ A_{17} = A_{18} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

and $b_s = 0$ for all $s \in \{1,\dots,18\}$. The local asymptotic stability of the origin of (36) can be analyzed by using the proposed PWQ-LF approach. By applying Theorem 19, we found a PWQ-LF for the PWA dynamics in the star-shape region contained in the feasibility domain. In the same region the no-sliding conditions (21) are satisfied. From Proposition 2.1 in [20] and Proposition 3.3 in [26] one can deduce that the star-shape region in Fig. 1 is an invariant set and hence the origin is asymptotically stable for any initial condition belonging to that region. Fig. 2 shows a state trajectory and some PWQ-LF level curves which confirm the stability result.

5.2 Stability with saturating control

Consider the Lur'e system with saturation feedback represented as

$$\dot{x} = Ax + bu, \tag{37a}$$

$$u = \mathrm{sat}(f^\top x), \quad -1 \le u \le 1. \tag{37b}$$

The state space can be divided into the three regions $\Omega_1 = \{x \in \mathbb{R}^n : f^\top x \ge 1\}$, $\Omega_2 = \{x \in \mathbb{R}^n : -1 \le f^\top x \le 1\}$, $\Omega_3 = \{x \in \mathbb{R}^n : f^\top x \le -1\}$. For each saturated

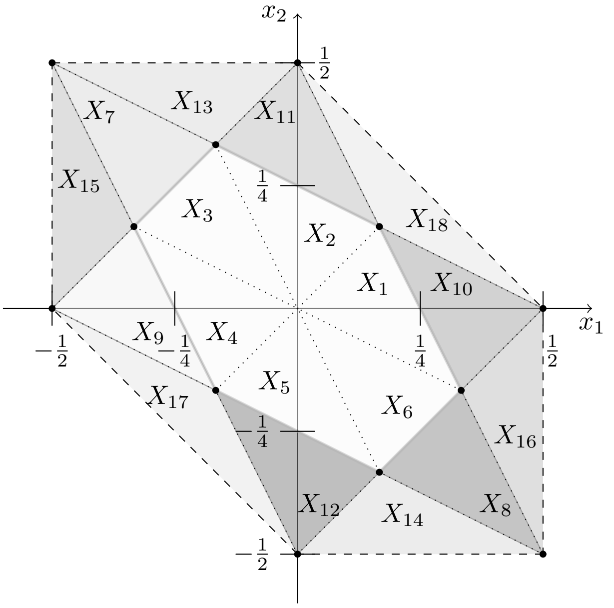

Fig. 1. State space polyhedral partition for (36) with $d = 0.5$, $\alpha = 0.5$ and $i = 1, 2$.

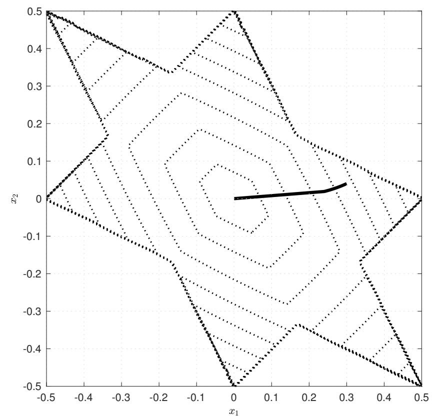

Fig. 2. State space for the model (36): a state trajectory (black line) and some level curves of the PWQ-LF (dotted lines).

region, the model can be written in the form (17) with $A_1 = A_3 = A$, $A_2 = A + bf^\top$, $b_1 = -b_3 = b$, $b_2 = 0$. The continuity of the vector fields implies that the system does not exhibit sliding behavior.

Consider the choice analyzed in [19]:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 5 \end{pmatrix}, \; f = \begin{pmatrix} -2 \\ -1 \end{pmatrix}. \tag{38}$$

By exploiting Remark 22 and by applying Theorem 19 we obtained a PWQ-LF with the partition in 10 polyhedra of the (bounded) region surrounding the origin, see Fig. 3. In particular, the local asymptotic stability

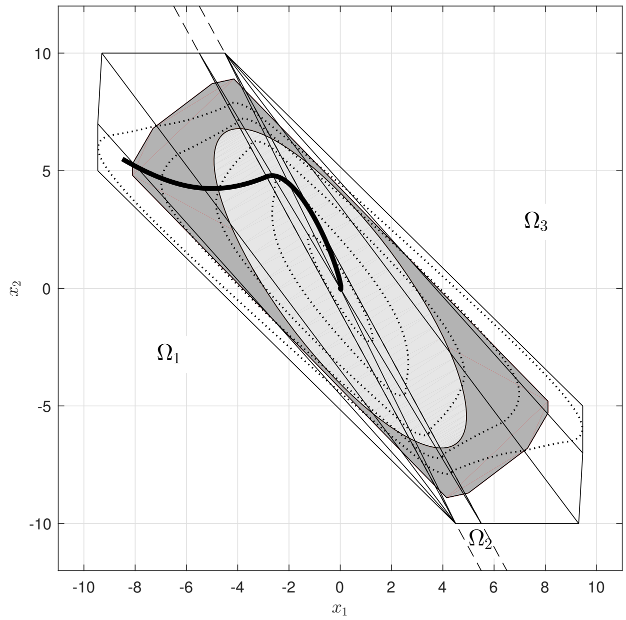

Fig. 3. State space for the system (37)–(38): the polyhedral partition (black lines) of the (bounded) region of analysis, a state trajectory (thick black line), some level curves of the PWQ-LF (dotted lines), the regions of attraction reported in [19] (light gray and light gray+gray areas).

of the origin is guaranteed for all initial conditions belonging to the region delimited by the thick red line, i.e., to the largest level set of the PWQ-LF contained in the region of analysis. Our approach provides a stability region whose area is about 9% wider than the one obtained with the approach in [19], where a PWA function with 30 symmetrical facets is employed.

The validity of Theorem 19 for global stability is checked by considering the saturating control system described by Example 1 in [4] which can be represented in the form (37) with the following matrices

$$A = \begin{pmatrix} -2 & 1 \\ -3 & 1 \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \; f = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{39}$$

By exploiting Remark 22 and by applying Theorem 19, we obtained a solution to the LMIs (26)–(27), i.e., a PWQ-LF, and then the global asymptotic stability of the origin by considering a partition of the state space in 12 polyhedra, see Fig. 4.

6 Conclusions

PWQ functions have been analyzed as candidate Lyapunov functions for the stability analysis of PWA systems. We have shown that the cone-copositivity approach and the homogenization procedure allow to exploit the advantages of having both quadratic and linear

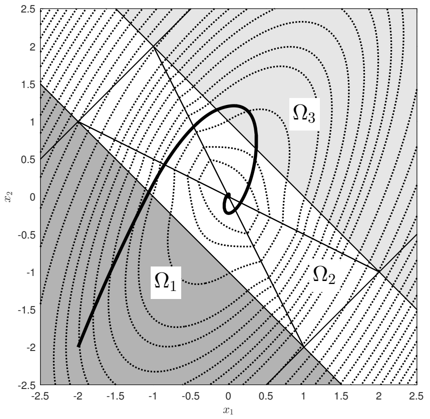

Fig. 4. State space for the system (37) with (39): the polyhedral partition (black lines) of $\mathbb{R}^2$, a state trajectory (thick black line), some level curves of the PWQ-LF (dotted lines).

terms in the Lyapunov function. By using the cone-copositivity idea we derived conditions for the existence of a continuous PWQ-LF and hence for the asymptotic stability of PWA systems. The stability conditions are expressed with LMIs constrained by continuity conditions. As a byproduct the result can be applied for searching PWQ-LFs with a state space partition further refined with respect to that originally dictated by the PWA system structure. The proposed approach has been shown to be successful for predicting the region of asymptotic stability of a Hegselmann-Krause consensus model and an input saturation system, and for the global asymptotic stability of a saturating control system. Possible directions for future research are a dual reformulation of the stability results based on the polyhedra $\mathcal{H}$-representation, the inclusion of sliding modes and conditions for the existence of a PWQ-LF for any asymptotically stable PWA system.

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